Introduction to Riemannian manifolds

All manifolds will be connected, Hausdorff and second countable.

Terminology. Let M be a smooth manifold. Denote the tangent space at $x \in M$ by T_xM. If $f: M \to N$ is a smooth map between smooth manifolds, denote the associated map on T_xM by $(Df)_x: T_xM \to T_{f(x)}N$. If I is an open interval in $\mathbb{R}$ and $\alpha: I \to M$ is a smooth path, then for $t \in I$, $\alpha'(t)$ denotes $(D\alpha)_t(1) \in T_{\alpha(t)}M$.

Definition. A *Riemannian metric* on a smooth manifold M is a choice at each point $x \in M$ of a positive definite inner product $\langle\ ,\ \rangle$ on T_xM, the inner products varying smoothly with x. Then M is known as a *Riemannian manifold*. We will not give a formal definition of the phrase 'varying smoothly with x'.

Definition. A *local isometry* between two Riemannian manifolds M and N is a local diffeomorphism $h: M \to N$, such that, for all points $x \in M$ and all vectors v and w in T_xM,

$$\langle v, w\rangle = \langle (Dh)_x(v), (Dh)_x(w)\rangle.$$

A *(Riemannian) isometry* is a local isometry that is also a diffeomorphism.

Let M be a Riemannian manifold and let x be a point in M. The Riemannian metric allows one to define for a vector $v \in T_xM$ the length $||v|| = \langle v, v\rangle^{1/2}$ and also the angle between two non-zero vectors v and w in T_xM:

$$\cos(\mathrm{Angle}(v, w)) = \frac{\langle v, w\rangle}{||v||\,||w||}.$$

The lengths $||\ ||$ determine the inner product: if $v, w \in T_xM$, then

$$\langle v, w\rangle = (||v + w||^2 - ||v||^2 - ||w||^2)/2.$$

So, a diffeomorphism which preserves the lengths of vectors is necessarily a Riemannian isometry.

Smooth paths $\alpha: [0, T] \to M$ inherit a length, given by

$$\mathrm{Length}(\alpha) = \int_0^T ||\alpha'(t)||\, dt.$$

This is independent of its parametrisation - in other words, if $\beta: [0, T_1] \to [0, T]$ is a diffeomorphism, then $\mathrm{Length}(\alpha \circ \beta) = \mathrm{Length}(\alpha)$. This is just a consequence of the fact that we can change the variable in the integration. A *piecewise smooth* path $\alpha\colon [0, T] \to M$ is a path that is smooth at all but finitely many points. Piecewise smooth paths also inherit a length. We construct a metric d on M: if x and y are points in M, then

$$d(x, y) = \inf\{\text{Length}(\alpha) : \alpha \text{ is a piecewise smooth path from } x \text{ to } y\}.$$

Proposition 1.1. *This does give a metric on M. The topology induced by this metric coincides with the original topology on M.*

Notation. If x is a point in a metric space M and $\epsilon > 0$, denote $\{y \in M : d(x, y) < \epsilon\}$ by $B_\epsilon(x)$.

Crucial in the study of Riemannian manifolds is the notion of a geodesic. Here's a non-standard definition, which is equivalent to the usual one.

Definition. A *geodesic* (with speed $s \in \mathbb{R}_{\geq 0}$) is a smooth map $\alpha\colon I \to M$ (where I is an interval in $\mathbb{R}$) such that $||\alpha'(t)|| = s$ for all $t \in I$ and which is 'locally length minimising'. This means that for all $t \in I$, there is an $\epsilon > 0$, such that for all t_1 and t_2 in $(t - \epsilon, t + \epsilon) \cap I$,

$$d(\alpha(t_1), \alpha(t_2)) = s|t_1 - t_2|.$$

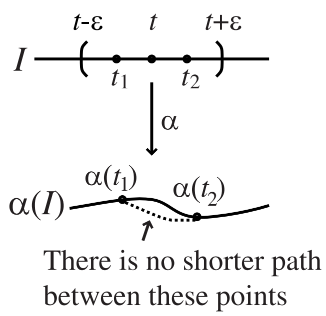

Figure 1.

Exercise. 1. The geodesics in $\mathbb{R}^n$ are straight lines.

2. The geodesics in S^2 are great circles.

Remarks. 1. This demonstrates that geodesics need not be globally length-minimising. In other words, it need not be true that

$$d(\alpha(t_1), \alpha(t_2)) = s|t_1 - t_2|$$

for all $t_1, t_2 \in I$. For example, great circles in S^2. This example also demonstrates that there need not be a unique shortest path between two points.

2. The maximal interval $I \subset \mathbb{R}$ on which a geodesic is defined need not be the whole of $\mathbb{R}$. For example, consider geodesics in the open unit disc in $\mathbb{R}^2$.

3. There need not be a shortest path between two points. For example, consider the points $(-1, 0)$ and $(1, 0)$ in $\mathbb{R}^2 - \{(0, 0)\}$. But if there is a shortest path between two points, then we may find one which has constant speed. This is necessarily globally length minimising and hence a geodesic.

4. A local isometry between Riemannian manifolds (for example, the inclusion of an open subset) preserves geodesics.

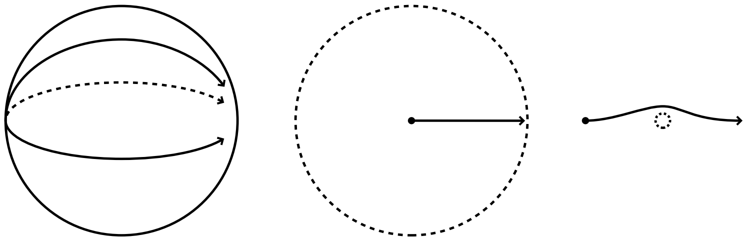

Figure 2.

A fundamental result from differential geometry is the following.

Theorem 1.2. [Existence and uniqueness of geodesics] *For all points $x \in M$ and for all $v \in T_xM$, there is a unique maximal interval $I \subset \mathbb{R}$ containing a neighbourhood of 0, and a unique geodesic $\alpha: I \to M$, such that $\alpha(0) = x$ and $\alpha'(0) = v$.*

Idea of proof. Pick a chart $\phi: U \to \mathbb{R}^n$ around x. For each path $\alpha: [-T, T] \to U$, consider $\phi \circ \alpha: [-T, T] \to \mathbb{R}^n$. One shows that α is a geodesic if and only if the n co-ordinates of $\phi \circ \alpha$ satisfy certain second order differential equations. These differential equations have a solution for small enough T, which is unique given the initial conditions $\alpha(0) = x$ and $\alpha'(0) = v$. □

Definition. The *exponential map* at a point $x \in M$ is the map $\exp_x$ from a subset of T_xM to a subset of M which takes a vector $v \in T_xM$ to $\alpha(1)$, where $\alpha: I \to M$

is the geodesic from Theorem 1.2 with $\alpha(0) = x$ and $\alpha'(0) = v$, *providing* $1 \in I$.

Proposition 1.3. *For each point* $x \in M$, $\exp_x$ *is a smooth map, whose domain is an open neighbourhood of* 0. *For small enough* $\epsilon > 0$, $\exp_x$ *maps* $B_\epsilon(0) \subset T_xM$ *diffeomorphically onto* $B_\epsilon(x) \subset M$.

Idea of proof. As in Theorem 1.2, one relates geodesics to certain second order differential equations, and then one uses the fact that their solutions are smooth and depend smoothly on the initial conditions. For the second part, one first determines the derivative of $\exp_\mathrm{x}$ and discovers that it has maximal rank. Hence, the inverse function theorem gives that $\exp_x$ sends $B_\epsilon(0) \subset T_xM$ diffeomorphically onto its image in M, which is clearly $B_\epsilon(x) \subset M$. □

Proposition 1.4. *If* $h: M \to N$ *is a local isometry between Riemannian manifolds, and* $x \in M$, *then the following diagram commutes (where the maps are defined):*

$$\begin{array}{ccc} T_xM & \xrightarrow{(Dh)_x} & T_{h(x)}N \\ \downarrow \exp_x & & \downarrow \exp_{h(x)} \\ M & \xrightarrow{h} & N \end{array}$$

Proof. Pick $v \in T_xM$. Let α be the unique geodesic in M with $\alpha(0) = x$ and $\alpha'(0) = v$. Since h is a local isometry, it preserves geodesics and so $h \circ \alpha$ is a geodesic in N. But $(h \circ \alpha)(0) = h(x)$ and $(h \circ \alpha)'(0) = (Dh)_x(v)$. Therefore, the uniqueness part of Theorem 1.2 gives that $h(\exp_x(v)) = (h \circ \alpha)(1) = \exp_{h(x)}((Dh)_x(v))$. □

Theorem 1.5. *Let* M *and* N *be Riemannian manifolds, with* M *connected. Let* $h: M \to N$ *and* $k: M \to N$ *be local isometries onto their images. Suppose that for some* $x \in M$, $h(x) = k(x)$ *and* $(Dh)_x = (Dk)_x$. *Then* $h = k$.

Proof. Consider the set

$$U = \{y \in M : h(y) = k(y) \text{ and } (Dh)_y = (Dk)_y : T_yM \to T_{h(y)}N\}.$$

We first show that U is open. Pick $y \in U$. By Proposition 1.4, the following diagram commutes:

$$\begin{array}{ccccc} T_{h(y)}M & \xleftarrow{(Dh)_y} & T_yM & \xrightarrow{(Dk)_y} & T_{k(y)}M \\ \downarrow \exp_{h(y)} & & \downarrow \exp_y & & \downarrow \exp_{k(y)} \\ N & \xleftarrow{h} & M & \xrightarrow{k} & N \end{array}$$

But $h(y) = k(y)$ and $(Dk)_y = (Dh)_y$. Therefore $h = k$ on the image of $\exp_y$, which is a neighbourhood of y by Proposition 1.3. If $h = k$ on an open set, then $(Dh) = (Dk)$ there. Therefore, U is open.

Now, we show that U is closed. Let $\{y_i : i \in \mathbb{N}\}$ be a sequence of points in U, tending to some point y in M. Then $h(y) = \lim_{i\to\infty} h(y_i) = \lim_{i\to\infty} k(y_i) = k(y)$. Similarly, $(Dh)_y = (Dk)_y$. So, $y \in U$. Therefore, U is closed. Since U is open, closed and non-empty, and M is connected, $U = M$. Therefore $h = k$. □

Remark. A Riemannian manifold M has a (possibly infinite) volume. For each $x \in M$, the paralleliped in T_xM spanned by n orthonormal vectors is defined to have volume 1. By integrating over M, this determines its volume. Compact Riemannian manifolds always have finite volume.