EVERY FINITE GROUP ADMITS A JUST FINITE PRESENTATION

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Abstract. A finite presentation $\langle X \mid R\rangle$ of a finite group G is called "just finite" if removing any relation from R results in a presentation for an infinite group. It has been an open question (Kourovka Notebook, Problem 21.10) whether every finite group admits such a presentation. We resolve this conjecture in the affirmative.

1. Introduction

Inspired by the extensive theory of just infinite groups [5, 6, 12, 13, 2, 8, 9], Barnea introduced a notion that is, in some sense, dual [1]. He defined a *just finite* presentation of a finite group G, which is a presentation $\langle X \mid R\rangle$ for the group with the property that, for every $r \in R$, the group $\langle X \mid R \setminus \{r\}\rangle$ is infinite. In Problem 21.10 of the Kourovka Notebook [4], Barnea asked whether every finite group admits a just finite presentation. The goal of this note to prove that this is indeed the case.

Theorem 1. *Every finite group admits a finite, just finite presentation.*

This is actually a consequence of a stronger result. Recall that a group G has *Property (FA)* if any action of G on a tree has a fixed point. It is a fundamental theorem of Serre [10] that a finitely generated group has Property (FA) if and only if it is neither a non-trivial amalgamated free product nor an HNN extension. We say that a presentation $\langle X \mid R\rangle$ for a group with Property (FA) is *just-(FA)* if, for every $r \in R$, the group $\langle X \mid R \setminus \{r\}\rangle$ does not have Property (FA).

Theorem 2. *Every finitely presented group with Property (FA) admits a finite just-(FA) presentation.*

This implies Theorem 1, since a finite group G clearly has Property (FA). Hence, it admits a finite just-(FA) presentation, which is therefore just-finite.

We also note the following consequence, concerning groups G with Kazhdan's Property (T) [3]. We say that a presentation $\langle X \mid R\rangle$ for such a group

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is *just-(T)* if, for every $r \in R$, the group $\langle X \mid R \backslash \{r\} \rangle$ does not have Property (T). Since Property (T) implies Property (FA) for countable discrete groups [11], Theorem 2 gives the following result.

Theorem 3. *Every finitely presented group with Property (T) admits a finite just-(T) presentation.*

2. The Construction

The aim is to start with a finite presentation $\langle X \mid R \rangle$ of the given group G with Property (FA) and modify it to a just-(FA) presentation. Each relation will be substituted for two new relations that have a form based on the following lemma.

Lemma 4 (B.H. Neumann [7])**.** *Let H be a group with elements $u, v \in H$ satisfying $u^{-1}vu = v^2$ and $v^{-1}uv = u^2$. Then $u = 1$ and $v = 1$.*

Proof. From the second relation, $v^{-1}uv = u^2$, we can write the commutator $[u, v] = u^{-1}(v^{-1}uv) = u^{-1}u^2 = u$. From the first relation, $u^{-1}vu = v^2$, we can invert it to get $u^{-1}v^{-1}u = v^{-2}$. We can then write the commutator as $[u, v] = (u^{-1}v^{-1}u)v = v^{-2}v = v^{-1}$. Equating the two expressions for the commutator, we have $u = v^{-1}$. Substituting this into the first relation $u^{-1}vu = v^2$ yields $(v^{-1})^{-1}v(v^{-1}) = v^2$, which simplifies to $v = v^2$. This implies $v = 1$, and therefore $u = 1$. □

The following type of semi-direct product of cyclic groups plays a technical role in our proof.

Lemma 5. *For $k > 1$, the presentation $\langle x, y \mid x^{-1}yx = y^2, x^k = 1 \rangle$ specifies the group $\mathbb{Z}_{2^k-1} \rtimes \mathbb{Z}_k$, where $\langle x \rangle = \mathbb{Z}_k$ and $\langle y \rangle = \mathbb{Z}_{2^k-1}$ and the conjugation action of x on $\langle y \rangle$ is $y \mapsto y^2$.*

Proof. There is a homomorphism $\langle x, y \mid x^{-1}yx = y^2, x^k = 1 \rangle \to \mathbb{Z}_{2^k-1} \rtimes \mathbb{Z}_k$, since the two relations are satisfied in the image group. It is surjective because the images of x and y generate $\mathbb{Z}_{2^k-1} \rtimes \mathbb{Z}_k$. Any element in $\langle x, y \mid x^{-1}yx = y^2, x^k = 1 \rangle$ can be written as $x^m y^n$ for some integers m and n. Using $x^k = 1$, we may assume $0 \le m < k$. Conjugating y by x a total of k times gives y^{2^k}. Since $x^k = 1$, we deduce that $y^{2^k} = y$. So, we may also assume $0 \le n < 2^k - 1$. So $\langle x, y \mid x^{-1}yx = y^2, x^k = 1 \rangle$ has at most $k(2^k - 1)$ elements. However, $\mathbb{Z}_{2^k-1} \rtimes \mathbb{Z}_k$ has exactly this many elements, and therefore the above surjective homomorphism is an isomorphism. □

We are now in a position to prove our main theorem.

Proof of Theorem 2. Let G be a finitely presented group with Property (FA). If G is cyclic, G must be a finite cyclic group $\mathbb{Z}_n$ and so admits the balanced, just-(FA) presentation $\langle x \mid x^n = 1 \rangle$.

Assume G is non-cyclic. Let $\langle X \mid R \rangle$ be an irredundant finite presentation for G (that is, for any $r \in R$, the group $H_r = \langle X \mid R \setminus \{r\} \rangle$ is not isomorphic to G, implying $r \neq 1$ in H_r).

We define a new presentation $P' = \langle X' \mid R' \rangle$ as follows:

- $X' = X \cup \{b_r \mid r \in R\}$
- $R' = \{r^{-1}b_r r = b_r^2 \mid r \in R\} \cup \{b_r^{-1} r b_r = r^2 \mid r \in R\}$

By Lemma 4, these relations force $b_r = 1$ and $r = 1$ for all $r \in R$. (See Section 3 for a specific example of such a presentation P'.)

We claim that P' presents G. Consider what happens to the group $G = \langle X \mid R \rangle$ when a new generator b_r is added, the relation r is removed and then replaced by the two new relations $r^{-1}b_r r = b_r^2$ and $b_r^{-1} r b_r = r^2$. These force $b_r = 1$ and $r = 1$. Conversely $b_r = 1$ and $r = 1$ imply the two new relations $r^{-1}b_r r = b_r^2$ and $b_r^{-1} r b_r = r^2$. Thus, instead of adding these two new relations, we could have simply set $b_r = 1$ and $r = 1$. In other words, the new group is obtained from $G = \langle X \mid R \rangle$ by adding the new generator b_r and a new relation $b_r = 1$. Hence, this does not change the group. Repeating this for each relation in R, we deduce that P' does present G.

We must show P' is just-(FA). Let $r \in R$. We show that removing either of the corresponding relations yields a group without Property (FA).

Case 1: Remove $b_r^{-1} r b_r = r^2$. Let K_1 be the resulting group. As above, let H_r be the group $\langle X \mid R \setminus \{r\} \rangle$. This also has a presentation obtained from P' by removing the generator b_r and the relations $r^{-1}b_r r = b_r^2$ and $b_r^{-1} r b_r = r^2$. Let $k \in \mathbb{N} \cup \{\infty\}$ be the order of r in H_r. Since R is irredundant, $k > 1$. Thus K_1 can be decomposed as an amalgamated free product:

$$K_1 \cong H_r *_{\langle r \rangle = \langle x \rangle} B'$$

where $B' = \langle x, b_r \mid x^{-1}b_r x = b_r^2, x^k = 1 \rangle$. When $k = \infty$, we do not include the relation $x^k = 1$. Thus, when $k = \infty$, B' is the Baumslag-Solitar group $B(1, 2)$. When k is finite, Lemma 5 gives that the group B' is the semi-direct product $\mathbb{Z}_{2^k-1} \rtimes \mathbb{Z}_k$, and the index of the amalgamating subgroup $\langle x \rangle$ in B' is $2^k - 1$. Thus, when $k = \infty$ or $k \geq 2$, this index is at least 3.

An amalgamated free product $A *_C B$ is non-trivial when $C \neq A$ and $C \neq B$. Since $[B' : \langle x \rangle] \geq 3 > 1$, K_1 is a non-trivial amalgamated free product unless $H_r = \langle r \rangle$. In this case, H_r is a cyclic group. However, G is the quotient of H_r by the normal closure of r. Since any quotient of a cyclic group is cyclic, this contradicts the assumption that G is non-cyclic. Therefore, $H_r \neq \langle r \rangle$, and K_1 is a non-trivial amalgamated free product.

Case 2: Remove $r^{-1}b_r r = b_r^2$. Let K_2 be the resulting group. The remaining relation is $b_r^{-1} r b_r = r^2$. We define a homomorphism $\phi : K_2 \to \mathbb{Z} = \langle t \rangle$ by setting $\phi(x) = 1$ for all $x \in X$, $\phi(b_{r'}) = 1$ for $r' \neq r$, and $\phi(b_r) = t$. Note that since X maps to the identity, all original relations in $R \setminus \{r\}$ are trivially satisfied. Under this map, the remaining relation maps to $t^{-1} \cdot 1 \cdot t = 1$. This homomorphism is well-defined and maps onto the infinite group $\mathbb{Z}$. Hence, K_2 does not have Property (FA).

Thus, removing any relation from P' yields a group without Property (FA), and so P' is a just-(FA) presentation for G. $\square$

We remark, as a consequence of the above proof, the following result. Recall that the *deficiency* of a presentation $\langle X \mid R\rangle$ is $|X| - |R|$.

Theorem 6. *Let P be any finite irredundant presentation of a non-cyclic group G with Property (FA). Then G admits a finite, just-(FA) presentation P' where the deficiency of P' is equal to that of P, and where the generators of P' represent the same elements of G that the generators of P did, plus multiple copies of the identity element.*

3. Examples

The following is a presentation of the dihedral group of order 8:

$$\langle \sigma, \tau \mid \sigma^4 = 1, \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^3\rangle.$$

When the first relation is removed, this gives a semi-direct product $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$, by an argument analogous to the one in Lemma 5. When the second relation is removed, the resulting group admits a surjective homomorphism onto $\mathbb{Z} = \langle t\rangle$, sending σ to 1 and τ to t. When the third relation is removed, we have the free product $\mathbb{Z}_4 * \mathbb{Z}_2$. So this presentation is irredundant but not just finite. When we apply the above procedure to it, we obtain the just finite presentation

$$\left\langle \sigma, \tau, a, b, c \;\middle|\; \begin{array}{c} \sigma^{-4}a\sigma^4 = a^2, a^{-1}\sigma^4 a = \sigma^8, \\ \tau^{-2}b\tau^2 = b^2, b^{-1}\tau^2 b = \tau^4, \\ (\tau^{-1}\sigma\tau\sigma^{-3})^{-1}c(\tau^{-1}\sigma\tau\sigma^{-3}) = c^2, \\ c^{-1}(\tau^{-1}\sigma\tau\sigma^{-3})c = (\tau^{-1}\sigma\tau\sigma^{-3})^2 \end{array} \right\rangle.$$

However, the following presentation for the dihedral group of order 8 is already just finite:

$$\langle \sigma, \tau \mid \sigma^4 = 1, \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle.$$

Removing the first relation gives the infinite dihedral group.

4. Methodology

This paper was produced using the Co-Mathematician tool developed by Google-DeepMind. The author posed Problem 21.10 from the Kourovka Notebook to the Co-Mathematician. GDM Co-Mathematician proposed a solution, which was essentially the proof given in this paper, but then found a flaw in its argument which it was unable to resolve. The issue was that it was uncertain how to proceed in the case where $H_r = \langle r\rangle$ in Case 1 of the proof of Theorem 2. By analysing the argument carefully, the author was able to find a fix for it, which the Co-Mathematician then confirmed was correct and wrote up. The author then made some minor edits and clarifications. The author then introduced the notions of just-(FA) and just-(T) presentations, and made the necessary minor adjustments to the proofs, resulting in this paper. The paper was then checked and corrected by the Co-Mathematician tool.

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