Knot theory and machine learning

Marc Lackenby
Joint work with Alex Davies, András Juhász, Nenad Tomasev

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The branches of knot theory

Knot theory is divided into three quite distinct subfields:

- hyperbolic knot theory
- gauge/Floer theory
- quantum topology

Invited speakers

- Ian Agol (U. C. Berkeley)
- Martin Bridson (Oxford U.)
- Jeff Brock (Yale U.)
- Ted Chinburg (U. Pennsylvania)
- Michelle Chu (U.I. Chicago)
- Jeff Danciger (U.T. Austin)
- Cameron Gordon (U.T. Austin)
- Ursula Hamenstadt (U. Bonn)
- Neil Hoffman (O.S.U.)
- Autumn Kent (U.W. Madison)
- Darren Long (U.C. Santa Barbara)
- Alex Lubotzky (Hebrew U.)
- Bruno Martelli (U. Pisa)
- Gaven Martin (Massey U.)
- Priyam Patel (U. Utah)
- Kate Petersen (U.M. Duluth)
- Jessica Purcell (Monash U.)
- Peter Sarnak (Princeton U.)
- Matt Stover (Temple U.)
- Sam Taylor (Temple U.)
- Genevieve Walsh (Tufts U.)
- Will Worden (Rice U.)

Alan Reid's conference

Topics:

- Interplay of 3-dimensional and 4-dimensional Topology
- Floer homology theories and associated invariants
- Khovanov homology
- Geometric and analytic aspects of gauge theoretic equations

Speakers:

D. GABAI, Princeton University, USA
L. GOETTSCHE, ICTP, Italy
L. GUTH, MIT, USA
J. HOM, Georgia Tech, USA
C. HUGELMEYER, Princeton University, USA
*P. KRONHEIMER, Harvard University, USA
F. LIN, Princeton University, USA
R. LIPSHITZ, University of Oregon, USA
P. LISCA, Università di Pisa, Italy
C. MANOLESCU, Stanford University, USA
G. MATIC, University of Georgia, USA
R. MAZZEO, Stanford University, USA
M. MILLER, Stanford University, USA
E. MURPHY, Princeton University, USA
*J. PARDON, Princeton University, USA
L. PICCIRILLO, MIT, USA
J. PINZON CAICEDO, University of Notre Dame, USA
J. RASMUSSEN, Cambridge, UK
D. RUBERMAN, Brandeis University, USA
A. STIPSICZ, Central European University, Hungary
Z. SZABO, Princeton University, USA
*C.TAUBES, Harvard University, USA
D. WANG SUNY, Stony Brook, USA
J. WANG, Harvard University, USA
C. ZIBROWIUS, University of Regensburg, Germany

Trieste conference

Finding connections between these fields

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Each field has plenty of knot invariants:

Hyperbolic invariants:

- Volume
- Cusp shape and volume
- Length spectrum
- Trace field ...

3/4-dimensional invariants:

- Heegaard Floer homology
- Instanton Floer homology
- s, τ, ϵ, Υ, ...

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Goal: Find new connections between these invariants

Knot signature

The 3/4-dimensional invariant that we focused on was the signature.

This is defined by starting with a Seifert surface S for the knot K.

The symmetrised Seifert form for S is the bilinear form

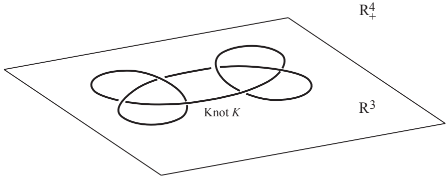
$$\begin{aligned} H_1(S) \times H_1(S) &\to \mathbb{Z} \\ (\ell_1, \ell_2) &\mapsto \mathrm{lk}(\ell_1, \ell_2^+) + \mathrm{lk}(\ell_2, \ell_1^+) \end{aligned}$$

where ℓ_2^+ is the push-off of ℓ_2 in the positive normal direction from S.

The signature $\sigma(K)$ is the signature of this bilinear form.

Connections with dimension 4

View $\mathbb{R}^3$ as the boundary of $\mathbb{R}^4_+ = \{(x_1, x_2, x_3, x_4) : x_4 \geq 0\}$.



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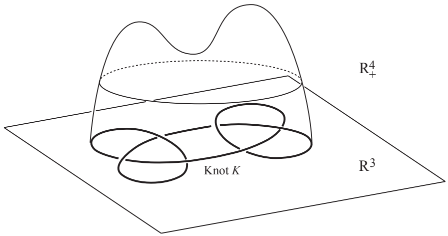

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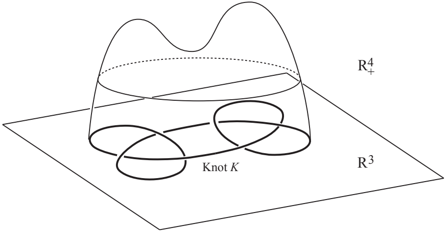

The 4-ball genus of a knot K is the minimal genus of a (topological locally-flat) surface in $\mathbb{R}^4_+$ with boundary equal to K.

Theorem: [Murasugi 1965] $g_4(K) \geq |\sigma(K)|/2$.

Using machine learning

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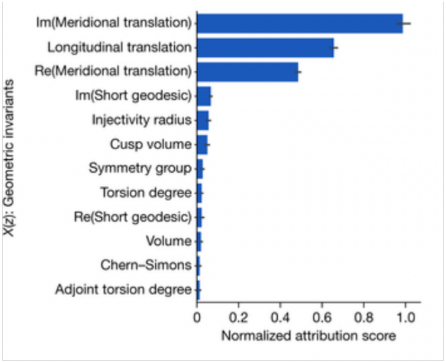
- We trained a neural network to predict the signature from the hyperbolic invariants.
- We then tested this network using the test set.
- The network could predict the signature with impressive accuracy.

Saliency

The main hyperbolic invariants that were used to predict signature:

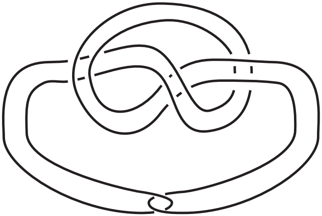


Hyperbolic structures

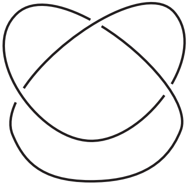
A hyperbolic structure on a knot complement is a complete finite-volume Riemannian metric of constant curvature -1.

By Mostow rigidity, if such a metric exists, it is a unique up to isometry.

Thurston's theorem: The complement of a non-trivial knot K has a hyperbolic structure if and only if K is not a torus knot or a satellite knot.



satellite knot



torus knot

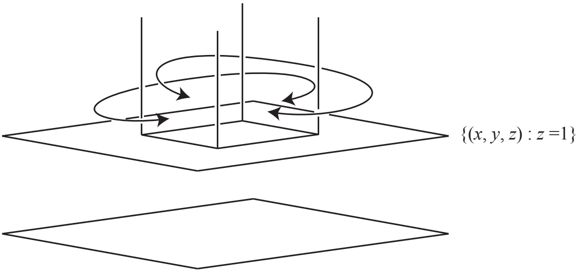
Cusp geometry

Any knot complement has an end of the form $T^2 \times [1, \infty)$.

When the knot is hyperbolic, this has a canonical geometry and is called a cusp.

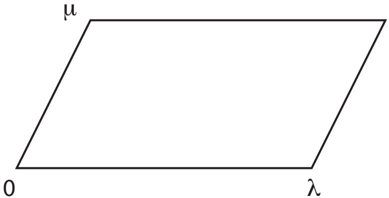
Let $\mathbb{H}^3$ be upper-half space $\{(x, y, z) : z > 0\}$. Let H be the horoball $\{z \geq 1\}$.

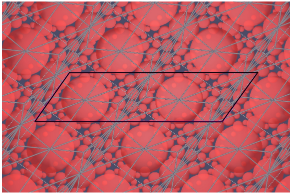
Then the cusp is formed $H / \langle$group of Euclidean translations$\rangle$.



The cusp boundary

The boundary of the cusp is a Euclidean torus $\mathbb{C}/\Lambda$ for a lattice Λ. We normalise Λ so that the longitude λ is real and positive, and the meridian μ has positive imaginary part.





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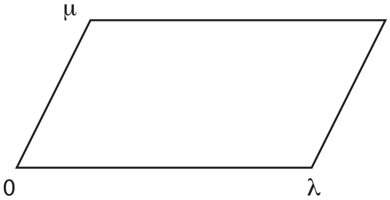
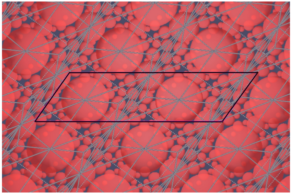
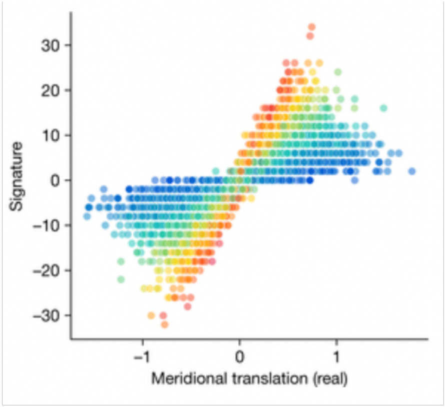




Cusp torus for 6_1

The three main features that the machine learning algorithms used to predict signature were λ, $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$.

Signature and cusp geometry



A plot of signature against $\mathrm{Re}(\mu)$ coloured by λ

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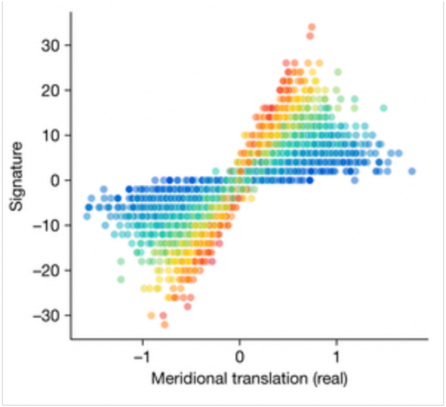

A plot of signature against $\mathrm{Re}(\mu)$ coloured by λ

Initial observation: the signs of the signature and $\mathrm{Re}(\mu)$ are highly correlated.

The natural slope

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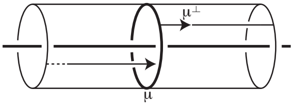

- Pick a geodesic representative μ for the meridian.
- Fire a geodesic $\mu^{\perp}$ orthogonally from it.
- Eventually, it will return to the meridian.
- In that time, it will have gone along one longitude and some number s of meridians.

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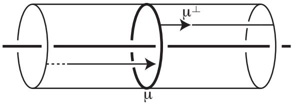

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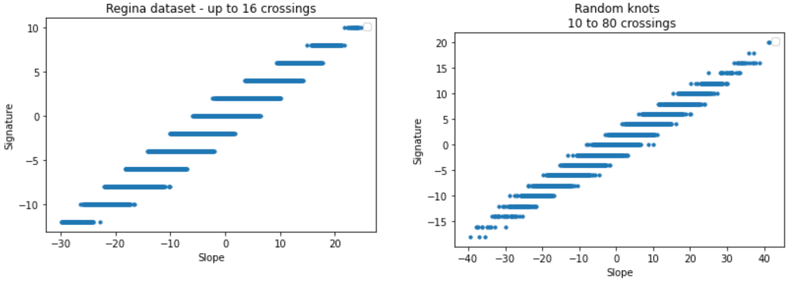
- Eventually, it will return to the meridian.
- In that time, it will have gone along one longitude and some number s of meridians.
- Define the natural slope to be $-s$.

$$\mathrm{slope}(K) = \mathrm{Re}(\lambda/\mu).$$

Slope and signature



First conjectures

Conjecture: There is a constant c_0 such that

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Conjecture: There are constants c_0 and c_1 such that

$$|\sigma(K) - c_0 \,\mathrm{slope}(K)| \leq c_1 \,\mathrm{vol}(K).$$

Highly twisted knots

Theorem: Let K be a knot, and let $C_1, \ldots, C_n$ be curves in the complement that bound disjoint discs in S^3. Suppose $K \cup C_1 \cup \cdots \cup C_n$ is hyperbolic. Let $K(q_1, \ldots, q_n)$ be the knot obtained from K by adding q_i full twists along each C_i.

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$$\left| \mathrm{slope}(K(q_1, \ldots, q_n)) + \sum_{i=1}^{n} \ell_i^2 q_i \right| \leq k$$

$$\left| \sigma(K(q_1, \ldots, q_n)) + \left(\frac{1}{2} \sum_{i=1}^{m} \ell_i^2 q_i + \frac{1}{2} \sum_{i=m+1}^{n} (\ell_i^2 - 1) q_i \right) \right| \leq k$$

$$\mathrm{vol}(K(q_1, \ldots, q_n)) \leq k.$$

So the conjectures are false!

Theorems

Theorem 1: There is a constant c_1 such that

$$|\sigma(K) - (1/2)\,\mathrm{slope}(K)| \leq c_1 \,\mathrm{vol}(K)\,\mathrm{inj}(K)^{-3}.$$

Here, $\mathrm{inj}(K)$ is $\inf\{\mathrm{inj}_x(S^3 - K) : x \in (S^3 - K) - \text{cusp}\}$.

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Theorem 2: $\sigma(K)$ and

$$(1/2)\,\mathrm{slope}(K) + \sum_{\gamma \in \mathrm{OddGeo}} \kappa(\gamma)$$

differ by at most $c_2\mathrm{vol}(K)$ for some constant c_2.

Here, OddGeo is the set of geodesics with length at most 0.1 and that have odd linking number with K, and $\kappa(\gamma)$ is a correction term defined in terms of the complex length of γ.

Consequence: 4-ball genus

Corollary:

$$g_4(K) \geq |\mathrm{slope}(K)|/4 - (c_1/4)\,\mathrm{vol}(K)\,\mathrm{inj}(K)^{-3}.$$

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Each slope q/p has a length $\ell(q/p)$ as measured in the Euclidean metric on the torus.

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Theorem: [Agol, L] If $\ell(q/p) > 6$, then q/p is not exceptional.

Lemma: $\ell(q/p) > |p \, \mathrm{slope}(K) + q|$. Hence if q/p is exceptional, then

$$q/p \in [-\mathrm{slope}(K) - 6/p, -\mathrm{slope}(K) + 6/p].$$

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Hence, q/p is 'close' to $-2\sigma(K)$.

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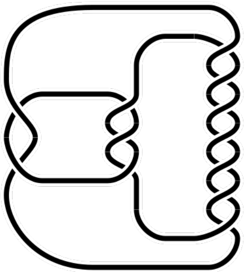
Example: (-2,3,7) pretzel knot.



Its exceptional slopes are
16, 17, 18, 37/2, 19, 20.
$\mathrm{slope}(K) \simeq -18.215$
$\sigma(K) = -8$.

Signature and spanning surfaces

Let S be an unoriented spanning surface for K.
Its Goeritz form is

$$G_S \colon H_1(S) \times H_1(S) \to \mathbb{Z}$$
$$(\ell_1, \ell_2) \mapsto \mathrm{lk}(\ell_1, \ell_2')$$

where ℓ_2' is the double push-off of ℓ_2.

Theorem: [Gordon-Litherland]

$$\sigma(K) = \sigma(G_S) + e(S)/2,$$

where $e(S)$ is the framing of ∂S.

Building a triangulation

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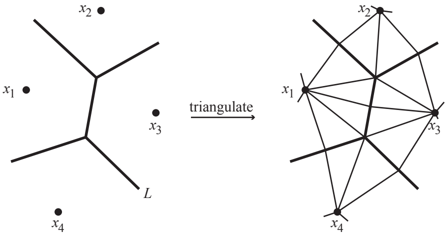
Proof:

Let $\epsilon = \min\{\mathrm{inj}(M)/4, 1\}$.

Pick a maximal set of points P in M, no two of which are closer than $\epsilon/4$.

$|P| \leq \mathrm{vol}(M)/\mathrm{vol}(B(\epsilon/8))$.



Form the associated Voronoi domain.
Add a vertex to each face and cone off the face.
Triangulate each polyhedron of the domain as a cone on its boundary.

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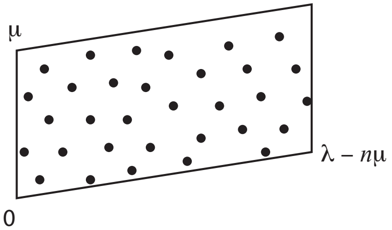
Start with a maximal set of points in ∂(cusp), no two of which are closer than $\epsilon/4$, and then extend this to a maximal set in $M - \mathrm{int}$(cusp).



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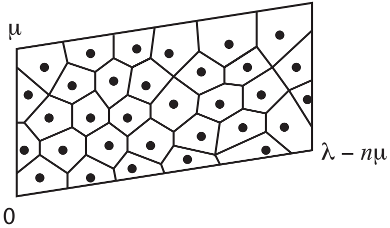

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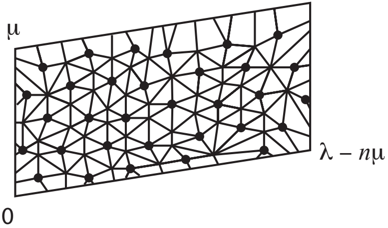

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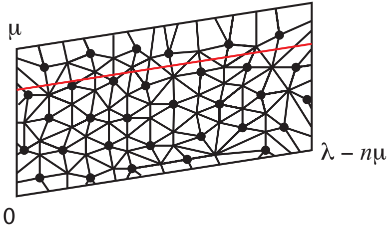

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Lemma: Let C be a normal curve in ∂M that intersects each triangle in at most one arc and that is trivial in $H_1(M; \mathbb{Z}_2)$. Then C extends to a normal surface F in M that intersects each tetrahedron in at most one triangle or square.

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Proof: C bounds an unoriented surface S in M.

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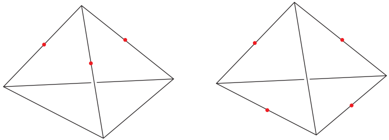

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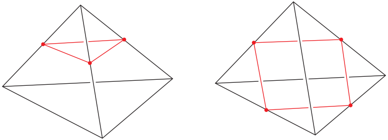

Join these points.

Proof of Theorem 1

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So $|\sigma(K) - \mathrm{slope}(K)/2| \leq |\sigma(G_F)| + 1 \leq O(\mathrm{vol}(M)\mathrm{inj}(M)^{-3})$.

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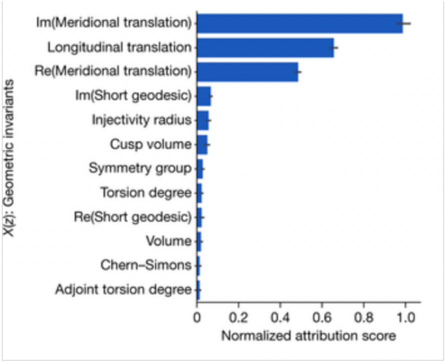

Items 4 and 5 are the terms appearing in Theorems 1 and 2.