DOUBLE BRANCHED COVERS OF KNOTOIDS

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ABSTRACT. By using double branched covers, we prove that there is a 1-1 correspondence between the set of knotoids in $S^2$ and knots with a strong inversion, up to conjugacy. This correspondence allows us to study knotoids through tools and invariants coming from knot theory. In particular, with our construction we are able to detect both the trivial knotoid in $S^2$ and the trivial planar knotoid.

1. INTRODUCTION

Knotoids were recently defined by V.Turaev [32] as a generalisation of knots in $S^3$. More precisely, knotoids are defined as equivalence classes of diagrams of arcs in $S^2$ or in $D^2$ up to an appropriate set of moves and isotopies. Two examples of knotoids are shown in Figure 1.1

Several invariants for knotoids have been adapted from classical knot theory, such as various versions of the bracket polynomial (see e.g. [32], [13]), and there are several well defined maps that associate a classical knot to a knotoid (see e.g. [32], [13], [20]). Often, non-equivalent knotoids share the same image under these maps, and it is possible to exhibit examples of non-equivalent knotoids with the same bracket polynomials. Here we present a complete invariant, that associates a knot to a knotoid through a double branched cover construction. Knotoids admit a 3-dimensional interpretation as equivalence classes of embedded arcs in $\mathbb{R}^3$ with endpoints lying on two fixed vertical lines. In this setting, given a knotoid $k$, its preimage in the double cover of $\mathbb{R}^3$ branched along these lines is a simple closed curve, that can be viewed as giving a classical knot $K$ in $S^3$. Knots arising as pre-images of knotoids are strongly invertible, that is, there exists an involution $\tau$ of $S^3$ mapping the knot to itself, preserving the orientation of $S^3$ and reversing the one of the knot. We exploit properties of strongly invertible knots to prove our main result in Section 5.

**Theorem 1.1.** There is a 1-1 correspondence between unoriented knotoids and knots $K$ with a strong inversion $\tau$ up to conjugacy.

As a corollary of Theorem 1.1 we have the following.
Corollary 1.2. Given any torus knot $K_t$ there is exactly one prime proper unoriented knotoid associated to it, and given any strongly invertible hyperbolic knot $K_h$ there either are 1 or 2 prime, proper and unoriented knotoids associated to it, depending on whether or not $K_h$ is periodic with period 2. In general, given any strongly invertible knot $K$ there are only finitely many knotoids associated to it.

In particular, Corollary 1.2 implies that there are at most 2 prime, proper, oriented knotoids associated to every torus knot, and at most 4 associated to every hyperbolic knot. Indeed, we can detect invertibility in the case of knotoids lifting to hyperbolic knots.

Theorem 1.3. A hyperbolic, oriented knotoid $k \in \mathbb{K} (S^2)$ is invertible if and only if $\gamma_S(k)$ has period 2.

As a consequence of this result we obtain the following.

Corollary 1.4. Given any strongly invertible hyperbolic knot $K_h$ there are exactly two oriented knotoids associated to it. If $K_h$ does not have period 2, then these oriented knotoids are one the reverse of the other.

We emphasise that Theorem 1.1 and Corollary 1.2 provide a powerful link between knot theory and knotoids, allowing us to borrow all the sophisticated tools developed to distinguish knots to study knotoids. In particular, we can extends the concepts from geometrisation to knotoids: we will call hyperbolic (respectively torus) knotoids those lifting to hyperbolic (respectively torus) knots. As there is an algorithm to decide whether two hyperbolic knots are equivalent (see [26] and [21]) and there is an algorithm to decide whether two involutions of a hyperbolic knot complement are conjugate (see e.g. Theorems 8.2 and 8.3 of [23], Theorem 1.1 implies the following stronger result.

Theorem 1.5. Given two hyperbolic and unoriented knotoids $k_1$ and $k_2$, there is an algorithm to determine whether $k_1$ and $k_2$ are equivalent.

Structure of the paper. The paper is structured as follows. After recalling some basics on knotoids in Section 2, we present the map defined by the double branched cover in Section 3. We recall the 1-1 correspondence between knotoids and isotopy classes of simple $\theta$-curves, following [32], in Section 3. In Section 4, results from [3] are translated in terms of knotoids. In particular, Theorem 4.4 allows us to detect the trivial knotoid $k_0$ among all the other knotoids. In Section 4.2, we prove a slightly more powerful version of Theorem 4.4, Theorem 4.5, enabling the detection of the trivial planar knotoid $k_{pl}^0$ (see Section 2 for the precise definitions). Section 5 is devoted to the proof of Theorems 1.1 and 1.5. In Section 6, we prove Theorem 1.3 together with two improvements of Corollary 1.2. In Section 7, we show that our construction can be used to distinguish between planar knotoids that are equivalent in $S^2$. Finally, in Section 8, we describe an algorithm that produces the Gauss code for the lift of a knotoid $k$ given the Gauss code for $k$.

All maps and manifolds are assumed to be smooth, and we use the following notation:
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\[ K(X) \text{ and } \mathbb{K}(X) \text{ are the sets of oriented and unoriented knotoid diagrams up to equivalence in } X, \text{ where } X = S^2 \text{ or } \mathbb{R}^2; \]

\[ \mathcal{K}(Y) \text{ is the set of knots in } Y, \text{ where } Y = S^3 \text{ or } S^1 \times D^2; \]

\[ \mathcal{K}_{S.I.}(S^3) \text{ is the set of strongly invertible knots in } S^3. \]

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2. Preliminares

A knotoid diagram in \( S^2 \) is a generic immersion of the interval \([0,1]\) in \( S^2 \) with finitely many transverse double points endowed with over/under-crossing data. The images of the points 0 and 1 are distinct from the other points and from each other. The endpoints of a knotoid diagram are called the tail and the head respectively. Knotoid diagrams are oriented from the tail to the head.

Definition 2.1. A knotoid is an equivalence class of knotoid diagrams on the sphere considered up to isotopies of \( S^2 \) and the three classical Reidemeister moves (see Figure 2.1), performed away from the endpoints.

It is not permitted to pull the strand adjacent to an endpoint over/under a transversal strand as shown in Figure 2.2. Notice that allowing such moves produces a trivial theory: namely, any knotoid diagram can be transformed into the crossingless one by a finite sequence of forbidden moves.

\[
\begin{align*}
\Omega_1 \quad \Omega_2 \quad \Omega_3
\end{align*}
\]

Figure 2.1. The classical Reidemeister moves.

The trivial knotoid \( k_0 \) is the equivalence class of the crossingless knotoid diagram.

\[
\rightarrow \rightarrow \leftrightarrow \rightarrow
\]

Figure 2.2. Forbidden moves.

Any knotoid \( k \) in \( S^2 \) can be represented by several knotoid diagrams in \( \mathbb{R}^2 \), by choosing different stereographic projections. Two knotoid diagrams
in $\mathbb{R}^2$ are said to be equivalent if they are related by planar isotopies and a finite sequence of Reidemeister moves, performed away from the endpoints. We will denote by $k_0^{4i}$ the equivalence class in $\mathbb{R}^2$ of the crossingless knotoid diagram.

Let us denote the set of knotoids in the plane and in the sphere by $\mathbb{K}(\mathbb{R}^2)$ and $\mathbb{K}(S^2)$ respectively. We can define the map

$$\iota : \mathbb{K}(\mathbb{R}^2) \rightarrow \mathbb{K}(S^2)$$

induced by the inclusion $\mathbb{R}^2 \hookrightarrow S^2 = \mathbb{R}^2 \cup \infty$. The map $\iota$ is surjective but not injective (see Figure 2.3 for an example, or [13] for more details).

There is a natural way to associate two different knots to any knotoid through the underpass closure map and the overpass closure map, defined in [32] and [13] and denoted by $\omega_-$ and $\omega_+$ respectively. Given a diagram representing a knotoid $k$, a diagram representing $\omega_-(k)$ (respectively $\omega_+(k)$) is obtained by connecting the two endpoints by an arc embedded in $S^2$ which is declared to go under (respectively over) each strand it meets during the connection.

**Remark 2.2.** Different knotoids may have the same image under $\omega_+$ and $\omega_-$, see for example the knotoids in Figure 2.4. In Section 3 we are going to present a more subtle way to associate a knot to a knotoid, which allows for a finer classification.

Moreover, every knot $K \subset S^3$ arises as the image under $\omega_+$ of a knotoid diagram. Indeed, choose any diagram representing $K$, and cut an arc that does not contain any crossings, or that contains only crossings which are overcrossings (respectively undercrossings). This results in creating a knotoid diagram, whose image under $\omega_+$ (respectively $\omega_-$) is the starting knot $K$. It is important to notice that different choices of arcs in $K$ may result in non-equivalent knotoid diagrams. However, choosing the arc to be crossingless induces a well defined injective map $\alpha$ from the set of knots in $S^3$ to $\mathbb{K}(S^2)$ (see [32], [13] for more details).
Definition 2.3. Knotoids in $\mathbb{K}(S^2)$ that are contained in the image of $\alpha$ are called knot-type knotoids. Equivalently, a knotoid is a knot-type knotoid if and only if it admits a diagram in which the endpoints lie in the same region (see the left side of Figure 2.4). The other knotoids are called proper knotoids (see the right side of Figure 2.4).

There is a 1-1 correspondence between knot-type knotoids and classical knots: knot-type knotoids may be thought as long knots, and (see e.g. [5]) closing the endpoints of a long knot produces a classical knot carrying the same knotting information. Thus, we can conclude that a knot-type knotoid can be considered the same as the knot it represents.

2.1. Multiplication of Knotoids. In [32] an analogue for the connected sum of knots is defined: the multiplication of knotoids. Note that each endpoint of a knotoid diagram $k$ in $S^2$ admits a neighbourhood $D$ such that $k$ intersects it in exactly one arc (a radius) of $D$. Such a neighbourhood is called a regular neighbourhood of the endpoint. Given two diagrams in $S^2$ representing the knotoids $k_1$ and $k_2$, equipped with a regular neighbourhood $D_1$ for the head of $k_1$ and $D_2$ for the tail of $k_2$, the product knotoid $k = k_1 \cdot k_2$ is defined as the equivalence class in $\mathbb{K}(S^2)$ of the diagram obtained by gluing $S^2 \setminus \text{int}(D_1)$ to $S^2 \setminus \text{int}(D_2)$ through an orientation-reversing homeomorphism $\partial D_1 \to \partial D_2$ mapping the only point in $\partial D_1 \cap k_1$ to the only point in $\partial D_2 \cap k_2$. Note that this operation is not commutative (see [32], Section 4).

Definition 2.4. A knotoid $k$ in $\mathbb{K}(S^2)$ is called prime if it is not the trivial knotoid and $k = k_1 \cdot k_2$ implies that either $k_1$ or $k_2$ is the trivial knotoid.

This multiplication operation has been extensively studied in [32], where the following result on prime decomposition is proven.

Theorem 2.5 (Theorem 4.2, [32]). Every knotoid $k$ in $\mathbb{K}(S^2)$ expands as a product of prime knotoids.

Moreover, the expansion as a product is unique up to an identity described in [32], and the multiplication operation turns $\mathbb{K}(S^2)$ into a semigroup.

Remark 2.6. Since the surface in which the diagram of $k = k_0 \cdot k_1$ lies is the 2-sphere obtained as the connected sum between the 2-spheres containing the diagrams of $k_0$ and $k_1$, the operation of multiplication is well defined only in $\mathbb{K}(S^2)$. A diagram in the plane for $k$ can be obtained by drawing the tail of $k_1$ in the external region of the diagram, as shown in Figure 2.5.

Note that the orientation is required in order to define the multiplication operation. In particular, given a knotoid $k \in \mathbb{K}(S^2)$, call $-k$ the knotoid represented by the same diagrams as $k$, but with opposite orientation. Then, the following relations hold.

- $k_1 \cdot k_2 = -(-k_2 \cdot -k_1)$
- $-k_1 \cdot k_2 = -(-k_2 \cdot k_1)$
- $k_1 \cdot -k_2 = -(k_2 \cdot -k_1)$
- $k_2 \cdot k_1 = -(k_1 \cdot k_2)$

We will however often find useful to forget about the orientation. To this end, we will call $\mathbb{K}(S^2)$ and $\mathbb{K}(\mathbb{R}^2)$ the sets of unoriented knotoids in the
Figure 2.5. On the bottom line, a diagram representing the product $k_0 \cdot k_1$ of the knotoids in the upper line.

sphere and in the plane, respectively. Note that, in general, the products $k_1 \cdot k_2$, $-k_1 \cdot k_2$, $k_1 \cdot -k_2$ and $k_2 \cdot k_1$ represent non-equivalent classes of unoriented knotoids.

2.2. Bracket polynomial. The bracket polynomial of oriented knotoids in $\mathbb{K}(S^2)$ or in $\mathbb{K}(\mathbb{R})^2$ was defined in [32], by extending the state expansion of the bracket polynomial of knots. The definition can be given in terms of a skein relation, with the appropriate normalisations, as for the bracket polynomial of knots. A normalisation of the bracket polynomial of knotoids gives rise to a knotoid invariant generalising the Jones polynomial of knots (after a change of variable). A version of the bracket polynomial (extended bracket polynomial, defined in [32]) is used in [2] to distinguish knotoids taken from a list containing diagrams with up to 5 crossings.

Even if bracket polynomials are useful invariants, it is fairly simple to produce examples of oriented knotoids that cannot be distinguished by them. One way to construct such examples is by using the concept of mutation. Recall that the mutation of an oriented knot $K$ can be described as follows. Consider a diagram for $K$, and a 2-tangle $R$ as in Figure 2.6. New knots $K_i'$ can be formed by replacing the tangle $R$ with the tangle $R' = \rho_i(R)$ given by rotating $R$ by $\pi$ in one of three ways described on the right side of Figure 2.6. Each of these three knots is called a mutant of $K$.

Figure 2.6. A portion of a knot diagram for $K$ contained in the 2-tangle $R$. By rotating $R$ as in the right side of the picture, we obtain new knots $K_1'$, $K_2'$ and $K_3'$. Each of these knots is called a mutant of $K$. 
The probably best known example of non-equivalent mutant knots is the Conway and Kinoshita-Teresaka pair shown in Figure 2.7.

Figure 2.7. The Kinoshita-Teresaka knot $KT$ (on the left) and the Conway knot $C$ (on the right). It was shown by Gabai [11] that the genus of $KT$ is 2 while $C$ has genus 3.

Mutation can be generalised to knotoids, by requiring that both the end-points of a knotoid $k$ lie outside of the tangle $R$ that is rotated.

Remark 2.7. It is well known that mutant knots share the same bracket and Jones polynomial. The same result also holds for knotoids, and a proof can be produced in exactly the same way as for knots by using Conway’s linear skein theory (see e.g. [25] and [24]).

Consider the knot-type knotoids associated to $KT$ and $C$; by construction they are non-equivalent, and Remark 2.7 implies that they share the same bracket polynomials. By taking the products $k_1 = KT \cdot k$ and $k_2 = C \cdot k$, where $k$ is any proper knotoid (see Figure 2.8) we obtain two proper knotoids with the same bracket polynomials. We will prove in Section 4 that $k_1$ and $k_2$ are non-equivalent.

3. Double Branched Covers

In this section, unless otherwise specified, we will assume that the knotoids are unoriented. As described in [13], it is possible to give a 3-dimensional definition of knotoids, as embedded arcs in $\mathbb{R}^3$, up to a particular isotopy notion.

3.1. Knotoids as embedded arcs. Consider a knotoid diagram $k$ in $\mathbb{R}^2$, and identify the plane of the diagram with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. We can embed $k$ in $\mathbb{R}^3$ by pushing the overpasses of the diagram into the upper half-space, and the underpasses into the lower one. The endpoints of $k$ are attached to two lines $t \times \mathbb{R}, h \times \mathbb{R}$ perpendicular to the $xy$ plane.

Two open embedded arcs in $\mathbb{R}^3$ with endpoints lying on these two lines are said to be line isotopic if there is a smooth ambient isotopy of the pair $(\mathbb{R}^3, t \times \mathbb{R} \cup h \times \mathbb{R})$ taking one curve to the other, endpoints to endpoints, and leaving each one of the special lines invariant. Conversely, an embedded curve in $\mathbb{R}^3$ whose projection on the $xy$-plane is generic (plus the additional data of over and underpassings) defines a knotoid diagram (see Figure 3.1).

\footnote{As a consequence of [32], Theorem 4.2, the product of two knotoids $k_1$ and $k_2$ is a knot-type knotoid if and only if both $k_1$ and $k_2$ are knot-type knotoids.}
Figure 2.8. The mutant knotoids $k_1 = KT \cdot k$ and $k_2 = C \cdot k$ share the same bracket polynomials.

Figure 3.1. On the top, the curve in $\mathbb{R}^3$ obtained from the knotoid diagram on the bottom.

There is a 1-1 correspondence (see [13], Theorem 2.1 and Corollary 2.2) between the set of knotoids in $\mathbb{R}^2$ and the set of line-isotopy classes of smooth oriented arcs in $\mathbb{R}^3$, with endpoints attached to two lines perpendicular to the $xy$ plane.

Similarly, given a knotoid $k$ in $\mathbb{K}(S^2)$ we can construct an embedded arc in $S^2 \times I$ with the same procedure. Now the endpoints are attached to two lines perpendicular to the sphere $S^2 \times \{pt\}$. Theorem 2.1 and Corollary 2.2 in [13] extend naturally to this setting.
3.1.1. θ-curves. Consider an embedded curve in $S^2 \times I$, with endpoints attached to the two special lines. We can compactify the manifold by collapsing $S^2 \times \partial I$ to two points, obtaining an embedded curve in $S^3$ with endpoints lying on an unknotted circle, as in Figure 3.2.

![Figure 3.2](image)

**Figure 3.2.** On the left, a knotoid seen as an embedded curve in $S^2 \times I$, with endpoints lying on the dotted lines. By collapsing $S^2 \times \partial I$ to two points, we obtain an embedded arc in $S^3$, with endpoints lying on a dotted circle (the projection of the dotted lines).

The union of the embedded curve with this unknotted circle is a θ-curve.

**Definition 3.1.** A labeled θ-curve is a graph embedded in $S^3$ with 2 vertices, $v_0$ and $v_1$, and 3 edges, $e_+$, $e_-$ and $e_0$, each of which joins $v_0$ to $v_1$. The curves $e_0 \cup e_-$, $e_- \cup e_+$ and $e_0 \cup e_+$ are called the constituent knots of the θ-curve. We will call two labeled θ-curves isotopic if they are related by an ambient isotopy preserving the labels of the vertices and the edges. A θ-curve is called simple if one of its constituent knots is the trivial knot.

Thus, we can associate a simple labeled θ-curve to a knotoid $k \in \mathbb{K}(S^2)$, whose vertices are the endpoints of $k$ and with $e_0 = k$. We will call the unknotted circle $e_- \cup e_+$ the preferred constituent unknot. It is shown in [32] that this operation induces a well defined map $\tilde{t}$ between the set of oriented knotoids $\mathbb{K}(S^2)$ and the set $\Theta^s$ of isotopy classes of simple labeled θ-curves. Moreover, $\Theta^s$ endowed with the vertex-multiplication operation (for a definition of vertex-multiplication see e.g. [30]) is a semigroup, and the following theorem holds.

**Theorem 3.2** (Theorem 6.2 in [32]). The map $\tilde{t} : \mathbb{K}(S^2) \longrightarrow \Theta^s$ is a semigroup isomorphism.

Call $\Theta^s/\sim$ the set of simple θ-curves up to relabelling the two vertices. The isomorphism $\tilde{t}$ of Theorem 3.2 gives a bijection

$$t : \mathbb{K}(S^2) \longrightarrow \Theta^s/\sim$$

between unoriented knotoids and elements of $\Theta^s/\sim$. We will simply refer to elements of $\Theta^s/\sim$ as θ-curves.

3.2. Double branched covers. Consider a planar knotoid $k$, thought of as an embedded arc in the cylinder $D^2 \times I$. The double cover of $D^2 \times I$
branched along the special vertical arcs is the solid torus \((S^1 \times I) \times I\). Denote the branched covering map by
\[
p : (S^1 \times I) \times I \longrightarrow D^2 \times I
\]
The pre-image \(p^{-1}(k)\) of the knotoid in the double branched cover is a knot inside the solid torus \(S^1 \times D^2\). The knot type of this branched cover is a knotoid invariant; in particular by composing the branched covering construction with any knot invariant we obtain a new knotoid invariant. Note that by definition, the lifts of line-isotopic embedded arcs are ambient isotopic knots, since isotopies of \(k\) preserving the branching set lift to equivariant isotopies.

Remark 3.3. From the knotoid diagram obtained by projecting \(k\), it is possible to construct a diagram in the annulus \(S^1 \times I\) for \(p^{-1}(k)\) by taking the double cover of the disk \(D^2 \times \{pt\}\) branched over the endpoints of the diagram, as shown in Figure 3.3.

Similarly, given a knotoid \(k \in \mathbb{K}(S^2)\) and the associated \(\theta\)-curve \(t(k)\) in \(S^3\), the pre-image of \(k\) under the double cover of \(S^3\) branched along the preferred constituent unknot of \(t(k)\) is a knot in \(S^3\). Double branched covers of simple \(\theta\)-curves have been extensively studied in \[3\], whose main results are discussed and used in Section 4.

Remark 3.4. Consider a diagram representing \(k \in \mathbb{K}(S^2)\): we can obtain a diagram for the lift of \(k\) by taking the double cover of \(S^2\) branched along the endpoints.

Figure 3.3. The two-fold branched cover of \(D^2\) by \(S^1 \times I\) can be described by “cuts” (see e.g. \[28\], Chapter 10.5) for a reference). The red arc in the disk lifts to a longitude of the solid torus.

Call \(\mathcal{K}(S^1 \times D^2)\) and \(\mathcal{K}(S^3)\) the sets of knots in the solid torus and in \(S^3\) respectively, taken up to the appropriate ambient isotopies. Thus, we have the following maps induced by the double branched covers:
\[
\gamma_T : \mathbb{K}(\mathbb{R}^2) \longrightarrow \mathcal{K}(S^1 \times D^2)
\]
\[
\gamma_S : \mathbb{K}(S^2) \longrightarrow \mathcal{K}(S^3)
\]
Consider an arc in the boundary of $D^2 \times \{pt\}$, as the red one in Figure 3.3. This lifts to a longitude of the solid torus. We can then define a natural embedding $e$ of the solid torus in $S^3$ by sending this longitude to the preferred longitude of the solid torus in $S^3$ arising as the neighbourhood of the standard unknot. By composing $\gamma_T$ with $e$ we can associate to a knotoid in $\mathbb{K}(\mathbb{R}^2)$ a knot in $S^3$.

**Proposition 3.5.** Given a knotoid $k$ in $\mathbb{K}(\mathbb{R}^2)$, $e(\gamma_T(k)) = \gamma_S(\iota(k))$. Similarly, given $k \in \mathbb{K}(S^2)$ take any planar representative $k^{pl}$ of $k$. Then, the knot type of $e(\gamma_T(k^{pl}))$ does not depend on the particular choice of $k^{pl}$.

In other words, the knot type in $S^3$ of the lift a planar knotoid $k$ depends only on its class $\iota(k) \in \mathbb{K}(S^2)$.

**Proof.** Consider the diagram for $k$ arising from the projection onto $D^2 \times \{pt\}$. The 2-fold cover of the disk branched along the endpoints can be viewed as the restriction of the 2-fold cover of a 2-sphere branched along the endpoints, see Figure 3.4. Thus, isotopies on the sphere below translate into isotopies on the sphere for the lifted diagram.

\[\square\]

**Remark 3.6.** Note that, as shown in Figure 3.5, non-equivalent knotoids in $\mathbb{K}(\mathbb{R}^2)$ that are equivalent in $\mathbb{K}(S^2)$ might lift to different knots in the solid torus.

For knot type knotoids the behaviour under the maps $\gamma_S$ and $\gamma_T$ is unsurprisingly trivial.

**Proposition 3.7.** Consider an oriented knot-type knotoid $K$. The lift $\gamma_S(K)$ is the connected sum $K' \# r K'$, where $K'$ is the knot naturally associated to $K$ (with orientation induced by $K$) and $r K'$ is its reverse.
Figure 3.5. Two knotoid diagrams $k_1$ and $k_2$ that represent different classes in $\mathbb{K}(\mathbb{R}^2)$ such that $\iota(k_1) = \iota(k_2)$ lift to different knots in the solid torus. $\gamma_T(k_2)$ is the core of the solid torus, while $\gamma_T(k_1)$ is the $2_3$ knot in Gabrovsek and Mroczkowski’s table for knots in the solid torus (see [12]), with winding number equal to 3. However, $e(\gamma_T(k_1)) = e(\gamma_T(k_2)) = \emptyset$.

**Proof.** Thanks to Proposition 3.3 we can choose a planar diagram for $k$ in which the endpoints lie in the external region of the disk, so that there are no intersections (apart from the endpoints themselves) between the diagram and the arc which define the cuts, and the statement is trivially true. □

3.3. Knotoids and strongly invertible knots. Consider a knotoid $k \in \mathbb{K}(S^2)$ and its lift $\gamma_S(k)$ in $S^3$. The fact that $S^3$ is the double cover of itself branched along the preferred constituent unknot $U$ of $t(k)$ defines an orientation preserving involution $\tau$ of $S^3$, whose fixed point set is precisely the unknot $U$. The involution $\tau$ reverses the orientation of $\gamma_S(k)$, and the the fixed point set intersects $\gamma_S(k)$ in exactly two points (the lifts of the endpoints of $k$). A knot with this property is called a strongly invertible knot (a precise definition will be given in Section 5).

Since not every knot in $S^3$ is strongly invertible, this in particular implies that the maps $\gamma_S$ and $\gamma_T$ are not surjective.

**Remark 3.8.** We could have inferred the non-surjectivity of the map $\gamma_T$ from the following observation: the winding number $[\gamma_T(k)] \in H_1(S^1 \times D^2; \mathbb{Z})$ of the lift $\gamma_T(k)$ of a knotoid $k$ is always odd. This is true since by construction the lifted knot intersects the meridian disk containing the lifted branching points an odd number of times.

In Section 5 we will use classical results on symmetry groups of knots to better understand the map $\gamma_S$ and to prove the 1-1 correspondence of Theorem 1.1.

3.4. Behaviour under forbidden moves. A band surgery is an operation which deforms a link into another link.

**Definition 3.9.** Let $L$ be a link and $b : I \times I \rightarrow S^3$ an embedding such that $L \cap b(I^2) = b(I \times \partial I)$. The link $L_1 = (L \setminus b(I \times \partial I)) \cup b(\partial I \times I)$ is said to be obtained from $L$ by a band surgery along the band $B = b(I \times I)$, see Figure 3.6.
Performing a band surgery on a link $L$ may change its number of components; band surgeries which leave unchanged the number of components are called $H(2)$-moves (see e.g. [1], [16]).

![Figure 3.6. Band surgery.](image)

An $H(2)$-move is an unknotting operation, that is, any knot may be transformed into the trivial knot by a finite sequence of $H(2)$-moves. Consider two knotoids that differ by a forbidden move, as on the top of Figure 3.7, it is easy to see that their lifts are related by a single $H(2)$-move (see the bottom part of Figure 3.7).

![Figure 3.7. Two knotoids that differ by a forbidden move have lifts related by a single band surgery.](image)

4. Multiplication and trivial knotoid detection

In this section we will first discuss the behaviour of $\gamma_S$ under multiplication of knotoids. We will then prove two different results on the detection of the trivial knotoid.

4.1. Behaviour under multiplication. Double branched covers of simple $\theta$-curves have been extensively studied in [3].

**Definition 4.1.** A $\theta$-curve is said to be prime if:

- it is non-trivial;
- it is not the connected sum of a non trivial knot and a $\theta$-curve (see the top part of Figure 4.1);
- it is not the result of a vertex-multiplication (for a definition of the vertex-multiplication operation see e.g. [32], Section 5) of two non-trivial $\theta$-curves (see the bottom part of Figure 4.1).

According to Definition 4.1 if $K$ is a knot-type knotoid, then $t(K)$ is the vertex multiplication of a non trivial knot and a $\theta$-curve, thus, it is not prime. The following result is attributed to Thurston by Moriuchi ([27], Theorem 4.1), and it has been proven in [3].
Figure 4.1. On the top, the result of a connected sum between a knot and a $\theta$-curve. On the bottom, the result of a vertex-multiplication of two $\theta$-curves.

**Theorem 4.2** (Main Theorem in [3]). Consider a simple $\theta$-curve $a$, with unknotted constituent knot $a_1$, and let $K$ be the closure of the pre-image of $a \setminus a_1$ under the double cover of $S^3$ branched along $a_1$. Then $K$ is prime if and only if $a$ is prime.

With a little abuse of notation, we will call $k_1 \cdot k_2$ the unoriented knotoid-class of the multiplication between two oriented knotoids. Theorem 4.2 together with Theorem 3.2 directly imply the following result on knotoids.

**Theorem 4.3.** The lift $\gamma_S(k)$ of a proper knotoid $k$ is prime if and only if $k$ is prime. In particular $\gamma_S(k_1 \cdot k_2) = \gamma_S(k_1) \# \gamma_S(k_2)$.

Note that even if the products $k_1 \cdot k_2$ and $k_2 \cdot k_1$ are in general distinct both as oriented and unoriented knotoids (see the relations described in Section 3.2), their lift are equivalent as knots in $S^3$. This seems to imply that $\gamma_S$ can not tell apart $k_1 \cdot k_2$ and $k_2 \cdot k_1$. Indeed, to distinguish them it is necessary to use the information on the involution defined by the double branched cover construction, as we will see in Section 5.

Consider the mutant knotoids $k_1$ and $k_2$ of Figure 2.8; Proposition 3.7 and Theorem 4.3 imply that:

$$\gamma_S(k_1) = KT \# KT \# \gamma_S(k)$$
$$\gamma_S(k_2) = C \# C \# \gamma_S(k)$$

Since $\gamma_S(k)$ is isotopic to the trefoil knot $3_1$ (see Figure 8.4), and since the genus of a knot is additive under connected sum, it follows that:

$$g(\gamma_S(k_1)) = 2 + 2 + 1 = 5 \quad g(\gamma_S(k_2)) = 3 + 3 + 1 = 7$$

Thus, $\gamma_S(k_1)$ and $\gamma_S(k_2)$ are different knots. Moreover, by letting $k$ vary in the set of proper knotoids, we obtain an infinite family of pairs of knotoids sharing the same polynomial invariants whose images under $\gamma_S$ are different.

**4.2. Trivial knotoid detection.** The double branched cover of knotoids provides a way to detect the trivial knotoid, thanks to the following result.

**Theorem 4.4** (Lemma 2.3 in [3]). A knotoid $k \in K(S^2)$ lifts to the trivial knot in $S^3$ if and only if $k$ is the trivial knotoid $k_0$ in $K(S^2)$. 
Theorem 4.4 is proven for $\theta$-curves. In the setting of knotoids, a slightly more powerful version of this result holds, allowing for the detection of the trivial planar knotoid $k_{pl}^0 \in K(\mathbb{R}^2)$ as well.

**Theorem 4.5.** A knotoid $k \in K(\mathbb{R}^2)$ lifts to a knot isotopic to the core of the solid torus if and only if $k = k_{pl}^0$ in $K(\mathbb{R}^2)$.

**Proof.** If $k$ is the trivial knotoid, then its lift is a knot isotopic to the core of the solid torus (see e.g. the right side of Figure 3.5). Conversely, suppose that $\gamma_T(k)$ is isotopic to the core $C$ of the solid torus $S^1 \times D^2$. Then, its complement in the solid torus is homeomorphic to the product $T^2 \times I$. Since $T^2 \times I$ arises as a double branched cover, there is an involution $\tau$ of $T^2 \times I$ with 4 disjoint arcs as fixed set (see Figure 4.2).

![Figure 4.2](image)

**Figure 4.2.** $T^2 \times I$ admits an involution with fixed set the union of 4 arcs. These arcs are the intersection between the lines defining the cover and the complement of a tubular neighbourhood of the lifted knot (the core of the solid torus).

Thanks to the following result we know that the involution defined by the double branched cover respects the product structure on $T^2 \times I$.

**Theorem 4.6** (Theorem A of [17]). Let $h$ be a PL involution of $F \times I$, where $F$ is a compact surface, such that $h(F \times \partial I) = F \times \partial I$. Then, there exist an involution $g$ of $F$ such that $h$ is equivalent (up to conjugation with homeomorphisms) to the involution of $F \times I$ defined by $(x,t) \mapsto (g(x), \lambda(t))$ for $(x,t) \in F \times I$, and where $\lambda : I \to I$ is either the identity or $t \mapsto 1 - t$.

The intersection between the fixed set $Fix(\tau)$ and every torus $T^2 \times \{pt\}$ consists of 4 isolated points, as highlighted in Figure 4.2. Involutions of closed surfaces are completely classified; the following result is well known, and it probably should be attributed to [18], but we refer to [7] for a more modern and complete survey.

**Theorem 4.7** (Theorem 1.11 of [7]). There is only one involution $\bar{\tau}$, up to conjugation with homeomorphisms, for the torus $S^1 \times S^1$ with 4 isolated fixed points. This involution is shown in Figure 4.3: it is orientation preserving and it is induced by a rotation of $\pi$ about the dotted line indicated in the picture.

With an abuse of notation, call $\bar{\tau}$ the involution of $T^2 \times I$ obtained as the product $\tau \times Id_I$. Since conjugated involutions produce homeomorphic quotient spaces, thanks to the previous two results we can say that the
complement of the trivial knot in the solid torus projects to a homeomorphic copy of the complement of the trivial knotoid in the three-ball. In other words, our quotient space $T^2 \times I/\tau$ is homeomorphic to $T^2 \times I/\bar{\tau}$, the last one being precisely the complement of the trivial knotoid, as in Figure 4.4.

We will be done once we prove that the line isotopy class of the curve in Figure 4.4 is not affected by the action of homeomorphisms; this is a consequence of the following proposition. Let $Y$ be the cylinder $D^2 \times I$, and call $\text{MCG}(Y; p, q)$ the group of isotopy-classes of automorphisms of $Y$ that leave $p \times I$ and $q \times I$ invariant, where $p, q$ are points in the interior of $D^2 \times \{pt\}$.

Proposition 4.8. $\text{MCG}(Y; p, q)$ is isomorphic to $\mathbb{Z}$, and it is generated by a Dehn-twist along the blue disk in Figure 4.5.

Figure 4.3. The involution of the torus with 4 fixed points, indicated with the red stars.

Figure 4.4. The quotient space under the involution is homeomorphic to the complement of the trivial knotoid in the cylinder.

Figure 4.5. $\text{MCG}(Y; p, q)$ is generated by a Dehn twist along the blue disk.
The proof of Proposition 4.8 requires a couple of preliminary results. First, note that removing the two lines yields a 3-dimensional genus 2 handlebody $H$. The homeomorphisms of a handlebody are determined by their behaviour on the boundary; more precisely, the mapping class group of a handlebody can be identified with the subgroup of the mapping class group of its boundary, consisting of homeomorphisms that can be extended to the handlebody due to the following lemma.

**Lemma 4.9.** Let $H$ be a genus 2 handlebody. Any homeomorphism $\phi : H \rightarrow H$ such that $\phi|_{\partial H}$ is isotopic to $Id_{\partial H}$ is isotopic to $Id_H$.

The previous lemma is well known, and a proof may be found e.g. in Chapter 3 of [10].

**Remark 4.10.** Recall that a self homeomorphism of the boundary of a handlebody can be extended to the handlebody if and only if the image of the boundary of every meridian disc is contractible in the handlebody. In particular, Dehn twists along the blue curves in Figure 4.6 do not extend to the handlebody.

Now, cutting the boundary of the handlebody $H$ along the blue curves, as in Figure 4.6 produces a sphere with 4 holes $S$.

![Figure 4.6](image)

**Figure 4.6.** Cutting the boundary of the handlebody along the blue curves gives back the sphere with 4 holes $S$.

A proof for the following lemma can be deduced from e.g. the proof of Proposition 2.7, Chapter 2, [9]. Given a surface $S$ with boundary, denote by $MCG(S, \partial S)$ the group of isotopy classes of orientation-preserving homeomorphisms of $S$ that leave each boundary component invariant.

**Lemma 4.11.** Let $S$ be the sphere with 4 holes. Then, $MCG(S, \partial S)$ is isomorphic to a subgroup of $MCG(T^2)/\sim = PSL(2, \mathbb{Z})$.

Thus, we defined a homomorphism $MCG(Y; p, q) \rightarrow MCG(S, \partial S)$, and an injective homomorphism $MCG(S, \partial S) \rightarrow PSL(2, \mathbb{Z})$. An element in the kernel of the composition of these two homomorphisms is then an automorphism of $\partial H$ that leaves the two blue curves in the left side of Figure 4.6 invariant and that is isotopic to the identity on $S$. Such an element is a product of Dehn twists about the two blue curves of Figure 4.6, but thanks to Remark 4.10, the only element in $MCG(Y; p, q)$ of that form is the trivial element. Moreover, Lemma 4.11 is proven by exhibiting a bijection between homotopy classes of essential closed curves in $T^2$ and in $S$, and
this in particular implies that any homeomorphism $\phi : S \to S$ leaving each component of $\partial S$ invariant is completely determined by the images of the curves $\nu$ and $\lambda$ in Figure 4.7. Now, $\phi(\nu) = \nu$, since $\nu$ is the only essential closed curve in $S$ which is trivial in $H_1(H)$; on the other hand, Remark 4.10 implies that $\phi(\lambda)$ is the curve that results from $\lambda$ by applying a Dehn twist along $\nu$. Putting all together, we obtain a proof for Proposition 4.8 and Theorem 4.5, as wanted.

5. Knotoids and strongly invertible knots

5.1. Proof of the main theorem. This section is devoted to the proof of the main result, Theorem 1.1. We should point out that the correspondence between knotoids and strongly invertible knots is partially inspired by the construction in [34], Section 2.2. We begin by giving a precise definition of what a strongly invertible knot is. Recall that $\text{Sym}(S^3, K)$ denotes the symmetry group of a knot $K$, that is, the group of diffeomorphisms of the pair $(S^3, K)$ modulo isotopies, and $\text{Sym}^+(S^3, K)$ is the subgroup of $\text{Sym}(S^3, K)$ of diffeomorphisms preserving the orientation of $S^3$.

Definition 5.1. A strongly invertible knot is a pair $(K, \tau)$, where $\tau \in \text{Sym}(S^3, K)$ is called a strong inversion, and it is an orientation preserving involution of $S^3$ that reverses the orientation of $K$, taken up to conjugacy in $\text{Sym}^+(S^3, K)$. Thus, two strongly invertible knots $(K_1, \tau_1)$ and $(K_2, \tau_2)$ are equivalent if there is an orientation preserving homeomorphism $f : S^3 \to S^3$ satisfying $f(K_1) = K_2$ and $f\tau_1f^{-1} = \tau_2$.

Call $\mathcal{KSI}(S^3)$ the set of strongly invertible knots $(K, \tau)$ in $S^3$, up to equivalence, and $\mathcal{K}_{S.I.}(S^3)$ the subset of $\mathcal{K}(S^3)$ consisting of knots that admit a strong inversion. There is then a natural forgetful map $\mathcal{KSI}(S^3) \to \mathcal{K}_{S.I.}(S^3)$. As we saw in Section 3.3 the lift of a knotoid through the double branched cover of $S^3$ is a strongly invertible knot, thus, $\gamma_S(\mathcal{K}(S^2)) \subset \mathcal{K}_{S.I.}(S^3)$. More precisely, the double branched cover determines an involution $\tau$. Thus, we can promote $\gamma_S$ to a map $\gamma_S : \mathcal{K}(S^2) \to \mathcal{KSI}(S^3)$. On the other hand, given a strongly invertible knot it is possible to produce a knotoid through the construction explained below. Consider a strongly invertible knot $(K, \tau) \in \mathcal{KSI}(S^3)$: the fixed point set of $\tau$ is an unknotted circle (thanks to the positive resolution of the Smith conjecture, [33]).
Moreover, $\tau$ defines the projection

$$p : S^3 \rightarrow S^3/\tau \cong S^3$$

that can be interpreted as the double cover of $S^3$ branched along $Fix(\tau)$.

**Figure 5.1.** The trefoil is a strongly invertible knot. Up to isotopy, we can represent the fixed point set as the $z$ axis in $\mathbb{R}^3$. On the right, the $\theta$-curve obtained from the projection. The unknotted component is again represented as the $z$ axis.

From $(K, \tau)$ we can construct the $\theta$-curve $\theta(K, \tau) = p(Fix(\tau)) \cup p(K)$, where $p(Fix(\tau)) = e_- \cup e_+$ and $p(K) = e_0$, as explained in [29] and as shown in Figure 5.1. Equivalent strongly invertible knots project to isotopic $\theta$-curves. Thus, we have a well defined map

$$\beta : KSI(S^3) \rightarrow \Theta^* / \sim$$

**Figure 5.2.** Up to isotopy we may assume that the vertices of $\theta(K, \tau)$ lie in $\mathbb{R}^2 \times \{0\}$ in $S^3 = \mathbb{R}^3 \cup \infty$, and that the edges $e_-$ and $e_+$ lie in the in the lower and upper half-space respectively, and that they project bijectively to the same embedded arc $a \in \mathbb{R}^2$. We can then isotope $p(K)$ such that it intersect transversally the disk $D$ bounded by $p(Fix(\tau)) = e_- \cup e_+$ and such that its projection on $\mathbb{R}^2$ is regular. The projection (with the additional data of over/under passings) is a knotoid $t^{-1}(\theta(K, \tau))$. The knotoid $t^{-1}(\theta(K, \tau))$ only depends on the isotopy class of $\theta(K, \tau)$.

The image of $\theta(K, \tau)$ under the inverse $t^{-1}$ of the bijection given by Theorem 3.2 is the knotoid $t^{-1}(\theta(K, \tau))$, see Figure 5.2 for an example. Thus, we have a well defined map

$$\Pi = \beta \circ t^{-1}$$
from the set of strongly invertible knots to $K(S^2)$, and since the preferred constituent unknot of $t(t^{-1}(\theta(K,\tau))) = \theta(K,\tau)$ is clearly $p(Fix(\tau))$, $\Pi$ is the inverse of $\gamma_S$. From this and the discussion in Section 3.3 we obtain that

$$\gamma_S : K(S^2) \rightarrow KSI(S^3)$$

is a bijection, and Theorem 1.1 is proven.

Call $k_1$ the unoriented knotoid of Figure 5.2 and consider the product $k_1 \cdot k_1$, whose image through $\gamma_S$ is the connected sum $3_1 \# 3_1$. Recall that (see Proposition 3.7) the knot-type knotoid $k$ associated to the trefoil knot lifts to $3_1 \# 3_1$ as well (since the trefoil is invertible, thus $3_1 \sim -3_1$). Then, Theorem 1.1 implies that $3_1 \# 3_1$ admits at least two non-equivalent involutions, associated to $k_1 \cdot k_1$ and $k$ respectively. These involutions are shown in Figure 5.3.

![Figure 5.3](image1)

**Figure 5.3.** The fixed point sets of two non-equivalent involutions are shown here. The one corresponding to the vertical line associates the knot $3_1 \# 3_1$ to $k_1 \cdot k_1$. The quotient under the involution corresponding to the horizontal line gives the knot-type knotoid associated to the trefoil.

Similarly, Figure 5.4 shows two different involutions of the composite knot $3_1 \# 8_{20}$, defining different composite knotoids.

![Figure 5.4](image2)

**Figure 5.4.** Two different involutions of the composite knot $3_1 \# 8_{20}$, associated to the composite knotoids $k_1 \cdot k_2$ and $k_2 \cdot k_1$. 
5.2. **Strong inversions.** It is a classical result [19] that every knot admits a finite number of non equivalent strong inversions. For torus and hyperbolic knots a stronger result holds. Recall that we say that a knot $K$ admits period 2 if it is fixed by an orientation preserving involution which also preserves the knot orientation; in other words, $K$ has period 2 if there exist a non-trivial $\phi \in \text{Sym}^+(S^3, K)$ with $\phi^2 = \text{id}$ that preserves the orientation on $K$.

**Theorem 5.2** (Proposition 3.1, [29]). A torus knot admits exactly one strong inversion. If a hyperbolic knot is strongly invertible, then it admits either 1 or 2 non equivalent inversions, and it admits exactly 2 if and only if it admits period 2.

The previous result together with Theorem 1.1 proves Corollary 1.2. Thus, to every torus knot there is a single knotoid associated, and to every hyperbolic knot there are at most two. We give the following definition, borrowed from classical knot theory.

**Definition 5.3.** We will call torus knotoids the knotoids in $\mathbb{K}(S^2)$ whose lifts are torus knots. Similarly, we will call hyperbolic knotoids those lifting to hyperbolic knots.

More generally, only finitely many knotoids are associated with a single knot type. Hence it is natural to ask the following.

**Question 5.4.** Is there an algorithm to decide whether two knotoids $k_1$ and $k_2$ in $\mathbb{K}(S^2)$ are equivalent?

Since there are now several known ways to solve the knot recognition problem (for a survey, see e.g [22] and [8]), to answer Question 5.4 positively it would be enough to be able to decide whether two given involutions of a knot complement are conjugate homeomorphisms. As stated in the introduction, using the the solution to the equivalence problem for hyperbolic knots ([26] and [21]), since there is an algorithm to decide whether two involutions of a hyperbolic knot complement are conjugate ([29]), this can be done in the hyperbolic case. We can then state the following result.

**Theorem 5.5.** Given two hyperbolic knotoids $k_1$ and $k_2$, there is an algorithm to determine whether $k_1$ and $k_2$ are equivalent.

**Remark 5.6.** Note that Question 5.4 can be answered positively using the correspondence between knotoids and $\theta$-curves (Theorem 3.2). Indeed, given two $\theta$-curves, we can consider their complements in $S^3$, together with the data of the meridians of the three edges. We could then let Haken’s algorithm (see [14], [31]) run to decide whether or not the obtained 3-manifolds are equivalent. However, the algorithm of Theorem 1.5 is actually practical, whereas Haken’s algorithm is not.

### 6. Amphichirality and invertibility

In this section we show how Corollary 1.2 can be improved in two different ways, by considering properties of the symmetry groups of hyperbolic knots.
6.1. Invertible knotoids. Let’s turn our attention back to oriented knotoids. Even if the maps $\gamma_S$ and $\gamma_T$ can not distinguish between two knotoids differing only in the orientation, using $\gamma_S$ it is possible to tell whether a hyperbolic knotoid is invertible or not. We begin by giving the following definition.

**Definition 6.1.** An oriented knotoid $k \in \mathcal{K}(S^2)$ is called invertible if it is unchanged when its orientation is reverse. That is, if it is line-isotopic to its reverse.

The following lemma follows from Theorem 3.2.

**Proposition 6.2.** An oriented knotoid $k \in \mathcal{K}(S^2)$ is invertible if and only if there is an isotopy taking the $\theta$-curve $t(k)$ back to itself, that preserves each edge but swaps the two vertices.

Furthermore, we can prove that the isotopy of the previous lemma has order 2.

**Lemma 6.3.** A hyperbolic, oriented knotoid $k \in \mathcal{K}(S^2)$ is invertible if and only if there is an order two homeomorphism of $S^3$ taking the $\theta$-curve $t(k)$ back to itself, that preserves each edge but swaps the two vertices.

**Proof.** One direction is clear. So suppose that $k$ is a hyperbolic knotoid that is invertible. Let $t(k)$ be the corresponding $\theta$-curve, with $e_-, e_+$ and $e_0$, where $e_- \cup e_+$ is the unknot. By hypothesis, $\gamma_S(k)$ is hyperbolic. The involution on the hyperbolic manifold $S^3 \setminus \text{int}(N(\gamma_S(k)))$ is realised by a hyperbolic isometry $\tau$. The quotient $(S^3 \setminus \text{int}(N(\gamma_S(k))))/\tau$ is therefore a hyperbolic orbifold $O$. Its underlying space is the 3-ball $S^3 \setminus \text{int}(N(e_0))$, and its singular locus is the intersection with $e_- \cup e_+$. Now, for the other direction, we are assuming that $k$ is invertible. Hence, by Lemma 6.3, there is a homeomorphism $\rho$ of $S^3$ taking $t(k)$ back to itself, that preserves each edge but swaps the two vertices. This therefore induces a homeomorphism of $O$ that preserves its singular locus. By Mostow rigidity, this is isotopic to an isometry $\overline{\rho}$ of $O$. The action of $\overline{\rho}^2$ on $\partial O$ is isotopic to the identity, via an isotopy that preserves the singular points throughout. Hence, because it is an isometry of a Euclidean pillowcase orbifold, $\overline{\rho}^2$ is the identity. Therefore, $\overline{\rho}^2$ is actually equal to the identity on $O$. This extends to the required order two homeomorphism of $S^3$, taking $t(k)$ back to itself, that preserves each edge but swaps the two vertices. \qed

The last ingredient we need in order to characterise hyperbolic, invertible knotoids is the following theorem. This result is well known, and a proof can be found e.g. in [29].

**Theorem 6.4 ([29], Lemma 3.3).** Let $K$ be a hyperbolic knot. Then the orientation-preserving symmetry group of $K$ can be realised as a finite group of homeomorphisms of $S^3$ that preserve $K$. This group acts faithfully on $K$. Hence, the symmetry group is a dihedral group $D_n$, where $n$ is even if and only if $K$ has period 2.

**Theorem 6.5.** A hyperbolic, oriented knotoid $k \in \mathcal{K}(S^2)$ is invertible if and only if $\gamma_S(k)$ has period 2.
Proof. Suppose first $\gamma_S(k)$ has period 2. Then there exists an orientation-preserving homeomorphism $\rho: S^3 \rightarrow S^3$ of order 2 that preserves $\gamma_S(k)$ and preserves its orientation. It acts on the circle $\gamma_S(k)$ like a rotation of order $\pi$. Also, since $\gamma_S(k)$ is strongly invertible, there is an orientation preserving-homeomorphism $\tau: S^3 \rightarrow S^3$ that preserves $\gamma_S(k)$ and reverses its orientation. Because $\gamma_S(k)$ is hyperbolic, its symmetry group $G$ is isomorphic to a dihedral group, which we can view as a group of symmetries of the circle $\gamma_S(k)$, by Theorem 6.4. Both $\tau$ and $\rho$ can be viewed as elements of this symmetry group. Now, $\rho$ is rotation by $\pi$ on the circle $\gamma_S(k)$, and hence it lies in the centre of $G$. In particular, it commutes with $\tau$. This implies that $\rho$ descends to an order two homeomorphism $S^3/\tau \rightarrow S^3/\tau$ that preserves the knotoid $k$ but reverses its orientation. Hence, $k$ is invertible. Suppose now that $k$ is invertible. Then by Proposition 6.2 there is an orientation-preserving order 2 homeomorphism of $S^3$ taking $\ell(k)$ back to itself, that preserves each edge but swaps the two vertices. This lifts to an order 2 homeomorphism $\rho: S^3 \rightarrow S^3$ preserving $\gamma_S(k)$. This swaps the two points of intersection between $\gamma_S(k)$ and the fixed-point set of $\tau$. The composition $\rho \circ \tau$ also has this property. One of $\rho$ or $\rho \circ \tau$ preserves the orientation of $\gamma_S(k)$ and hence acts as a rotation by $\pi$ on $\gamma_S(k)$. Therefore, $\gamma_S(k)$ has period 2.

By combining Theorem 6.5 with Theorem 5.2 we obtain the following improvement of Corollary 1.2, dealing with oriented knotoids.

**Corollary 6.6.** Given any strongly invertible hyperbolic knot $K_h$ there are exactly two oriented knotoids associated to it. If $K_h$ does not have period 2, then these oriented knotoids are one the reverse of the other.

![Figure 6.1.](image)

There is a unique unoriented knotoid associated to the knot $8_{20}$, which is not invertible.

In particular, the knotoid in Figure 6.1 is not invertible. In fact, its image through the double branched cover construction is the knot $8_{20}$, which is hyperbolic with symmetry group isomorphic to $D_1$ (thus, it does not admit period 2).
6.2. **Amphichiral strongly invertible knots.** It is possible to further improve Corollary 1.2 by considering amphichirality in hyperbolic knots.

**Definition 6.7.** A knot $K$ is called amphichiral if there exists an orientation-reversing homeomorphism of $S^3$ fixing the knot (setwise). Note that this implies that $K$ is equivalent to its mirror $K_m$.

Consider an invertible, hyperbolic, amphichiral knot $K$, and suppose that it admits period 2. From Theorem 5.2 it follows that $K$ admits two non-equivalent involutions $\tau_1$ and $\tau_2$. Let $\phi$ be the (isotopy class of) the orientation reversing homeomorphism of Definition 6.7 from [29, Propositions 3.4], we know that $\tau_1$ and $\tau_2$ are conjugated through $\phi$

$$\tau_2 = \phi \circ \tau_1 \circ \phi^{-1}$$

Thus, $(K, \tau_1)$ is equivalent to $m(K, \tau_2)$, where $m(K, \tau_2)$ is the strongly invertible knot obtained from $(K, \tau_2)$ by reversing the orientation of $S^3$, and the following holds.

**Proposition 6.8.** Given an invertible, hyperbolic, amphichiral knot $K$ admitting period 2, and let $\tau_1$ and $\tau_2$ be the two non-equivalent strong involutions of $K$. Then $\Pi(K, \tau_1)$ is the mirror image of $\Pi(K, \tau_2)$.

We refer to the mirror image of a knotoid $k$ as $k_m$. We emphasise that $\Pi(K, \tau_1)$ and $\Pi(K, \tau_2)$ are unoriented knotoids. However, thanks to Theorem 6.5 we know that these knotoids are both invertible. Thus, Proposition 6.8 implies that given a knotoid $k$ whose image through the double branched cover construction is an invertible, hyperbolic, amphichiral knot $K$ admitting period 2, the only other oriented knotoid associated to $K$ is $k_m$.

6.2.1. **Example: $4_1$.** We work out the case of the figure eight knot ($4_1$ in the Rolfsen table) as an example of Proposition 6.8. The $4_1$ knot is known to be hyperbolic, invertible, amphichiral and it admits period 2; thus, it admits two distinct inversions $\tau_1$ and $\tau_2$, shown in the upper part of Figure 6.2.

Figure 6.2. On the top, a diagram for $4_1$ with the fixed point sets of $\tau_1$ and $\tau_2$ represented as straight lines. On the bottom, the $\theta$-curves $\theta(4_1, \tau_1)$ and $\theta(4_1, \tau_2)$.

By considering the quotients under $\tau_1$ and $\tau_2$ we obtain the $\theta$-curves $\theta(4_1, \tau_1)$ and $\theta(4_1, \tau_2)$, shown in the bottom of Figure 6.2. Their constituent
knots are two unknots and the torus knot $5_1$, and two unknots and the mirror image of $5_1$ respectively. Since it is well known that $5_1 \sim m5_1$, it follows that $\theta(4_1, \tau_1) \sim \theta(4_1, \tau_2)$, in accordance with Theorem 1.1.

Figure 6.3. On the top: from $\theta(4_1, \tau_1)$ to $\Pi(4_1, \tau_1)$. On the bottom: from $\theta(4_1, \tau_2)$ to $\Pi(4_1, \tau_2)$. The two knotoids are one the mirror image of the other.

In figure 6.3 we show how to obtain the knotoids $\Pi(4_1, \tau_1)$ and $\Pi(4_1, \tau_2)$ as the image of the $\theta$-curves under the isomorphism $t^{-1}$ of Theorem 3.2. It is clear from the picture that these knotoids are one the mirror image of the other.

7. On the map $\gamma_T$: an example

It is often hard to distinguish non-equivalent planar knotoids which represent the same class in $\mathbb{K}(S^2)$. Important developments in this direction have been carried on in [6], where polynomial invariants are used to detect the planar knotoid types associated to open polymers. In what follows, we show how we can efficiently use the map $\gamma_T$ to this end. Consider the pair of knotoids $k_1$ and $k_2$ in $\mathbb{K}(\mathbb{R}^2)$ of Figure 7.1 on the top. They both represent the trivial knotoid in $\mathbb{K}(S^2)$.

The images of knotoids $k_1$ and $k_2$ under $\gamma_T$ are two knots in the solid torus. To distinguish them, we can consider the following construction. We can embed the solid torus in $S^3$ as done in Section 3.2, but this time after giving a full twist along the meridian of $S^1 \times D^2$. We then obtain two knots
Figure 7.1. On the top, the knotoids $k_1$ and $k_2$ in $\mathbb{R}^2$. On the bottom, their images under $\gamma_T$.

in $S^3$, shown in Figure 7.2 that can be easily shown to be the knots $9_{46}$ and $5_2$ by computing the Alexander and Jones polynomials. These invariants are in fact enough to distinguish them since, according to knotinfo [4], the knots $5_2$ and $9_{46}$ are uniquely determined by their Alexander and Jones polynomials among all knots up to 12 crossings. Note that this procedure may be applied to several similar cases, highlighting the power of the map $\gamma_T$.

We emphasise that the authors are not aware of any other invariant capable of distinguishing $k_1$ and $k_2$, and that this example was kindly suggested by Dimos Goundaroulis.

Figure 7.2. By embedding the solid torus in $S^3$ as in Section 3.2 after giving a full twist along the meridian, we obtain a pair of knots in $S^3$.

8. Gauss Code and Computations

The oriented Gauss code $GC(D)$ for a knot diagram $D$ is an pair $(C, S)$, where $C$ is a $2n$-tuple and $S$ a $n$-tuple, $n$ being the number of crossings of the diagram. Given a diagram $D$, $GC(D)$ is constructed as follows: assign a number between 1 and $n$ to each crossing, and pick a point $a$ in the diagram, which is not a double point. Start walking along the diagram from $a$, following the orientation, and record every crossing encountered (in order) by adding an entry to $C$ consisting of the corresponding number, together with a sign $+$ for overpassing and $-$ for underpassing. $S$ is the $n$-tuple whose $i$th entry is equal to 1 if the $i$th crossing is positive and $-1$ otherwise. As
an example, the Gauss code associated to the diagram in Figure 8.1 is equal to:

\[ GC(D) = ((1, -2, 3, -1, 2, -3), (1, 1, 1)) \]

**Figure 8.1.** Computing the Gauss code for a knot diagram.

Gauss codes can be easily extended to knotoid diagrams, see [13]. The procedure is basically the same, but in this case the starting point \( a \) coincides with the tail of the diagram. As an example, the Gauss code for the knotoid in Figure 8.2 is equal to:

\[ GC(D) = ((-1, 1, 2, -3, -2, 3), (1, 1, 1)) \]

**Figure 8.2.** Computing the Gauss code for a knotoid.

The information encoded in \( GC(D) \) is enough to reconstruct \( D \), both in the case of knot and knotoid diagrams.

### 8.1. Generalised Gauss code for knotoids

Gauss code for knotoid diagrams may be generalised to contain also the information about the intersection with the branching set. We will call the *generalised Gauss code* (indicated as \( gGC(D) \)) the pair \( (C, S) \) where \( S \) is the same as in \( GC(D) \), while \( C \) contains also entries equal to \( b \) every time every time the diagram intersects with the arcs connecting the branched points (i.e. the endpoints) with the boundary of the disk containing the diagram. For instance, the Gauss code for the knotoid in Figure 8.3 is:

\[ -1, b, b, 1, 2, -3, -2, 3/1, 1, 1 \]

### 8.2. Gauss code for the lifts

Given a diagram \( D \) representing a knotoid \( k \) with \( gGC(D) = (C, S) \) it is possible to compute \( GC(\gamma_S(D)) \), where \( \gamma_S(D) \) is the diagram representing \( \gamma_S(k) \) obtained with the “cuts” technique, as in Figure 3.3.

Consider the knotoid diagram \( D \) on the left-side of Figure 8.4 with \( gGC(D) = ((1, -2, b, -1, 2), (1, 1)) \). Label the crossings in \( \gamma_S(D) \) as shown on the right-side of Figure 8.4: every half of the annulus is a copy of the disk in which \( D \) lies, keep the same enumeration on the top-half and increase by 2 the labels...
in the bottom one. Now, start computing $GC(\gamma_S(D))$: notice that until we reach an intersection between the diagram and one of the arcs splitting the annulus, the entries added in $GC(\gamma_S(D))$ are equal to the first entries in $gGC(D)$. After an intersection point, the path continues on the bottom half of the annulus, and the next entries added in $GC(\gamma_S(D))$ are equal to the corresponding ones in $gGC(D)$, but with every label increased by 2. Once we reach the lift of the head, the path along the knot continues, and it is the same path we have just done, but in the opposite direction and on opposite halves of the annulus. Thus, the last entries added are a copy of the entries written so far, added in the opposite order and with labels corresponding to opposite halves of the annulus and thus:

$$GC(\gamma_S(D)) = ((1, -2, -3, 4, 2, -1, -4, 3), S)$$

To compute $S$, note that the sign of a crossing in the top-half is the same as its corresponding crossing in the bottom-half. Moreover, since the labels corresponding to each crossings in $gGC(D)$ appear once before the entry $b$ and once after, the signs of the first two crossings in the knot diagram are changed, and the complete Gauss code is

$$GC(\gamma_S(D)) = ((1, -2, -3, 4, 2, -1, -4, 3), (-1, -1, -1))$$

The previous procedure can be generalised to produce an algorithm. Thus, consider the diagram in Figure 8.5 and start walking along the knot from the lift the tail. Every time we pass from one half of the annulus to the other, the path on the diagram follows as in the knotoid diagram, but on a different half. Moreover, as before, once we reach the lift of the head of the knotoid, the path proceeds as the one just traced, in the opposite direction and on different halves as before. Now, suppose that on $gGC(D)$ the two appearances of the same label happen without the occurrence of a $b$ entry between them. This means that in $\gamma_S(D)$ we are going to reach
Figure 8.5. Computing the Gauss code of the lift $\gamma_S(D)$.

the top-lift of the crossing twice without passing to the other half (thus, without swapping the orientation), and the same holds for the bottom-lift of the crossings. In this case the signs of both the lifted crossings in $\gamma_S(D)$ are equal to the sign of the corresponding one in $D$. Putting everything together, we obtain the following algorithm, that can be easily implemented.

**Input:** generalised Gauss code of the knotoid, $n =$ number of crossings in the knotoid diagram:

- go through the knotoid code: copy the entries until you find a $b$;
- until you reach the point corresponding to the head of the knotoid:
  - after reaching a $b$
    - if the number of $b$-entries encountered is odd, add entries equals to the knotoid ones, but changing the labels by adding $n$ to them. Do that until you reach another $b$;
    - if the number of $b$-entries encountered is even, add entries equals to the knotoid ones. Do that until you reach another $b$.
- After reaching the head: copy the entries added so far, starting from the last one, and changing the labels by subtracting $n$ if they are greater than $n$, and adding $n$ otherwise;
- Consider the $k$ crossing in the knotoid diagram:
  - if the corresponding labels in the knotoid code appear twice with an even (or zero) number of $b$-entries between them, then the sign of the $k$ and $k+n$ crossings in the knot diagram are equal to the sign of the starting crossing;
  - if the corresponding labels in the knotoid code appear twice with an odd number of $b$-entries between them, then the sign of the $k$ and $k+n$ crossings in the knot diagram are opposite to the sign of the starting crossing.

**Output:** Gauss code for the lifted knot diagram.

**References**

15. Holzmann, W. H. Involutions with 1-or 2-dimensional fixed point sets on orientable torus bundles over a 1-sphere and on unions of orientable twisted I-bundles over a Klein bottle. Diss. University of British Columbia, 1984.