LERF and the Lubotzky-Sarnak Conjecture

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1   Introduction

We begin by recalling the definition of Property τ. Let X be a finite graph, and let $V(X)$ denote its vertex set. For any subset A of $V(X)$, let ∂A denote those edges with one endpoint in A and one not in A. Define the *Cheeger constant* of X to be

$$h(X) = \min\left\{\frac{|\partial A|}{|A|} : A \subset V(X) \text{ and } 0 < |A| \le |V(X)|/2\right\}.$$

Now let G be a group with a finite symmetric generating set S. For any subgroup G_i of G, let $X(G/G_i; S)$ be the Schreier coset graph of G/G_i with respect to S. Then G is said to have *Property τ with respect to a collection of finite index subgroups* $\{G_i\}$ if $\inf_i h(X(G/G_i; S)) > 0$. This turns out not to depend on the choice of finite generating set S. Also, G is said to have *Property τ* if it has Property τ with respect to the collection of all subgroups of finite index in G.

In the context of of finite volume hyperbolic manifolds, Lubotzky and Sarnak made the following conjecture.

Conjecture 1.1. *The fundamental group of any finite volume hyperbolic n-manifold does not have Property τ.*

It is easy to check that if a group G contains a finite index subgroup surjecting onto **Z**, then G does not have Property τ, and it is this that has attracted attention to the Lubotzky-Sarnak conjecture recently. This is particularly relevant in the context of hyperbolic 3-manifolds (see [14] and [16] for example), in part due to the connection to the virtual positive first Betti number conjecture from 3-manifold topology (see [16] for a discussion).

While it appears to be much weaker than the virtual positive first Betti number conjecture, it appears that there is no method known to show that the fundamental group of a finite volume hyperbolic n-manifold does not have Property τ without exhibiting a surjection onto **Z** from a finite index subgroup. The main result of this note provides a method for hyperbolic 3-manifolds.

To state the main result we require some additional terminology.

Let G be a finitely generated group and H a finitely generated subgroup. G is *H-separable* if H is closed in the profinite topology on G, and G is called *LERF or subgroup separable* if G is H-separable for every finitely generated subgroup $H < G$. We say that H is *engulfed in G* if there is a proper finite index subgroup $K < G$ with $H < K$. In the context of hyperbolic 3-manifolds, it turns out that these two notions are intimately related, see [17]. Another refinement of LERF for Kleinian groups is GFERF; namely if G is a Kleinian group, then G is called *GFERF* if G is H-separable for every geometrically finite subgroup H of G. This has a generalization when G is a word hyperbolic group; G is called *QCERF* if G is H-separable for every finitely generated, quasi-convex subgroup H of G.

We restrict attention to closed orientable hyperbolic 3-manifolds, since in this dimension, it is well-known that the fundamental group of a finite volume, non-compact hyperbolic 3-manifold or a non-orientable closed hyperbolic 3-manifold surjects onto **Z**.

Theorem 1.2. *Let $M = \mathbf{H}^3/\Gamma$ be a closed orientable hyperbolic 3-manifold. Assume that Γ has the property that every infinite index, geometrically finite subgroup of Γ is engulfed in Γ.*

Then the Lubotzky-Sarnak Conjecture holds for Γ.

An immediate corollary of this is (notation as in Theorem 1.2).

Corollary 1.3. *If Γ is LERF, then the Lubotzky-Sarnak Conjecture holds for Γ.*

There is now some evidence that the fundamental group of any finite volume hyperbolic 3-manifold is LERF (see [2], [3] and [13] to name a few). Moreover, Corollary 1.3 was previously known to hold, if in addition Γ contains a surface subgroup (see §3.2 where we give a proof for convenience).

It is also interesting to compare Corollary 1.3 with the result that if an arithmetic lattice in a semi-simple Lie group is LERF it cannot have the Congruence Subgroup Property (see [18] Chapter 4 for example). It is a consequence of Clozel's work [9] (which is the culmination of work of many authors) that if an arithmetic lattice in a semi-simple Lie group has the Congruence Subgroup Property it has Property τ.

Another interesting corollary follows from [16] (see §3 for a proof).

Corollary 1.4. *Assume that the fundamental group of every closed hyperbolic 3-manifold is GFERF. Then, if Γ is an arithmetic Kleinian group, Γ is large.*

It has recently been proved by Agol, Groves and Manning [2] that if every word hyperbolic group is residually finite, then every word hyperbolic group is QCERF. Combining this with above result, we obtain the following unexpected conclusion.

Corollary 1.5. *Assume that every word hyperbolic group is residually finite. Then every arithmetic Kleinian group is large.*

Finally we point out that while the Lubotzky-Sarnak Conjecture remains open, our results have the following consequence even in the absence of the LERF hypothesis. We let

$h(X)$ denote the Cheeger constant of a Riemannian manifold, possibly with infinite volume. When the manifold has finite volume, this is defined to be

$$h(X) = \inf_S \frac{\mathrm{Area}(S)}{\min\{\mathrm{vol}(X_1), \mathrm{vol}(X_2)\}}$$

where the infimum is taken over all smooth co-dimension one submanifolds S that separate X into submanifolds X_1 and X_2. When X has infinite volume, the Cheeger constant is defined to be

$$h(X) = \inf_S \frac{\mathrm{Area}(S)}{\mathrm{vol}(X_1)}$$

where the infimum is taken over all smooth co-dimension one submanifolds S that bound a compact submanifold X_1.

Theorem 1.6. *Let M be a closed hyperbolic 3-manifold.*
 Then there is a sequence of (possibly infinite) coverings M_i for which $h(M_i) \to 0$.

This result has recently been used by the first author [15] to show that nonelementary Kleinian groups which contain a finite noncyclic subgroup are either virtually free, or contain the fundamental group of a closed orientable surface of positive genus. In particular, co-compact arithmetic Kleinian groups contain surface subgroups.

2 Two preliminary propositions

Let N be a possibly noncompact complete Riemannian manifold and Δ the Laplace-Beltrami operator, with sign chosen so that this is a positive operator. Set

$$\lambda_0(N) = \inf\left(\frac{\int_N ||\nabla f||^2}{\int_N f^2}\right),$$

where the infimum is taken over smooth functions f of compact support. It is shown in [8] that $\lambda_0(N)$ is the greatest lower bound of the spectrum of Δ acting on $L^2(N)$.

Remark: When N is a closed Riemannian manifold, $\lambda_0(N) = 0$, and it is $\lambda_1(N)$ (the first non-zero eigenvalue of Δ) that is computed by the above infimum, except that f is required to be orthogonal to the constant functions.

Proposition 2.1. *Let M be a closed Riemannian manifold, H an infinite index finitely generated subgroup of $\pi_1(M)$ and N the cover of M corresponding to H. Suppose that $\pi_1(M)$ is H-separable.*
 Then given $\epsilon > 0$, there is a finite sheeted cover $\tilde{M}$ of M for which $\lambda_1(\tilde{M}) < \lambda_0(N) + \epsilon$.

Proof: Set $\delta = \epsilon/(1 + \epsilon + \lambda_0(N))$. By [8], we may fix some compactly supported function $f : N \longrightarrow \mathbb{R}$ for which

$$\lambda_0(N) + \delta > \frac{\int_N ||\nabla f||^2}{\int_N f^2}.$$

Choose some compact set $X \subset N$ so that support$(f) \subset$ interior(X).

Since $\pi_1(M)$ is H-separable, we may find a finite sheeted covering, $\widetilde{M}$ of M which is subordinate to N and for which the compact set X is embedded by the projection $N \longrightarrow \widetilde{M}$ (see [22]). By choosing a larger covering if necessary, we may arrange that

$$\frac{1}{\text{vol}(\widetilde{M})}\left(\int_X f\right)^2 < \delta \int_X f^2.$$

Define a function $g : \widetilde{M} \longrightarrow \mathbb{R}$ to be f on X and zero elsewhere. It follows that

$$\lambda_0(N) + \delta > \frac{\int_{\widetilde{M}} ||\nabla g||^2}{\int_{\widetilde{M}} g^2}.$$

We need to adjust the function g slightly, since it is not orthogonal to the constant functions. This is achieved by replacing g by $g^* = g - \alpha$ where α is the constant function whose value is $(\int_{\widetilde{M}} g)/\text{vol}(\widetilde{M})$. Then

$$\int_{\widetilde{M}}(g^*)^2 = \int_{\widetilde{M}} g^2 - 2\int_{\widetilde{M}} \alpha g + \int_{\widetilde{M}} \alpha^2 = \int_{\widetilde{M}} g^2 - \left(\int_{\widetilde{M}} g\right)^2/\text{vol}(\widetilde{M}).$$

Now by construction of g, $\int_{\widetilde{M}} g^2 = \int_X f^2$ and $(\int_{\widetilde{M}} g)^2 = (\int_X f)^2$, so that the right hand side of this expression satisfies

$$\int_{\widetilde{M}} g^2 - \left(\int_{\widetilde{M}} g\right)^2/\text{vol}(\widetilde{M}) = \int_X f^2 - \left(\int_X f\right)^2/\text{vol}(\widetilde{M}) > (1 - \delta)\int_X f^2 = (1 - \delta)\int_{\widetilde{M}} g^2$$

so that $(1 - \delta)^{-1} > (\int_{\widetilde{M}} g^2)/(\int_{\widetilde{M}}(g^*)^2)$.

Now, noting that $\nabla g^* = \nabla g$ we compute

$$\lambda_1(\widetilde{M}) \leq \int_{\widetilde{M}} ||\nabla g^*||^2 \Big/ \int_{\widetilde{M}}(g^*)^2 = \int_{\widetilde{M}} ||\nabla g||^2 \Big/ \int_{\widetilde{M}}(g^*)^2$$

$$< (\lambda_0(N) + \delta)\int_{\widetilde{M}} g^2 \Big/ \int_{\widetilde{M}}(g^*)^2 < (\lambda_0(N) + \delta)/(1 - \delta) = \lambda_0(N) + \epsilon$$

as required. $\square$

The following result is an analogue of the above proposition, but using Cheeger constants rather than the first eigenvalue of the Laplacian.

Proposition 2.2. *Let M be a closed Riemannian manifold, H an infinite index finitely generated subgroup of $\pi_1(M)$ and N the cover of M corresponding to H. Suppose that $\pi_1(M)$ is H-separable.*

Then given $\epsilon > 0$, there is a finite sheeted cover $\tilde{M}$ of M for which $h(\tilde{M}) < h(N) + \epsilon$.

Proof: Let X be some compact submanifold of N with zero codimension, and such that Area$(\partial X)/\text{vol}(X) < h(N) + \epsilon$.

Since $\pi_1(M)$ is H-separable, we may find a finite sheeted covering, $\widetilde{M}$ of M which is subordinate to N and for which the compact set X is embedded by the projection $N \longrightarrow \widetilde{M}$. By choosing a larger covering if necessary, we may arrange that vol$(\widetilde{M}) > 2$ vol(X). So,

$$h(\widetilde{M}) \leq \text{Area}(\partial X)/\text{vol}(X) < h(N) + \epsilon$$

as required. $\square$

3 Proof of Theorem 1.2

In the setting of the fundamental groups of closed Riemannian manifolds the definition of Property τ described in §1 is equivalent to the following (see [20] Chapter 4):

Let X be a closed Riemannian manifold and let $\Gamma = \pi_1(X)$. Then Γ or X has Property τ if there is a constant $C > 0$ such that $\lambda_1(N) > C$ for all finite sheeted covers N of X.

3.1

We need the following proposition. (For the definition of Hausdorff dimension we refer the reader to [23].)

Proposition 3.1. *Let $M = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-manifold. Then Γ contains an infinite sequence of finitely generated, free, convex cocompact subgroups $\{F_j\}$ such that $\lambda_0(\mathbf{H}^3/F_j) \to 0$.*

Proof: This is a consequence of results of Sullivan [23] and L. Bowen [4].

For, it is shown in [23] that if $N = \mathbf{H}^3/\Gamma$ is a geometrically finite hyperbolic 3-manifold and D the Hausdorff dimension of the limit set of Γ, then $\lambda_0(N) = 1$ if and only if $D \leq 1$ and otherwise $\lambda_0(N) = D(2 - D)$.

Now Bowen shows in [4] (actually he shows more than this, but this suffices for our purpose) that if $M = \mathbf{H}^3/\Gamma$ is a closed hyperbolic 3-manifold, then Γ contains an infinite sequence of finitely generated, free, convex cocompact subgroups $\{F_j\}$ such that the Hausdorff dimension of the limit sets of F_j tend to 2. □

Remarks:
1. By the solution to the Tameness Conjecture [1] and [7], all finitely generated free subgroups of a cocompact Kleinian group are convex cocompact. However, Bowen proves that the subgroups F_j are convex co-compact without appealing to this theorem (see Lemma 5.3 in [4]).

2. Note that it is a consequence of Sullivan's result above that if $N = \mathbf{H}^3/\Gamma$ is a geometrically finite hyperbolic 3-manifold and Γ_1 is a supergroup or subgroup of Γ of finite index then $\lambda_0(\mathbf{H}^3/\Gamma_1) = \lambda_0(N)$.

We can now complete the proof of our main result.

Proof of Theorem 1.2: We begin with a reduction. We can assume that all finitely generated subgroups F of Γ are geometrically finite. For, if not, then by the solution to the Tameness Conjecture, F is the fundamental group of a virtual fibre in a fibration over the circle. It is well known that the Lubotzky-Sarnak Conjecture holds in this case.

Given Proposition 3.1, the remarks following it and Proposition 2.1 or 2.2 it clearly suffices to prove the following: given a finitely generated, free, convex cocompact subgroup F in Γ then there is subgroup $F' < \Gamma$ such that $[F' : F] < \infty$ and Γ is F'-separable. This

follows immediately from the engulfing hypothesis using [17] Theorem 2.7. □

Proof of Corollary 1.4: This is seen as follows. Theorem 1.9 of [16] shows that if every compact 3-manifold with infinite fundamental group does not have Property τ, then arithmetic Kleinian groups are large.

Now assuming the Geometrization Conjecture, this is well known for compact 3-manifolds which are not hyperbolic (see e.g. the Appendix in [16]) and the remaining case is provided by Theorem 1.2 (since GFERF obviously implies the engulfing property for infinite index, geometrically finite subgroups). □

Finally, Theorem 1.6 follows from the non-compact version of Cheeger's inequality $\lambda_0(N) \geq h(N)^2/4$ applied to Proposition 3.1. □

3.2

We include the following argument for convenience, and to emphasize Corollary 1.4.

Theorem 3.2. *Let* $M = \mathbf{H}^3/\Gamma$ *be a closed hyperbolic 3-manifold, and assume that* Γ *contains the fundamental group of a closed surface of genus at least 2. Then if* Γ *is LERF, then* Γ *is large.*

Proof: The surface subgroup corresponds to a closed incompressible surface immersed in M. If S is geometrically infinite, then it is a virtual fiber in a fibration over the circle. By passing to a finite sheeted cover, it follows from [10] that Γ must also contain a closed quasi-Fuchsian surface subgroup. Thus we now work with F a quasi-Fuchsian surface subgroup of Γ.

Using the LERF assumption, we invoke Scott's result [22] to pass to a finite sheeted cover $M_1 = \mathbf{H}^3/\Gamma_1$ so that M_1 contains a closed embedded quasi-Fuchsian surface with covering group F. This determines a free product with amalgamation decomposition $A *_F B$ or HNN-extension $A*_F$ for Γ_1. One can now arrange a surjective homomorphism onto an amalgam of finite groups to finish the proof (see for example [19]). Note that, since F is quasi-Fuchsian, the amalgam cannot be $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$. □

4   Final Comments

1. It is important in Bowen's proof that there exist discrete, convex compact, free Kleinian groups, where the Hausdorff dimension of their limit set is arbitrarily close to 2. That there are examples of purely hyperbolic free subgroups whose limit set is the entire sphere at infinity seems to have first been established by Greenberg [12] (as points on the boundary of Schottky space which are limits of convex cocompact groups). That the Hausdorff dimension of the limit sets of these convex cocompact groups get arbitrarily close to 2 can be seen from Corollary 7.8 of [21] for example.

The existence of analogous subgroups in $\mathrm{SO}(n, 1)$ for $n \geq 4$ is as yet unknown. Their existence, together with the known generalization of Sullivan's result [23] to higher dimensions

would prove that LERF implies the Lubotzky-Sarnak Conjecture for higher dimensional hyperbolic manifolds.

2. It is interesting to contrast Proposition 3.1 with what happens, for instance for (free) subgroups of cocompact lattices in $\mathrm{Sp}(n, 1)$, $n \geq 2$. If Γ is such a lattice then it has Property T. It is shown in [5] (see Theorem 3), in contrast to Proposition 3.1, that if Δ is a subgroup of Γ, which is either finite or infinite index, there is a spectral gap for the smallest non-zero eigenvalue of the Laplacian.

A similar result was established in [11] for the Hausdorff dimension; namely that any infinite index convex cocompact subgroup of Γ (as above) has Hausdorff co-dimension of its limit set being at least 2.

Neither LERF nor the Congruence Subgroup Property are known for any example in this setting.

3. Other situations where LERF is used to imply large were recently given in [6].

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