The Geometry and Topology of Finite Index Subgroups, II

Last time:

Theorem 1.3: [L] Let Γ be a finitely presented group. Let $\{\Gamma_i\}$ be a nested sequence of finite index normal subgroups. Then one of the following must hold:

1. Γ_i is an amalgamated free product or HNN extension for all $i >> 0$;
2. $\inf_i h(X(\Gamma/\Gamma_i; S)) > 0$;

   $\uparrow$

   Cheeger constant which relates to Property (τ)
3. $\lim_{i\to\infty}(d(\Gamma_i) - 1)/[\Gamma : \Gamma_i] = 0$.

   $\uparrow$

   Rank gradient

Property (τ)

Let
Γ = a group
S = a finite generating set
$\{\Gamma_i\}$ = a collection of finite index (normal) subgroups

We say that Γ has Property (τ) w.r.t. $\{\Gamma_i\}$ if $\inf h(X(\Gamma/\Gamma_i; S)) > 0$.

We say that Γ has Property (τ) if it has Property (τ) w.r.t. all its finite index normal subgroups.

This was defined by [Lubotzky-Zimmer 1989]

If a group has Property (τ), this has lots of nice consequences.

If a group does not have Property (τ), this has lots of other nice consequences.

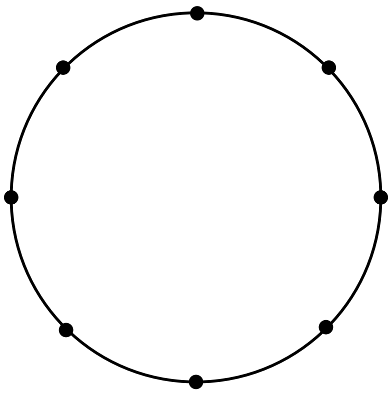
Which groups have (τ)?

- [Margulis 1975] Kazhdan's Property T $\Rightarrow$ Property (τ)
- So, if $n \geq 3$, then $SL(n,\mathbb{Z})$ has Property (τ) w.r.t. its congruence subgroups ie. if S is a set of generators for $SL(n,\mathbb{Z})$, then (for fixed n)
$$\inf_i h(X(SL(n,\mathbb{Z}/i\mathbb{Z});S)) > 0.$$
- [Selberg 1965] $SL(2,\mathbb{Z})$ has Property (τ) w.r.t. its congruence subgroups

Which groups don't have (τ)?

- We have already seen that $\mathbb{Z}$ does not have Property (τ):

  Let $S = \{1\}$. Then $X(\mathbb{Z}/n\mathbb{Z}; S) =$



$$h = \frac{2}{\lfloor n/2 \rfloor} \to 0$$

- If $vb_1(\Gamma) > 0$, then Γ does not have Property (τ)
- Conjecture: [Lubotzky-Sarnak] Any lattice in $\mathrm{PSL}(2, \mathbb{C})$ does not have Property (τ).

  This is a weaker form of the Thurston - Waldhausen conjecture and may be easier to prove.

Consequences of (τ)

Let X_n be a family of finite Cayley graphs, with bounded vertex valence, s.t. $\inf_n h(X_n) > 0$. Then:

- $\text{diameter}(X_n) \simeq \log(|V(X_n)|)$

  Sketch proof:

  $B(r)$ = ball of radius r about a vertex

  At least when $|B(r)| \leq |V(X)|/2$,

  $$|B(r+1)| - |B(r)| \geq (\text{const})\, |B(r)|$$

- Random walks on X_n 'converge quickly' to the uniform distribution
- Property (τ) has been used to solve many problems in the theory of error-correcting codes, and questions about algorithmic complexity.

Theorem 1.3: [L] Let Γ be a finitely presented group. Let $\{\Gamma_i\}$ be a nested sequence of finite index normal subgroups. Suppose that Γ does not have Property (τ) w.r.t. $\{\Gamma_i\}$. Then one of the following must hold:

1. Γ_i is an amalgamated free product or HNN extension for all $i >> 0$;
2. $(\Gamma, \{\Gamma_i\})$ has zero rank gradient.

How can one prove that a sequence has positive rank gradient?

Homology gradient

The mod p homology gradient of $(\Gamma, \{\Gamma_i\})$ is

$$\inf_i \frac{d_p(\Gamma_i) - 1}{[\Gamma : \Gamma_i]},$$

where

$$d_p(\Gamma_i) = \dim(H_1(\Gamma_i; \mathbb{F}_p)).$$

The mod p homology gradient is clearly a lower bound for the rank gradient.

One can sometimes prove that it is positive ...

Theorem 2.1: [L] Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{C})$. Suppose that either

1. Γ contains a torsion element, or
2. Γ is arithmetic.

Then Γ has a nested sequence of finite index subgroups Γ_i with positive mod p homology gradient, for some prime p.

Theorem 2.2: Let Γ be a finitely generated group, and let p be a prime. TFAE:

1. Γ has an infinite nested sequence of finite index subnormal subgroups, each with index a power of p, and with positive mod p homology gradient;
2. the pro-p completion of Γ has exponential subgroup growth.

(1) $\Rightarrow$ (2): Let Γ_i be a f.i. (subnormal) subgroup of Γ. The number of normal subgroups of Γ_i with index p is

$$\frac{p^{d_p(\Gamma_i)} - 1}{p - 1}.$$

(2) $\Rightarrow$ (1): Omitted.

Positive mod p homology gradient has stronger consequences than positive rank gradient:

Theorem 2.3: [L] Let Γ be a finitely presented group, and let

$$\Gamma \geq \Gamma_1 \triangleright \Gamma_2 \triangleright \dots$$

be a sequence of finite index subgroups such that each Γ_{i+1} is normal in Γ_i and has index a power of p. Suppose that the mod p homology gradient of this sequence is positive. Then

1. Γ is large, or
2. Γ has Property (τ) w.r.t. $\{\Gamma_i\}$.

+ve rank gradient	$\rightsquigarrow$	amalgam/HNN/(τ)
+ve homology gradient	$\rightsquigarrow$	large or (τ)

Corollary 2.4: [L] Under the hypotheses of Theorem 2.3, Γ has Property (τ) w.r.t. some nested sequence of finite index subgroups.

Corollary 2.5: [L] If Γ is finitely presented and $\hat{\Gamma}_p$ has exponential subgroup growth, then Γ has Property (τ) w.r.t. some nested sequence of finite index subgroups.

A key step in the proof of Theorem 2.3:

Theorem 2.6: [L] Let K be a cell complex, and let B and C be subcomplexes s.t. $K = B \cup C$. Suppose that the maps

$$H^1(B; \mathbb{F}_p) \to H^1(B \cap C; \mathbb{F}_p)$$
$$H^1(C; \mathbb{F}_p) \to H^1(B \cap C; \mathbb{F}_p)$$

induced by inclusion are not injections. In the case $p = 2$, suppose also that the kernel of at least one of these maps has dimension more than 1. Then $\pi_1(K)$ is large.

Define the mod p Cheeger constant $h_p(M)$ of a closed manifold M to be

$$\inf\left\{\frac{|H_1(S;\mathbb{F}_p)|}{\min\{|H_1(M_1;\mathbb{F}_p)|,|H_1(M_2;\mathbb{F}_p)|\}}\right\},$$

as S runs over all codimension 1 submanifolds S of M, dividing M into M_1 and M_2.

Corollary 2.7: [L] If

$$h_p(M) < \begin{cases} 1 & \text{if } p > 2; \\ 1/2 & \text{if } p = 2, \end{cases}$$

then $\pi_1(M)$ is large.

Let's assume Theorem 2.6 for the moment.

Outline of the proof of Theorem 2.3

Suppose that

1. Γ does not have Property (τ) w.r.t. $\{\Gamma_i\}$, and
2. $(\Gamma, \{\Gamma_i\})$ has positive mod p homology gradient.

We will show that Γ_i is large for all $i >> 0$, and hence so is Γ.

Proposition 2.8: Let Γ_i be a subnormal subgroup of Γ with index a power of p. Then

$$d_p(\Gamma_i) - 1 \leq [\Gamma : \Gamma_i](d_p(\Gamma) - 1).$$

compare: Reidermeister-Shreier

So, by replacing Γ by Γ_i for some big i, we may assume that $d_p(\Gamma_i)$ is approximately the mod p homology gradient of $(\Gamma, \{\Gamma_i\})$.

Let K = a finite 2-complex with $\pi_1(K) = \Gamma$, with a single 0-cell.
K_i = covering space corr. to Γ_i.
X_i = 1-skeleton of K_i.

Let A_i be a non-empty subset of $V(X_i)$ such that

$$|A_i| \leq |V(X_i)|/2 \text{ and } h(X_i) = |\partial A_i|/|A_i|.$$

Use A_i to decompose K_i into two sets B_i and C_i with small intersection.

$$d_p(B_i) \gtrapprox d_p(K_i)/4$$

$$d_p(C_i) \gtrapprox d_p(K_i)/4$$

$$d_p(B_i \cap C_i) \text{ is small.}$$

(This is not completely obvious.)

Corollary 2.7 $\Rightarrow \pi_1(K_i) = \Gamma_i$ is large. $\square$

Use cohomology with mod p coefficients

$$H^1(K; \mathbb{F}_p) = \frac{\text{1-cocycles}}{\text{1-coboundaries}}$$

A 1-cochain is an assignment to each oriented 1-cell of an integer mod p (its weight).

It is a 1-cocycle if its evaluation around each 2-cell is zero.

It is a 1-coboundary if its evaluation around every closed loop is zero.

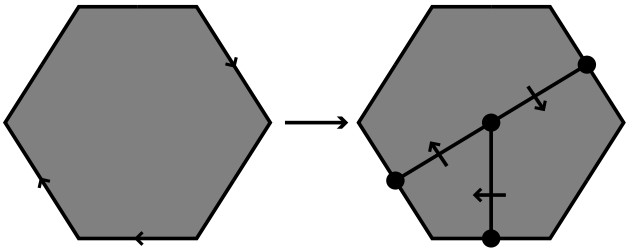
From a 1-cocycle, make a subgraph of K called a regular cocycle R:

Place a vertex of R at the midpoint each 1-cell with non-zero weight.

Place a vertex of R into the centre of each 2-cell that has at least one 1-cell with non-zero weight in its boundary.

Join these up with edges.

Assign weights and transverse orientations in the obvious way.



A regular cocycle specifies a collapsing map

$$K \to L(p)$$

where $L(p)$ is the 2-complex with
a single 0-cell,
a single 1-cell,
a single 2-cell, winding p times around the 1-cell.

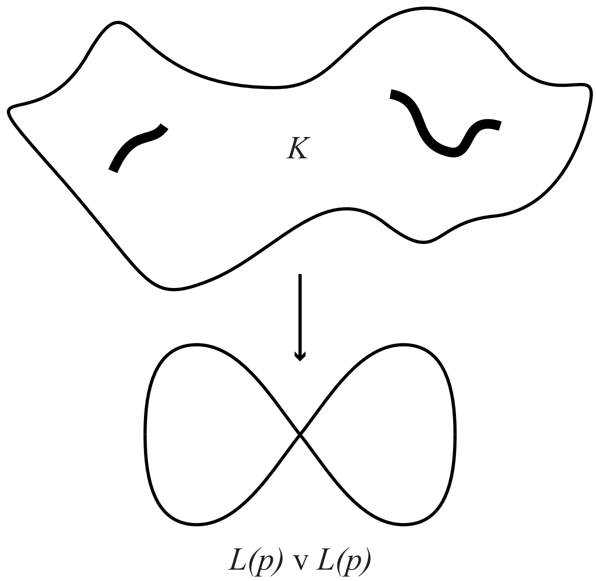
Proposition 2.9: Suppose that K contains two disjoint regular cocycles whose union is non-separating. Then

$$\pi_1(K) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/p\mathbb{Z}).$$

Hence, $\pi_1(K)$ is large if $p > 2$.

(There is a version when $p = 2$.)

Proof: We have a collapsing map $K \to L(p) \vee L(p)$. Since the cocycles are non-separating, the induced homomorphism is surjective. $\square$



Proof of Theorem 2.6.

Find a regular cocycle in B representing a non-trivial element in the kernel of

$$H^1(B; \mathbb{F}_p) \to H^1(B \cap C; \mathbb{F}_p).$$

Can arrange that this is disjoint from $B \cap C$ and non-separating.

Find a similar regular cocycle in C.

Their union is non-separating in K.

Proposition 2.9 $\Rightarrow$ $\pi_1(K)$ is large. $\square$

Theorem 2.10: [L] Let Γ be a finitely presented group. Then TFAE:

1. Γ is large;
2. there is a sequence
$$\Gamma \geq \Gamma_1 \geq \Gamma_2 \ldots$$
of finite index subgroups, each normal in Γ_1, such that

(i) Γ_i / Γ_{i+1} is abelian for each i;

(ii)
$$\frac{\log |\Gamma_i / \Gamma_{i+1}|}{[\Gamma : \Gamma_i]} \to \infty$$

(iii)
$$\limsup_i \frac{d(\Gamma_i / \Gamma_{i+1})}{[\Gamma : \Gamma_i]} > 0.$$

Let's assume that each $[\Gamma_1 : \Gamma_i]$ is a power of p.

Idea of proof: (1) $\Rightarrow$ (2) is easy.

(2) $\Rightarrow$ (1):

(i) and (ii) $\Rightarrow$ Γ does not have (τ) w.r.t. $\{\Gamma_i\}$ [Lubotzky-Weiss]

(iii) $\Rightarrow$ the mod p homology gradient is positive.

Theorem 2.3 $\Rightarrow$ Γ is large.

Comments

1. For finitely presented groups, largeness is detectable by the profinite completion.

2. This gives a new proof of a theorem of Baumslag and Pride:

   If $\Gamma = \langle X | R \rangle$ where $|X| - |R| > 1$, then Γ is large.

3. Theorem 2.3 is false in general for fin. gen. infinitely presented groups.

4. Condition (ii) can be dropped if each Γ_i/Γ_{i+1} is an elementary abelian p-group. This implies:

   Theorem 2.11. [L] If the derived p-series of a finitely presented group Γ has positive mod p homology gradient, then Γ is large.

   The proof uses the theory of error-correcting codes!