The Geometry and Topology of Finite Index Subgroups, III

Let M = finite volume hyperbolic 3-manifold

Let $\Gamma = \pi_1(M)$.

Recall:

Conjectures:

Γ has a finite index subgroup Γ_i s.t.

1. Γ_i is an amalgam/HNN
2. $b_1(\Gamma_i) > 0$
3. $b_1(\Gamma_i)$ is arbitrarily large
4. Γ_i has free non-abelian quotient.

Lemma: Let M be a compact orientable 3-manifold. Then

$$b_1(M) \geq b_1(\partial M)/2.$$

Proof: Use $\mathbb{Q}$ coefficients:

$$\begin{array}{ccccc} H^1(M) & \rightarrow & H^1(\partial M) & \rightarrow & H^2(M, \partial M) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_2(M, \partial M) & \rightarrow & H_1(\partial M) & \rightarrow & H_1(M) \end{array}$$

$$\begin{aligned} \mathrm{Ker}(H_1(\partial M) \rightarrow H_1(M)) &\cong \mathrm{Im}(H^1(M) \rightarrow H^1(\partial M)) \\ &\cong \mathrm{Im}(H_1(\partial M) \rightarrow H_1(M)) \end{aligned}$$

So, each has dimension $b_1(\partial M)/2$. $\square$

This works for any field coefficients, and so $d_p(M) \geq d_p(\partial M)/2$.

Theorem 3.1: [Cooper-Long-Reid]
Let M be a finite volume hyperbolic 3-manifold with non-empty boundary. Then $\pi_1(M)$ is large.

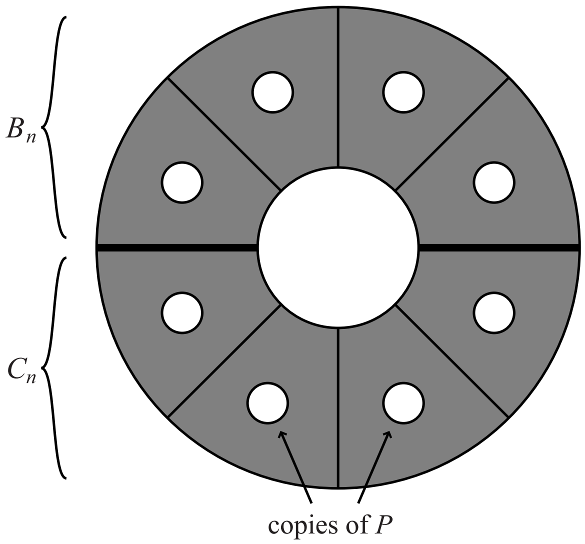
Proof: ∂M is a collection of tori. We may find a finite covering space $\tilde{M}$ with at least 3 boundary tori. Let P be one of these.

$$b_1(\tilde{M}) \geq b_1(\partial \tilde{M})/2 \geq 3$$

So, $i^*: H^1(\tilde{M}) \to H^1(P)$ has non-trivial kernel, containing a non-zero element α, say.

Let M_n be the n-fold cyclic cover of $\tilde{M}$ arising from α.

Decompose it into 2 halves: B_n and C_n.



$d_p(B_n)$ big (any prime p)
$d_p(C_n)$ big
$d_p(B_n \cap C_n)$ indep of n

Corollary 2.7 $\Rightarrow$
$\pi_1(M_n)$ large $\square$

Corollary 2.7: [L] If

$$h_p(M) < \begin{cases} 1 & \text{if } p > 2; \\ 1/2 & \text{if } p = 2, \end{cases}$$

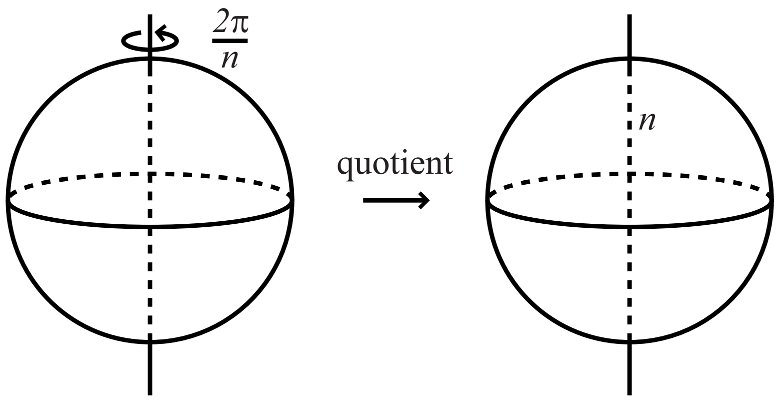
then $\pi_1(M)$ is large.

Orbifolds

Lattices in $\mathrm{PSL}(2,\mathbb{C})$ maybe with torsion	$\longleftrightarrow$	Finite volume hyperbolic 3-orbifolds

A 3-orbifold is a space locally modelled on

$$B^3/(\text{finite subgroup of } O(3)).$$



Basic principle

Easier		Harder
$\longleftarrow$	$\cdots\cdots\cdots\cdots\cdots\cdots$	$\longrightarrow$
$\uparrow$	$\uparrow$	$\uparrow$
3-manifolds with boundary	3-orbifolds	closed 3-manifolds

Theorem 3.2: [L] Let $O = \mathbb{H}^3/\Gamma$ be a compact orientable 3-orbifold. Suppose that the singular locus is a link L, and that the singularity order of each component of L is a prime p. Then

$$d_p(\Gamma) \geq |L|.$$

Sketch proof: Let M be $O - \mathrm{int}(N(L))$. This is a compact orientable 3-manifold. So,

$$d_p(M) \geq d_p(\partial M)/2 \geq |L|.$$

Γ is obtained from $\pi_1(M)$ by quotienting p^{th} powers of certain words (the meridians of the singular locus). This does not affect d_p. □

So, as far as mod p homology is concerned, these 3-orbifolds behave as though they have boundary.

Theorem 3.3: [L] Let Γ be a lattice in $\mathrm{PSL}(2, \mathbb{C})$ with torsion. Then Γ has a nested sequence of finite index subgroups with positive mod p homology gradient, for some prime p.

Idea of proof: Let O be the 3-orbifold $\mathbb{H}^3/\Gamma$.

By passing to a finite index subgroup of Γ, we may assume that the singular locus of O is a link L, and the singularity order of each component is a prime p . Fix a component K of L.

Find a sequence of finite covering spaces O_i of O, such that the inverse image of K is a disjoint union of copies of K. (details later)

Let Γ_i be the finite index subgroup of Γ corresponding to O_i. Then, by Theorem 3.2,

$$d_p(\Gamma_i) \geq |\mathrm{sing}(O_i)| \geq [\Gamma : \Gamma_i].$$

Let λ, μ be the meridian and longitude of K.

Covering spaces of O such that the inverse image of K is a disjoint union of copies of K.

$\updownarrow$

Subgroups of Γ containing $\langle\langle \lambda, \mu \rangle\rangle$

$\updownarrow$

Subgroups of $\Gamma / \langle\langle \lambda, \mu \rangle\rangle$

How do we know that this group contains infinitely many finite index subgroups?

Construct a presentation for $G = \Gamma / \langle\langle \lambda, \mu \rangle\rangle$:

1. Let $M = O - L$.
2. $\pi_1(M) = \langle X | R \rangle$, where $|R| - |X| = -1$.
3. $\Gamma = \langle X | R' \rangle$, where $|R'| - |X| = |L| - 1 \leq d_p(\Gamma) - 1$.
4. $G = \langle X | R'' \rangle$, where $|R''| - |X| = |L| + 1 \leq d_p(\Gamma) + 1 \leq d_p(G) + 3$.

Now use:

Theorem: (Golod-Shafarevich) Let $G = \langle X | R \rangle$. Suppose that

$$\frac{d_p(G)^2}{4} > |R| - |X| + d_p(G).$$

Then G has an infinite nested sequence of normal subgroups, each with index a power of p.

Recall:

Theorem 2.3: [L] Let Γ be a finitely presented group, and let

$$\Gamma \geq \Gamma_1 \triangleright \Gamma_2 \triangleright \dots$$

be a sequence of subgroups such that each Γ_{i+1} is normal in Γ_i and has index a power of p. Suppose that the mod p homology gradient of this sequence is positive. Then

1. Γ is large, or
2. Γ has Property (τ) w.r.t. $\{\Gamma_i\}$.

Key question: Does Γ have Property (τ) w.r.t. $\{\Gamma_i\}$?

If not, then Γ is large.

This would follow from:

GS-τ Conjecture: [Lubotzky - Zelmanov]

Let $G = \langle X | R \rangle$. Suppose that

$$\frac{d_p(G)^2}{4} > |R| - |X| + d_p(G).$$

Then G does not have Property (τ) (w.r.t some nested sequence of f.i. normal subgroups, each with index a power of p).

GS-τ Conjecture $\Rightarrow$ any lattice in $\mathrm{PSL}(2, \mathbb{C})$ with torsion is large.

Arithmetic 3-manifolds

An important class of hyperbolic 3-manifolds.

Theorem 3.4: [Reid] Any arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ is commensurable with one containing torsion.

So the above results for 3-orbifolds also apply to arithmetic 3-mflds:

1. Positive mod p homology gradient
2. Exponential subgroup growth
3. GS-$\tau \Rightarrow$ large. In fact:

   Theorem 3.5: [L-Long-Reid]
   The Geometrisation Conjecture and the Lubotzky-Sarnak Conjecture together imply that any arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ is large.

These techniques give many other results about arithmetic 3-manifolds, eg:

Theorem 3.6: [L-Long-Reid] Let Γ be an arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ such that $b_1(\Gamma_i) \geq 4$ for some finite index subgroup Γ_i of Γ. Then Γ is large.

Theorem 3.7: [L-Long-Reid] Let Γ be an arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$ containing one of the finite groups A_4, S_4 or A_5. Then Γ is large.

Still quite mysterious.

Theorem 3.8: [L] Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ has a nested sequence of finite index subgroups Γ_i such that

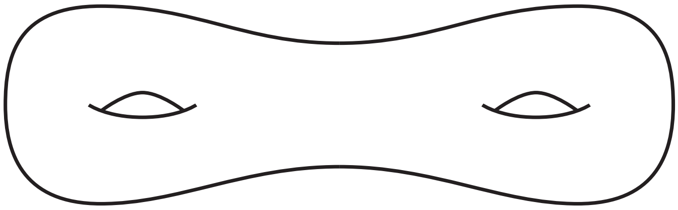
$$d_2(\Gamma_i) \geq \frac{[\Gamma : \Gamma_i]}{\sqrt{\log[\Gamma : \Gamma_i]}\,\log\log[\Gamma : \Gamma_i]}.$$

This is not quite linear. But this $\Rightarrow$ fast subgroup growth.

Maybe can't do better than this?? Probably too ambitious to try and prove positive mod p homology gradient.

Let's return to rank gradient, or more precisely a variant known as Heegaard gradient.

A handlebody is a 'thickened up' graph. Its genus is the genus of its boundary surface.



Gluing two handlebodies with the same genus g, via a homeomorphism between their boundaries, gives a closed orientable 3-manifold M. This is said to be a Heegaard splitting for M. The boundary of each handlebody is a Heegaard surface. Every closed orientable 3-manifold has a Heegaard splitting. The Heegaard genus $g(M)$ for M is the minimal genus of a Heegaard splitting.

The generators of π_1(one handlebody) gives a generating set for $\pi_1(M)$. So

$$d(\pi_1(M)) \leq g(M)$$

Let $\tilde{M} \to M$ be a finite-sheeted covering.

The inverse image of a Heegaard surface for M is a Heegaard surface for $\tilde{M}$. So

$$g(\tilde{M}) - 1 \leq (g(M) - 1)\deg(\tilde{M} \to M)$$

compare: Reidermeister-Shreier

Define the Heegaard gradient of a sequence M_i of finite-sheeted covers of M to be

$$\inf_i \frac{g(M_i) - 1}{\deg(M_i \to M)}$$

Heegaard gradient $\geq$ rank gradient

Theorem 3.9: [L] Let M be a closed orientable irreducible 3-manifold. Suppose that M has a nested sequence of finite-sheeted regular covers M_i such that

1. they have positive Heegaard gradient, and
2. $\pi_1(M)$ does not have Property (τ) w.r.t. $\pi_1(M_i)$.

Then M_i is Haken for all $i >> 0$.

Heegaard gradient conjecture: [L] If M_i does not have positive Heegaard gradient, then M has a finite-sheeted cover that fibres over S^1.

Heegaard grad conjecture	$+$	Lubotzky-Sarnak conjecture	$\Rightarrow$	Virtually Haken conjecture

Evidence for the Heegaard gradient conjecture:

Theorem 3.10: [L] Let $\{M_i \to M\}$ be a collection of finite regular covering spaces of a hyperbolic 3-manifold M. If $g(M_i) = o(\sqrt[4]{\deg(M_i \to M)})$, then M_i fibres over S^1 for all $i >> 0$.

Theorem 3.11: [L] Let $h(M)$ be the Cheeger constant of M. Then

$$h(M) \leq 2\pi \frac{g(M) - 1}{\text{Volume}(M)}$$

So, $\quad (\tau) \Rightarrow$ Positive Heegaard gradient

So, for any M, at least one of the Lub-Sarnak conjecture and the Heegaard gradient conjecture is true!

The proof of both of these use minimal surface theory: 'most' minimal genus Heegaard surfaces can be realised as a minimal surface.

Unsolved problems

1. Virtually Haken conjecture (and stronger forms)
2. GS-τ conjecture
3. Lubotzky-Sarnak conjecture
4. Heegaard gradient conjecture
5. If a finitely presented group does not have Property (τ), does it have a finite index subgroup with positive b_1?
6. If a finitely presented group G has positive mod p homology gradient for some nested sequence of finite index subgroups, must G be large?
7. Are there finitely presented, residually finite groups of intermediate growth?