The Poincaré Conjecture

Marc Lackenby

19 May 2012

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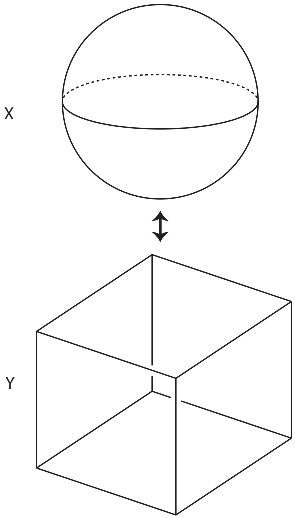
Topology is the study of spatial objects, called 'topological spaces' or 'spaces' for short.

Topology

A topologist views two spaces X and Y as the 'same' if one can be deformed into the other.



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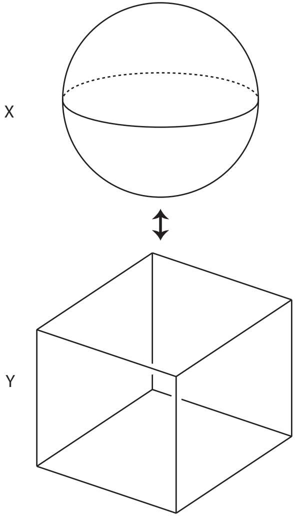

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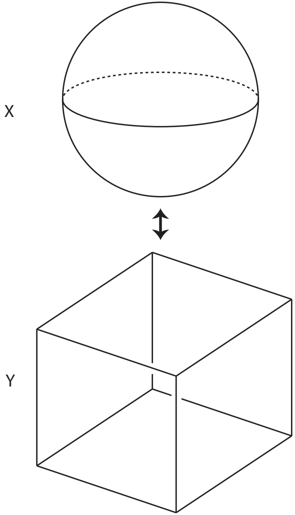
This correspondence is known as a *homeomorphism*.



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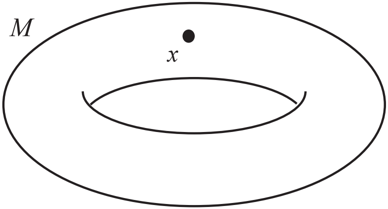
Poincaré investigated two key algebraic structures associated to a topological space M:

- its fundamental group $\pi_1(M)$,
- its homology groups $H_0(M), H_1(M), \ldots$

(Actually, the homology groups were first studied by Betti, and their group structure wasn't identified until Noether's work.)

The fundamental group

Pick a basepoint x in M.



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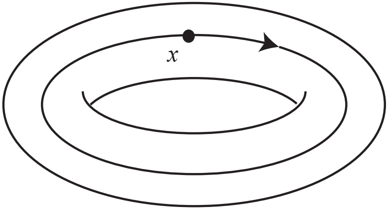

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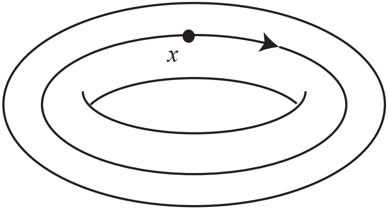

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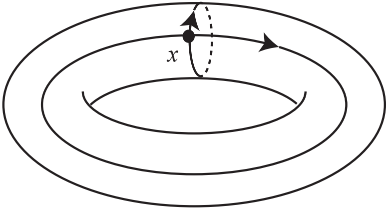
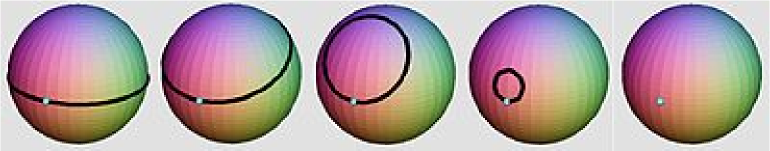
For example, $\pi_1(\mathrm{torus}) \cong \mathbb{Z} \times \mathbb{Z}$



The fundamental group of the 2-sphere

The 2-sphere has trivial fundamental group.

In other words, any loop on the sphere can be contracted to a point:



Poincaré's investigation of manifolds

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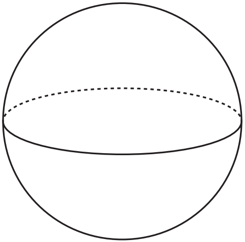
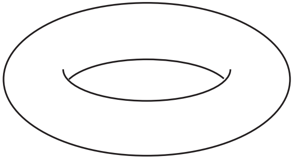
Example: the 2-sphere, the torus





2-manifolds

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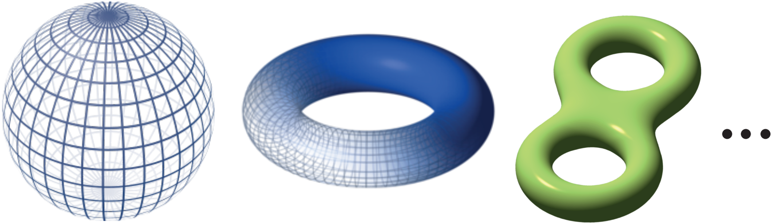
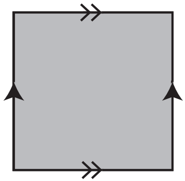
Theorem: Any closed 2-manifold is one of



or one of another list that includes the 'Klein bottle'.

How to build a torus

Start with a square.



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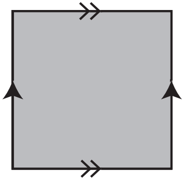
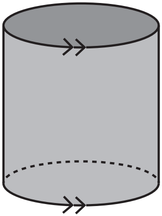


Glue the left side to the right.



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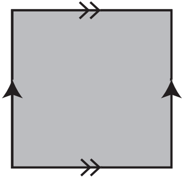

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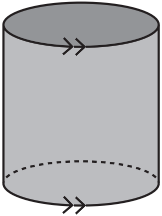
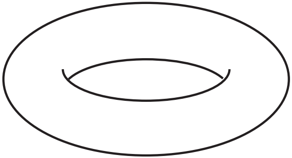


Glue the top to the bottom



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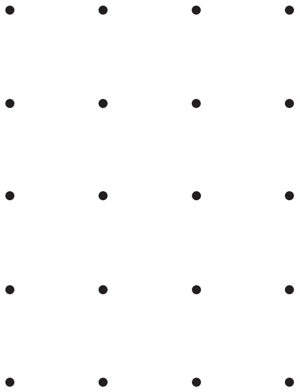

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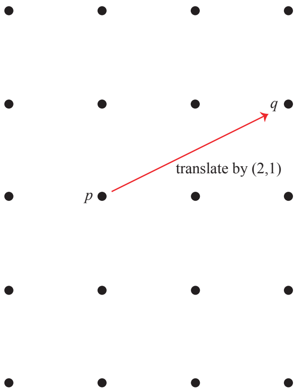

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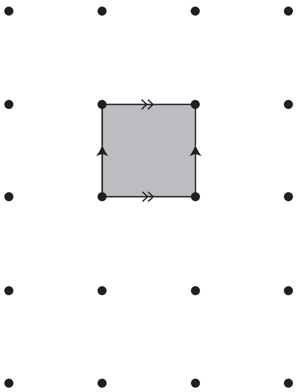
T is the torus.



3-manifolds

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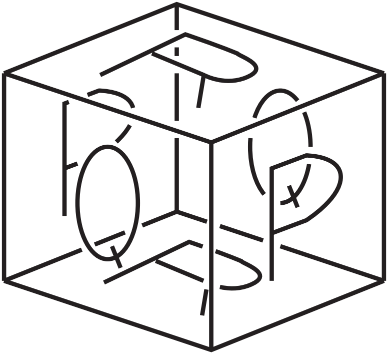
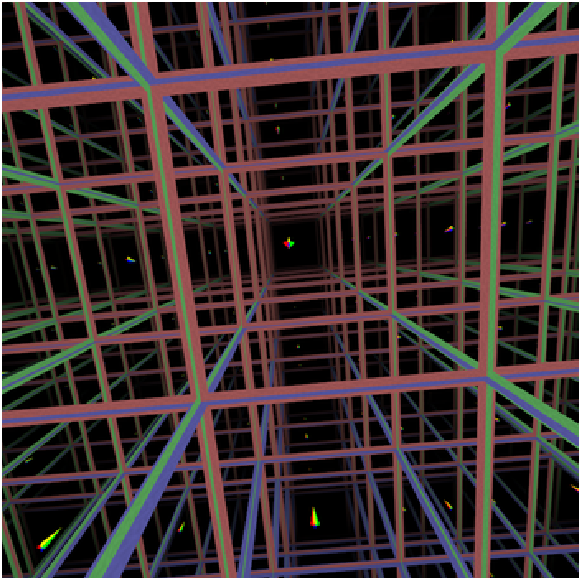


Inside the 3-torus



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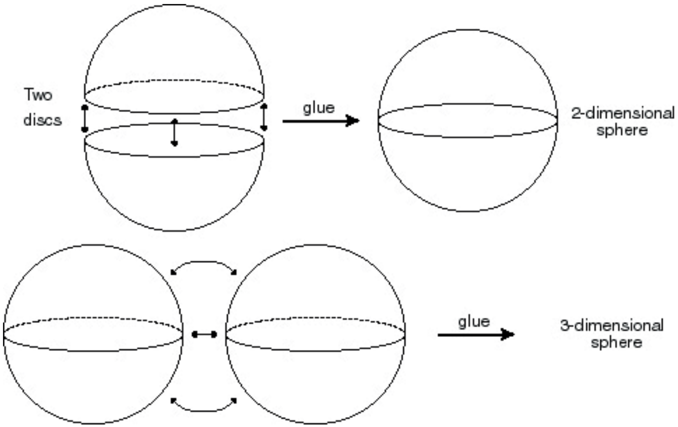

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But what about the fundamental group?

The Poincaré Conjecture

At the end of the 5th supplement to Analysis Situs, Poincaré asks:

> '*Consider a compact 3-dimensional manifold V without boundary. Is it possible that the fundamental group of V could be trivial, even though V is not homeomorphic to the 3-dimensional sphere?*'

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His closing words to the supplement are:

> '*However, this question would carry us too far away.*'

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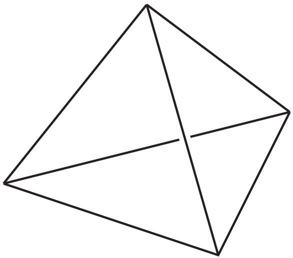

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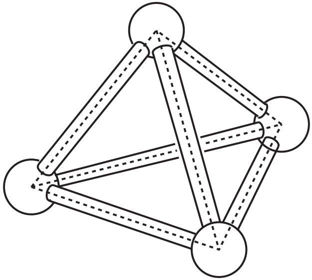
If we thicken up the vertices and edges, we get a 'handlebody'.



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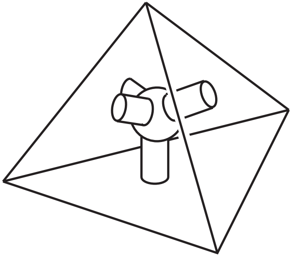

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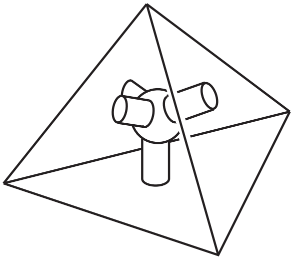
So, any closed orientable 3-manifold is obtained from two handlebodies by gluing their boundaries homeomorphically.

This is called a *Heegaard splitting*.



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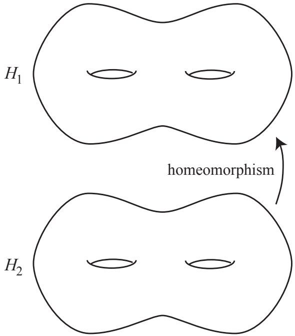

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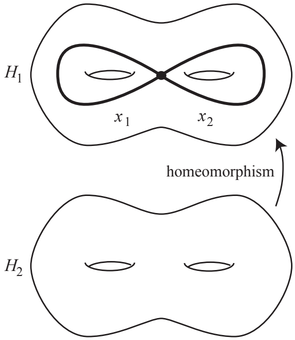

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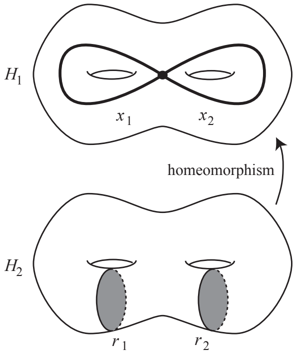

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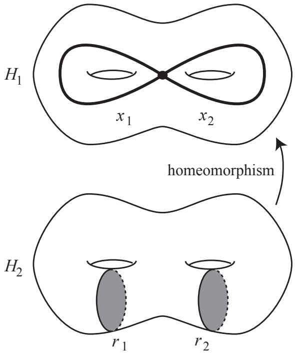

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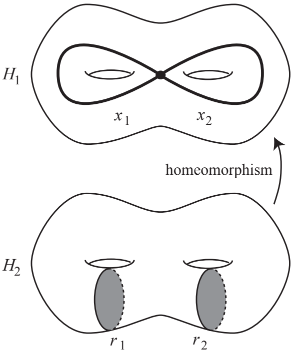
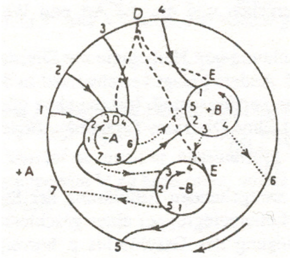
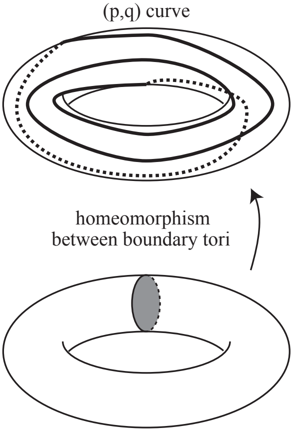




Lens spaces

When we glue two solid tori along their boundaries, the result is a *lens space L*.



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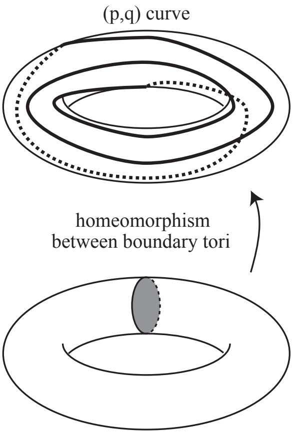

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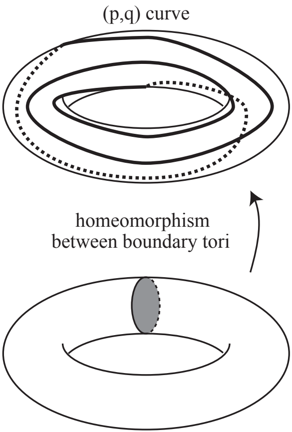

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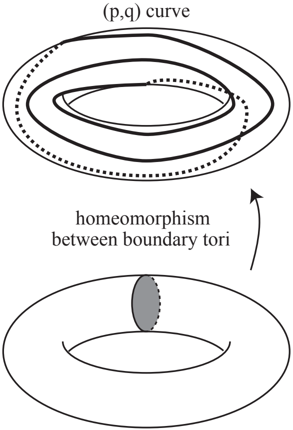

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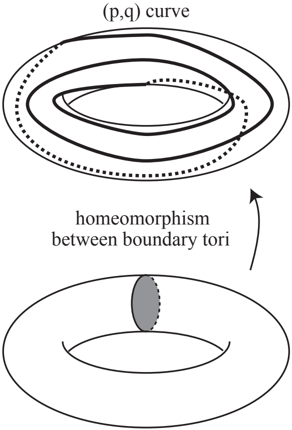
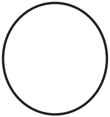
So, 3-manifolds are not determined by their π_1.

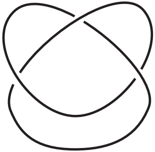


Dehn's lemma

A nice example of a 3-manifold is the complement of a knot K in $\mathbb{R}^3$.



unknot



trefoil

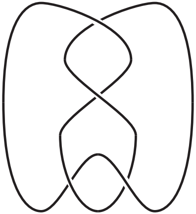


figure-eight

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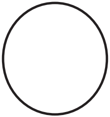

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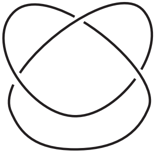

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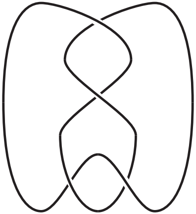

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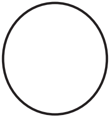

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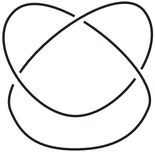

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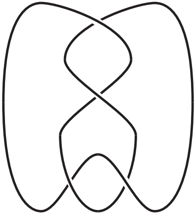

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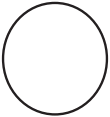
This was proved by Christos Papakyriakopoulos in 1957.



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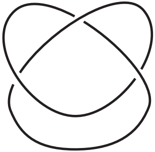

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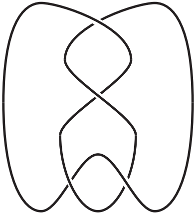
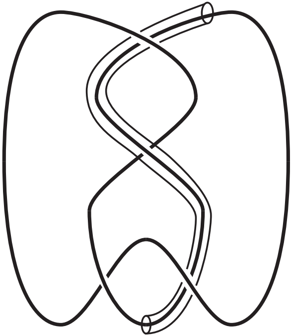


figure-eight

Dehn's lemma

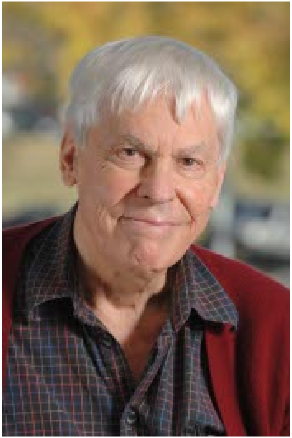
Theorem: If M is a compact 3-manifold with boundary, then $\pi_1(\partial M) \to \pi_1(M)$ is injective, unless there is an **embedded** disc D in M with boundary a non-trivial curve in ∂M.



The higher-dimensional Poincaré conjecture

In 1960, Stephen Smale proved:

Theorem: If M is a (smooth) n-dimensional manifold that has the same fundamental group and homology groups as the n-sphere, then it is homeomorphic to the n-sphere, provided $n \geq 5$.



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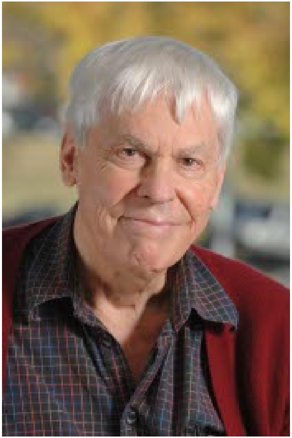
He received a Fields Medal in 1966.

His work was built upon by Stallings, Zeeman, and Newman.

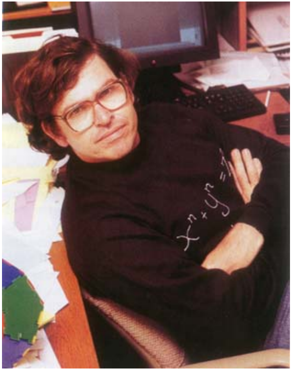


Geometrisation

A breakthrough came in the late 1970s.

Bill Thurston introduced his Geometrisation Conjecture.

This mostly relates to hyperbolic geometry . . .



Hyperbolic geometry

In the *Elements*, Euclid introduced the axioms for plane geometry.

For example: between any two distinct points in the plane, there is a unique straight line.

There was one axiom that was much less obvious than the others. This is usually now stated as the *Parallel Postulate*:

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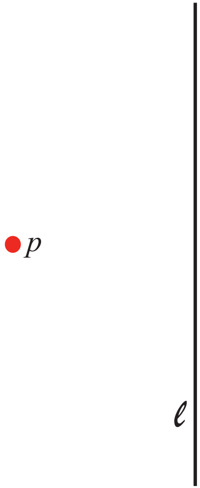

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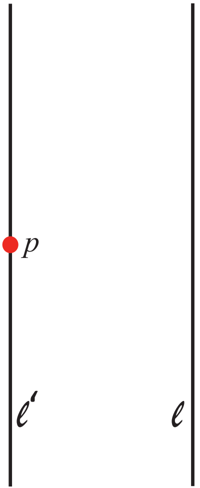
For each straight line ℓ, and each point p not on ℓ, there is a unique straight line ℓ' through p that is disjoint from ℓ.



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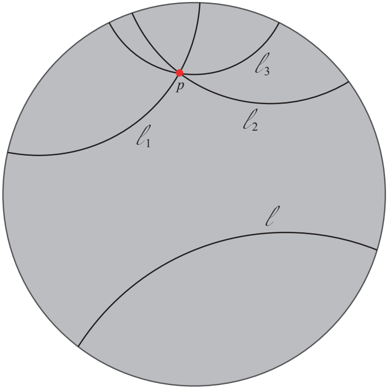
Poincaré introduced a useful model for the hyperbolic plane.

It is the open unit disc.

'Straight lines' are circles and diameters that intersect the boundary circle orthogonally.



Hyperbolic structures

We saw that the torus can be obtained as

Euclidean plane quotiented out by the action of $\mathbb{Z} \times \mathbb{Z}$ by translations

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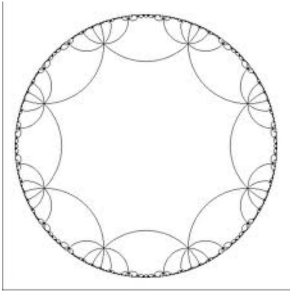

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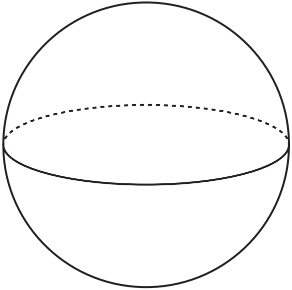
Hyperbolic 3-space

There is a natural generalisation of the hyperbolic plane to any dimension n.

For example, hyperbolic 3-space is the open unit ball.

Again, 'straight lines' are again circles perpendicular to the boundary sphere.

So, one can define a hyperbolic structure on a 3-manifold.



The Geometrisation Conjecture

Thurston's Geometrisation Conjecture: Any closed 3-manifold can be canonically decomposed into pieces, each of which admits a geometric structure.

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These 8 geometries are:

Euclidean, Hyperbolic, Spherical, ...

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The Poincaré Conjecture is just a special case of the Geometrisation Conjecture.

Thurston's work

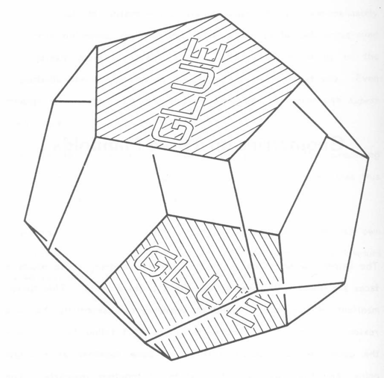
Thurston proved his Geometrisation Conjecture for many 3-manifolds, including

- all prime 3-manifolds M with infinite $H_1(M)$;
- all knot and link complements.

For this, he received a Fields Medal in 1982.

Inside a hyperbolic 3-manifold

The Seifert-Weber dodecahedral space



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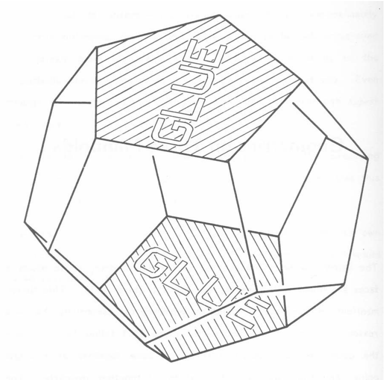

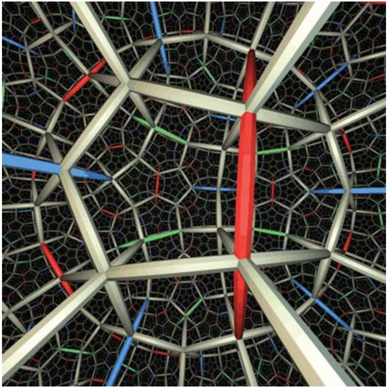

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At each point x of a Riemannian manifold and for each 2-dimensional plane P in the tangent space at x, there is a notion of curvature in that plane, called the *sectional curvature* of P.

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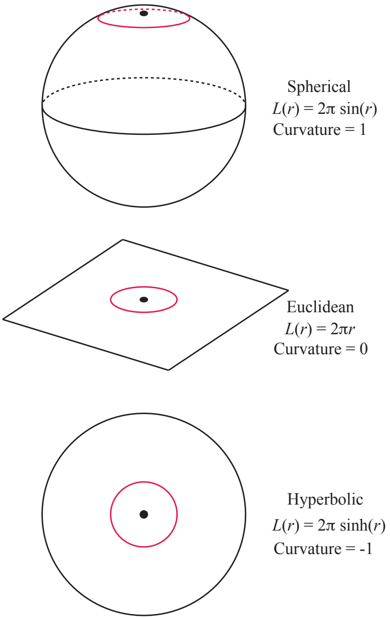
Fire out geodesics from x of length r in the direction of the plane P.

Their endpoints form a closed curve with length $L(r)$.



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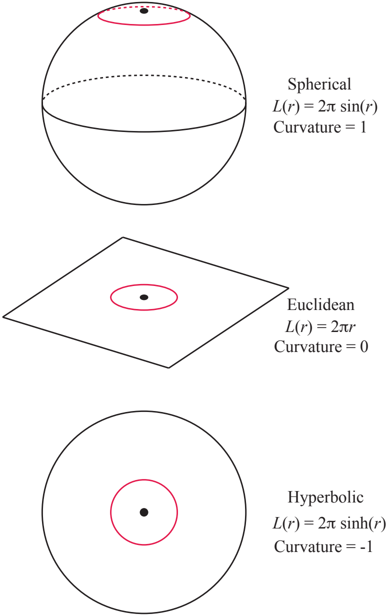
The *sectional curvature* of P is

$$\frac{1}{2\pi}\frac{d^3}{dr^3}L(r)\Big|_{r=0}$$



The Geometrisation Conjecture

An alternative formulation of the Geometrisation Conjecture:

> Any closed orientable 3-manifold which satisfies some necessary topological constraints admits a Riemannian metric of constant curvature.

An approach: start with any metric on the manifold, and improve it. Hope to get a constant curvature metric or deduce that the topological hypotheses fail.

Ricci flow

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If g_0 has constant curvature, then the metrics don't change.

Starting with an arbitrary metric g_0, do the metrics g_t 'tend to' a constant curvature metric?

Ricci flow

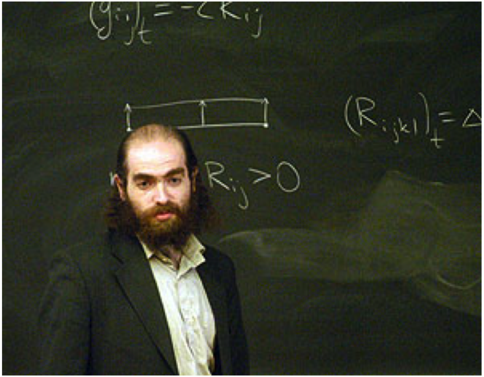
In 1982, Hamilton proved:

Theorem: Any closed 3-manifold with 'positive Ricci curvature' is homeomorphic to the 3-sphere.

He used the Ricci flow.

Perelman's work

In 2003, Grigori Perelman announced a solution to the Geometrisation Conjecture.



Perelman's work

His also work uses the Ricci flow, and employed a staggering mastery of geometric analysis.

His proof was distinctly sketchy.

arXiv:math/0303109v1 [math.DG] 10 Mar 2003

Ricci flow with surgery on three-manifolds

Grisha Perelman*

This is a technical paper, which is a continuation of [I]. Here we verify most of the assertions, made in [I, §13]; the exceptions are (1) the statement that a 3-manifold which collapses with local lower bound for sectional curvature is a graph manifold - this is deferred to a separate paper, as the proof has nothing to do with the Ricci flow, and (2) the claim about the lower bound for the volumes of the maximal horns and the smoothness of the solution from some time on, which turned out to be unjustified, and, on the other hand, irrelevant for the other conclusions.

The Ricci flow with surgery was considered by Hamilton [H 5,§4,5]; unfortunately, his argument, as written, contains an unjustified statement ($R_{MAX} = \Gamma$, on page 62, lines 7-10 from the bottom), which I was unable to fix. Our approach is somewhat different, and is aimed at eventually constructing a canonical Ricci flow, defined on a largest possible subset of space-time, - a goal, that has not been achieved yet in the present work. For this reason, we consider two scale bounds: the cutoff radius h, which is the radius of the necks, where the surgeries are performed, and the much larger radius r, such that the solution on the scales less than r has standard geometry. The point is to make h arbitrarily small while keeping r bounded away from zero.

Notation and terminology

$B(x,t,r)$ denotes the open metric ball of radius r, with respect to the metric at time t, centered at x.

$P(x,t,r,\triangle t)$ denotes a parabolic neighborhood, that is the set of all points (x',t') with $x' \in B(x,t,r)$ and $t' \in [t, t+\triangle t]$ or $t' \in [t+\triangle t, t]$, depending on the sign of $\triangle t$.

A ball $B(x,t,\epsilon^{-1}r)$ is called an ϵ-neck, if, after scaling the metric with factor r^{-2}, it is ϵ-close to the standard neck $\mathbb{S}^2 \times \mathbb{I}$, with the product metric, where $\mathbb{S}^2$ has constant scalar curvature one, and $\mathbb{I}$ has length $2\epsilon^{-1}$; here ϵ-close refers to C^N topology, with $N > \epsilon^{-1}$.

A parabolic neighborhood $P(x,t,\epsilon^{-1}r,r^2)$ is called a strong ϵ-neck, if, after scaling with factor r^{-2}, it is ϵ-close to the evolving standard neck, which at each

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Terence Tao:

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Perelman was awarded a Fields Medal in 2006 and the Clay Millenium Prize in 2010.

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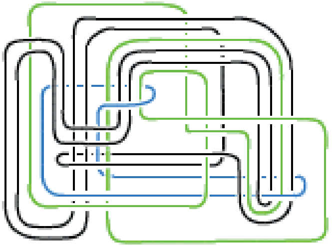

Figure 8: The two component corore link with $m = 1$.

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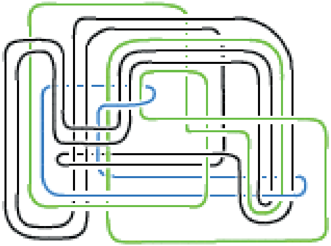
But we can't prove that they are counterexamples!



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