NEW LOWER BOUNDS ON SUBGROUP GROWTH AND HOMOLOGY GROWTH

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1. Introduction

Subgroup growth is an important new area of group theory. It attempts to quantify the number of finite index subgroups of a group, as a function of their index. In this paper, we will provide new, strong lower bounds on the subgroup growth of a variety of different groups. This will include the fundamental groups of all finite-volume hyperbolic 3-manifolds. By using the correspondence between subgroups and covering spaces, we will be able to address the following natural question: how many finite-sheeted covering spaces does a hyperbolic 3-manifold have, as a function of the covering degree?

We will see that, when analysing the subgroup growth of a group, it is helpful also to consider its 'homology growth'. This is concerned with the rank and order of the first homology of its finite index subgroups. Fast homology growth is a useful tool when establishing fast subgroup growth.

Our main result is a very general theorem, which places a lower bound on the rank of the first homology (with mod p coefficients) of a normal subgroup G_1 of a group G, when G/G_1 is a finite elementary abelian p-group. This homology can then be used to construct a finite index subgroup G_2 of G_1. Repeating this process, we obtain a nested sequence of finite index subgroups G_i with lower bounds on the rank of their first homology. This works best when there is an upper bound on the rank of the *second homology* of each G_i in terms of the rank of its first homology. Such a relationship is known to hold when G is the fundamental group of a closed 3-manifold, but it appears to be true in a much wider context. For example, we will be able to provide new information about the homology growth and subgroup growth of groups with deficiency at least 1, including free-by-cyclic groups, and the fundamental groups of closed 4-manifolds with non-positive Euler characteristic.

An interesting aspect to this paper is that the proofs of the main theorems are largely topological, despite the fact that their statements are entirely algebraic in nature.

We now give more precise statements of these results. Let p be a prime and let $\mathbb{F}_p$ be the field of order p. For $r = 1$ and 2, let $b_r(G; \mathbb{F}_p)$ be the dimension of the homology group $H_r(G; \mathbb{F}_p)$. Thus, $b_1(G; \mathbb{F}_p)$ is the dimension over $\mathbb{F}_p$ of $G/([G,G]G^p)$, and $b_2(G; \mathbb{F}_p)$ is the mod p Schur multiplier. We will be interested in groups satisfying the following conditions.

Definition. A group G has the $b_2 - b_1$ *property with respect to the prime p* if there is a uniform upper bound on

$$b_2(G_i; \mathbb{F}_p) - b_1(G_i; \mathbb{F}_p),$$

as G_i ranges over all finite index subgroups. A group has the b_2/b_1 *property with respect to p* if there is a uniform upper bound on

$$\frac{b_2(G_i; \mathbb{F}_p)}{b_1(G_i; \mathbb{F}_p) + 1}$$

as G_i ranges over all finite index subgroups.

Examples of groups satisfying the $b_2 - b_1$ condition are the fundamental groups of closed 3-manifolds and groups with deficiency at least 1. The fundamental group of any closed orientable 4-manifold with non-positive Euler characteristic is a b_2/b_1 group. (Section 8 contains a proof of these results.)

Let $s_n(G)$ denote the number of subgroups of G with index at most n. Recall that a subgroup K of a group G is *subnormal* (written $K \vartriangleleft\vartriangleleft G$) if there exists a finite sequence of subgroups $G = G_1 \geq G_2 \geq \ldots \geq G_r = K$ such that each G_i is normal in G_{i-1}. Let $s_n^{\vartriangleleft\vartriangleleft}(G)$ be the number of subnormal subgroups of G with index at most n.

The following is our main result on subgroup growth.

Theorem 1.1. *Let G be a finitely generated group that has the $b_2 - b_1$ property with respect to the prime 2. Suppose that*

$$\sup\{b_1(G_i; \mathbb{F}_2) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Then, for infinitely many n,

$$s_n(G) \geq s_n^{\lhd\lhd}(G) > 2^{n/(\sqrt{\log(n)}\log\log n)}.$$

This is a rather strong statement, since the lower bound that it places on $s_n(G)$ and $s_n^{\lhd\lhd}(G)$ is not far from the fastest possible subgroup growth of a finitely generated group. It is known, that for any finitely generated group G, there is a constant k such that, for all n,

$$s_n(G) \leq k^{n\log n}$$
$$s_n^{\lhd\lhd}(G) \leq k^n.$$

It will be obvious from the proof of Theorem 1.1 that the full hypotheses of the theorem are not required. In particular, one does not need to bound $b_2(G_i; \mathbb{F}_2) - b_1(G_i; \mathbb{F}_2)$ for all finite index subgroups G_i of G, merely for those in the derived 2-series of a certain finite index subgroup of G. Also, one may further weaker the $b_2 - b_1$ condition, by hypothesising that $b_2(G_i; \mathbb{F}_2) - b_1(G_i; \mathbb{F}_2)$ does not grow too fast as a function of $b_1(G_i; \mathbb{F}_2)$. One also does not need to assume that the supremum of $b_1(G_i; \mathbb{F}_2)$ is infinite. For this follows from the $b_2 - b_1$ hypothesis, provided that some $b_1(G_i; \mathbb{F}_2)$ is greater than some constant that can be estimated. (For example, for closed 3-manifold groups, this constant is 3.) We discuss these matters in Sections 6 and 8.

For b_2/b_1 groups, we can prove the following result. Although not as strong as Theorem 1.1, it nonetheless provides good lower bounds on subgroup growth.

Recall that for sequences f_n and g_n, the terminology $f_n = \Omega(g_n)$ means that $f_n/g_n \to \infty$ as $n \to \infty$.

Theorem 1.2. *Let G be a finitely generated group that has the b_2/b_1 property with respect to some prime p. Suppose that*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Then, there is a constant $k > 0$ such that

$$s_n(G) \geq s_n^{\lhd\lhd}(G) = \Omega(p^{n^k}).$$

We will find lower bounds on the constant k in the above result.

These theorems make the hypothesis that $b_1(G_i; \mathbb{F}_p)$, for suitable p, can be chosen to be arbitrarily large. Clearly, some sort of hypothesis along these lines is necessary. For example, finitely generated abelian groups satisfy the $b_2 - b_1$ condition but have only polynomial subgroup growth. However, this is a reasonably mild restriction, and is often satisfied in practice. For example, the following result is a well known consequence of the Lubotzky alternative and the Nori-Weisfeller strong approximation theorem (see Corollary 18 of Window 9 in [5].)

Theorem 1.3. *Let G be a finitely generated linear group. Then either G is virtually soluble, or, for any prime p,*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

In fact, it is often the case that if $b_1(G_1; \mathbb{F}_p)$ is bigger than some fixed constant, for some finite index subgroup G_1 of G, then the supremum of $b_1(G_i; \mathbb{F}_p)$ over all finite index subgroups G_i is infinite. To illustrate this point and to emphasise how wide-ranging Theorems 1.1 and 1.2 are, we give the following.

Theorem 1.4. *Let G be a group satisfying one of the following conditions:*

1. *G is a lattice in $\mathrm{PSL}(2, \mathbb{C})$;*

2. *G is the fundamental group of a closed 3-manifold and $b_1(G_i; \mathbb{F}_2) > 3$ for some finite index subgroup G_i of G;*

3. *G has deficiency at least 1 and $b_1(G_i; \mathbb{F}_2) > 2$ for some finite index subgroup G_i of G;*

4. *G is a (finitely generated free non-abelian)-by-cyclic group.*

Then

$$s_n(G) \geq s_n^{\lhd\lhd}(G) > 2^{n/(\sqrt{\log(n)}\,\log\log n)})$$

for infinitely many n.

Theorem 1.5. *Let G be a group satisfying one of the following conditions:*

1. *G is the fundamental group of a closed 3-manifold and $b_1(G_i; \mathbb{F}_p) > 3$ for some finite index subgroup G_i of G and some prime p;*

2. G has deficiency at least 1 and $b_1(G_i; \mathbb{F}_p) > 2$ for some finite index subgroup G_i of G and some prime p;

3. G is the fundamental group of a closed 4-manifold with non-positive Euler characteristic and $b_1(G_i; \mathbb{F}_p) > 4$ for some prime p and some finite index subgroup G_i of G.

Then there is a constant $k > 0$ such that

$$s_n(G) \geq s_n^{\lhd\lhd}(G) = \Omega(p^{n^k}).$$

The key piece of machinery that is the driving force behind this paper is the following result.

Theorem 1.6. *Let G be a finitely generated group, and let p be a prime such that $b_2(G; \mathbb{F}_p)$ is finite. Let K be a finite index normal subgroup such that G/K is an elementary abelian p-group of rank n. Then, for any integer ℓ between 0 and n,*

$$b_1(K; \mathbb{F}_p) \geq \sum_{r=2}^{\ell+1} \binom{n}{r}(r-1) + (b_1(G; \mathbb{F}_p) - n)\sum_{r=0}^{\ell} \binom{n}{r} - b_2(G; \mathbb{F}_p)\sum_{r=0}^{\ell-1} \binom{n}{r}.$$

Moreover, if $p = 2$,

$$b_1(K; \mathbb{F}_2) \geq b_1(G; \mathbb{F}_2)\sum_{r=0}^{\ell} \binom{n}{r} - \sum_{r=1}^{\ell+1} \binom{n}{r} - b_2(G; \mathbb{F}_2)\sum_{r=0}^{\ell-1} \binom{n}{r}.$$

In the case where $\ell = 0$, these formulas should be interpreted by taking the sums $\sum_{r=2}^{\ell+1}$ and $\sum_{r=0}^{\ell-1}$ to be zero.

Theorem 1.6 really is a collection of inequalities, one for each integer ℓ between 0 and n, known as the 'level'. In practice, one chooses ℓ to obtain the strongest possible inequality. For $p = 2$, the important point here is that the first summation runs up to $r = \ell$, whereas the third summation goes up only to $r = \ell - 1$.

By applying this result to the derived 2-series of some finite index subgroup of G, we obtain the following lower bounds on homology growth.

Theorem 1.7. *Let G be a finitely generated group that has the $b_2 - b_1$ property with respect to the prime 2. Suppose that*

$$\sup\{b_1(G_i; \mathbb{F}_2) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Then G has a nested sequence of finite index normal subgroups $\{G_i\}$, such that

$$b_1(G_i; \mathbb{F}_2) = \Omega\left(\frac{[G : G_i]}{\sqrt{\log[G : G_i]}\log\log[G : G_i]}\right).$$

This is very nearly the maximum possible growth rate of homology. For, $b_1(G_i; \mathbb{F}_2)$ is at most the rank of G_i, which, by the Reidemeister-Schreier process, is bounded above by a linear function of $[G : G_i]$.

The sequence $\{G_i\}$ provided by Theorem 1.7 is the derived 2-series of some finite index subgroup G_1 of G. Indeed, assuming that G is a finitely generated group with the $b_2 - b_1$ property with respect to the prime 2 and that $b_1(G_1; \mathbb{F}_2)$ is sufficiently large, then the derived 2-series of G_1 always has fast homology growth. More precisely, for any real number $\lambda < \sqrt{2/\pi}$, there is a constant N, with the following property. If G_1 is any finite index subgroup of G with $b_1(G_1; \mathbb{F}_2) \geq N$, and $\{G_i\}$ is the derived 2-series of G_1, then

$$b_1(G_{i+1}; \mathbb{F}_2) \geq \lambda 2^{b_1(G_i; \mathbb{F}_2)}\sqrt{b_1(G_i; \mathbb{F}_2)},$$

for all $i \geq 1$. This is proved in Section 6.

Theorem 1.1 follows rapidly from Theorem 1.7, because if G_i is a finite index normal (or just subnormal) subgroup of G, then for $n = 2[G : G_i]$,

$$s_n(G) \geq s_n^{\lhd\lhd}(G) \geq 2^{b_1(G_i; \mathbb{F}_2)}.$$

Of course, Theorem 1.7 applies to any of the groups in Theorem 1.4.

A weaker form of Theorem 1.7 holds for groups satisfying the b_2/b_1 condition. This applies, in particular, to any of the groups in Theorem 1.5.

Theorem 1.8. *Let G be a finitely generated group that has the b_2/b_1 property with respect to the prime p. Suppose that*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Then G has a nested sequence of finite index subgroups $G \triangleright G_1 \triangleright G_2 \triangleright \ldots$, where each G_i is normal in G_{i-1}, such that, for some $k > 0$,

$$b_1(G_i; \mathbb{F}_p) = \Omega([G : G_i]^k).$$

One might wonder why Theorems 1.1 and 1.7 are stated only for the prime 2, whereas Theorems 1.2 and 1.8 work for any prime. This is due to the asymptotics of the binomial coefficients. A full reason is given in Section 6. Of course, however, if a group satisfies the $b_2 - b_1$ condition with respect to an odd prime p, then it also satisfies the b_2/b_1 condition with respect to p. So, Theorems 1.2 and 1.8 provide lower bounds on its subgroup growth and homology growth.

Theorems 1.1 and 1.7 apply to the fundamental groups of finite-volume hyperbolic 3-manifolds. They significantly improve the previous known lower bounds on their subgroup growth (see Proposition 7.2.3 in [5]). They also suggest an interesting direction for future research. It is a major conjecture that the fundamental group of any closed hyperbolic 3-manifold should have a finite index subgroup with positive first Betti number. Even more ambitious is the conjecture that such groups have a finite index subgroup with a free non-abelian quotient. It has been shown that both these conclusions hold for a finitely presented group G provided it has an abelian p-series $G \triangleright G_1 \triangleright G_2 \triangleright \ldots$ with 'rapid descent'. This means that each quotient G_i/G_{i+1} is an elementary abelian p-group, and that $b_1(G_i/G_{i+1}; \mathbb{F}_p)/[G : G_i]$ is bounded away from zero. (See Theorem 1.1 of [3].) Thus, Theorem 1.7 may represent a first step towards a proof of these conjectures. In any case, a good understanding of how many covering spaces the manifold has and of their homology must surely be useful.

The paper is organised as follows. In Section 2, we establish a preliminary technical result which produces a presentation for a group having some useful properties. Section 3 contains the proof of Theorem 1.6, and is the heart of the paper. Section 4 gives an explanation of the link between Theorem 1.6, the lower central p-series and an exact sequence of Stallings. In Section 5, we prove, under hypotheses rather weaker than those in Theorems 1.1 and 1.2, that G has finite index normal subgroups G_i where $b_1(G_i; \mathbb{F}_p)$ is arbitrarily large. In Section 6, we use the lower bounds of Theorem 1.6 to deduce the existence of a sequence of

finite index subgroups with fast homology growth, giving Theorems 1.7 and 1.1. In Section 7, we deal with b_2/b_1 groups. In Section 8, we prove Theorems 1.4 and 1.5, which establish that our results apply to a wide variety of different groups.

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2. Choosing a group presentation

Our goal in this and the next section is to prove Theorem 1.6. Recall that we are assuming that G has a normal subgroup K such that G/K is an elementary abelian p-group of rank n. In this section, we prove that the group G has a presentation $\langle X_1, X_2, X_3 | R_1, R_2, R_3 \rangle$ having some useful technical properties. The sets X_1, X_2 and X_3 will be finite. The free group on the generators $X_1 \cup X_2 \cup X_3$ will be denoted by F. Then G is the quotient of F by a normal subgroup R, the relations of G. Let F_- be the free group on the generators $X_1 \cup X_2$.

We now introduce some terminology. For any group H and integer $m \geq 1$, let $\gamma_m(H)$ be the mth term of the lower central p-series for H. Recall that this is defined recursively by setting $\gamma_1(H) = H$ and $\gamma_{m+1}(H) = [\gamma_m(H), H](\gamma_m(H))^p$.

Suppose that we are given an integer $m \geq 2$. (We will fix m later.) We are aiming to ensure that the presentation $\langle X_1, X_2, X_3 | R_1, R_2, R_3 \rangle$ has the following properties:

 (i) X_1 forms a basis for G/K;

 (ii) $X_1 \cup X_2$ forms a basis for $H_1(G; \mathbb{F}_p)$;

(iii) every element of X_3 is trivial in $H_1(G; \mathbb{F}_p)$;

(iv) every element of R_2 lies in $\gamma_2(F)$;

 (v) every element of R_3 is of the form $x_3 = f(x_3)$, where $f(x_3)$ is the product of an element in $\gamma_2(F_-)$ and an element of $\gamma_m(F)$;

(vi) R_1 is a basis for $H_2(G; \mathbb{F}_p)$;

(vii) every element of R_1 lies in $\gamma_2(F_-)\gamma_m(F)$.

We will first prove that such a presentation can always be found. Afterwards,

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we will illustrate this proof with an example.

Ensuring properties (i), (ii) and (iii) is trivial.

We claim that, for each integer $m \geq 2$, $\gamma_2(F)R = \gamma_2(F_-)\gamma_m(F)R$. This is equivalent to the statement that $\gamma_2(G) = \gamma_2(F_-)\gamma_m(G)$. (This represents a slight abuse of terminology: we are confusing $\gamma_2(F_-)$ and its image in G.) Now, $G/\gamma_m(G)$ is a finite p-group. The Burnside basis theorem (Theorem 12.2.1 of [2]) states that in any finite p-group H, a set of elements generates H if and only if it generates $H/\gamma_2(H)$. Since $X_1 \cup X_2$ generates $G/\gamma_2(G)$, it therefore also generates $G/\gamma_m(G)$. When any element of $\gamma_2(G/\gamma_m(G))$ is expressed as a word in the generators $X_1 \cup X_2$, its total weight in each generator is a multiple of p. Thus, $\gamma_2(G) = \gamma_2(F_-)\gamma_m(G)$, as required. This proves the claim.

Thus, any element of X_3 is equal in G to the product of an element of $\gamma_2(F_-)$ and an element of $\gamma_m(F)$. When we use this fact, m will be some fixed integer at least 2, to be chosen later. For each $x_3 \in X_3$, let $f(x_3)$ be the product of an element of $\gamma_2(F_-)$ and an element of $\gamma_m(F)$, such that $x_3 = f(x_3)$ in G. Let R_3 be the relations $\{x_3^{-1}f(x_3) : x_3 \in X_3\}$.

We now construct a set of relations R_2, as follows: for each $r \in R$, replace each occurrence of every $x_3 \in X_3$ in r by $f(x_3)$. (Note that x_3 may still appear in these relations R_2, since x_3 may appear in the word $f(x_3)$, for example.) Clearly, the subgroup of F normally generated by R_2 and R_3 is R. That is, the relations $R_2 \cup R_3$ specify the same group G.

We claim that, for every relation in R_2, the total weight of each generator is multiple of p. This is because the total weight of every element of $X_1 \cup X_2$ in each relation is a multiple of p, because these generators form a basis for $H_1(G; \mathbb{F}_p)$. Each occurrence of an $x_3 \in X_3$ in $r \in R$ has been replaced by $f(x_3)$, which is an element of $\gamma_2(F)$. In every element of $\gamma_2(F)$, the total weight of each generator is a multiple of p. Thus, the claim is proved.

To summarise, we have constructed a presentation $\langle X_1, X_2, X_3 \mid R_2, R_3 \rangle$ for G, satisfying (i), (ii), (iii), (iv) and (v).

Now, the Hopf formula states that
$$H_2(G; \mathbb{F}_p) \cong \frac{R \cap ([F, F]F^p)}{[R, F]R^p}.$$

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Let R_1' be a basis for $H_2(G; \mathbb{F}_p)$, where we view each element of R_1' as lying in $R \cap ([F, F]F^p)$. From R_1', we will create a new set R_1 of elements of $R \cap ([F, F]F^p)$ representing the same basis for $H_2(G; \mathbb{F}_p)$, as follows. For each occurrence of an $x_3 \in X_3$ in an element of R_1', we replace it by $f(x_3)$. Let R_1 be the resulting set of relations. We claim that, when passing from R_1' to R_1, we have not changed the classes in $H_2(G; \mathbb{F}_p)$ that these relations represent. This will establish that R_1 is also a basis for $H_2(G; \mathbb{F}_p)$. To prove the claim, consider one such relation $r_1' \in R_1'$. Its total x_3 weight is a multiple of p, since $R_1' \subset [F, F]F^p$. Thus, at the level of $H_2(G; \mathbb{F}_p)$, we have simply added a multiple of p copies of the relation $x_3^{-1}f(x_3)$ to r_1'. We have performed such an operation for each $x_3 \in X_3$. Note that $x_3^{-1}f(x_3)$ lies in $[F, F]F^p$ and therefore represents an element of $H_2(G; \mathbb{F}_p)$. Since $H_2(G; \mathbb{F}_p)$ is an elementary abelian p-group, this operation therefore does not change the class in $H_2(G; \mathbb{F}_p)$, proving the claim.

We claim that every element of R_1 lies in $\gamma_2(F_-)\gamma_m(F)$. Note that in each $r_1' \in R_1'$, the total weight of each generator in $X_1 \cup X_2$ is a multiple of p. We have replaced each element of X_3 with the product of an element in $\gamma_2(F_-)$ and an element in $\gamma_m(F)$. Possibly changing each such element of $\gamma_m(F)$, we may move it to the end of r_1. Thus, r_1 is the product of a word in F_- where each generator has weight which is a multiple of p, and a word in $\gamma_m(F)$. Thus, $R_1 \subset \gamma_2(F_-)\gamma_m(F)$, as required.

Adding in this set of relations R_1 to the presentation above, we obtain a presentation

$$G = \langle X_1, X_2, X_3 \mid R_1, R_2, R_3 \rangle,$$

satisfying (i) - (vii) above, as required.

We now give an example that demonstrates the construction of such a presentation in practice. Let G be

$$\langle x_1, x_2, x_3 \mid x_3^{-1}[x_3, x_1], x_1^2 x_3^2 \rangle.$$

Then $H_1(G; \mathbb{F}_2)$ has rank 2, with x_1 and x_2 as generators. Let K be the subgroup of G generated by x_2 and $[G, G]G^2$. Thus, G/K is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let m be 3, say.

We start by setting $X_1 = \{x_1\}$, $X_2 = \{x_2\}$ and $X_3 = \{x_3\}$. Then, clearly, (i), (ii) and (iii) are satisfied. In the next stage of the procedure, we find $f(x_3)$, which

equals x_3 in G and which is the product of an element of $\gamma_2(F_-)$ and an element of $\gamma_3(F)$. Since $x_3 = [x_3, x_1]$ (by the first relation in the group), we may insert x_3 into this commutator to obtain the relation $x_3 = [[x_3, x_1], x_1]$. We set $f(x_3)$ to be $[[x_3, x_1], x_1]$, which is the product of an element of $\gamma_2(F_-)$ (the identity) and an element of $\gamma_3(F)$, as required. Thus,

$$R_3 = \{x_3^{-1}[[x_3, x_1], x_1]\}.$$

To construct R_2, we need to substitute every occurrence of x_3 in the relations with $f(x_3)$. It suffices to do this for the initial defining relations for the group. Thus, we may set

$$R_2 = \{[[x_3, x_1], x_1]^{-1}[[[x_3, x_1], x_1], x_1], x_1^2[[x_3, x_1], x_1]^2\}.$$

For the final stage of the procedure, we start with a basis for $H_2(G; \mathbb{F}_2)$. In this case, $H_2(G; \mathbb{F}_2) \cong \mathbb{F}_2$ with generator $x_1^2 x_3^2 \in R \cap ([F, F]F^2)$. Set R_1' to be this generator. To create R_1 from R_1', we substitute every occurrence of x_3 with $f(x_3)$. Thus,

$$R_1 = \{x_1^2[[x_3, x_1], x_1]^2\}.$$

We now explain how the integer m is chosen.

Lemma 2.1. *For some integer $m \geq 2$, $\gamma_m(F) \subset \gamma_2(K)$.*

Proof. Note that $\gamma_2(K)$ is a normal subgroup of F. Its index is a power of p, and so $F/\gamma_2(K)$ is a finite p-group. The lower central p-series of any finite p-group terminates. Hence, for all sufficiently large integers m, $\gamma_m(F) \subset \gamma_2(K)$. $\square$

We now fix m to be the integer given by the above lemma. (We can take m to be $\operatorname{rank}(G/K) + 2$, but we will not need this fact.)

Define subgroups of R recursively, by setting $R_{(1)} = R$ and letting $R_{(j+1)} = [R_{(j)}, F](R_{(j)})^p$, for each $j \geq 1$. An elementary induction establishes that each $R_{(j)}$ is normal in F.

Lemma 2.2. *Let $S = \langle\langle R_1, R_3 \rangle\rangle$, the subgroup of F normally generated by R_1 and R_3. Then, for each $j \geq 1$, $R = SR_{(j)}$.*

Proof. We first establish the inclusions

$$
\begin{aligned}
R &= \langle\!\langle R_1 \rangle\!\rangle \langle\!\langle R_2 \rangle\!\rangle \langle\!\langle R_3 \rangle\!\rangle \\
&\subseteq (R \cap ([F,F]F^p)) \langle\!\langle R_3 \rangle\!\rangle \\
&= \langle R_1 \rangle [R,F] R^p \langle\!\langle R_3 \rangle\!\rangle \\
&\subseteq S[R,F]R^p \\
&\subseteq R.
\end{aligned}
$$

In the second step above, we use are using properties (iv) and (vii) of the presentation. In the third step, we are using property (vi). We deduce that each of these inclusions is an equality.

We now prove the lemma by induction on j. For $j = 1$, this is trivial. Suppose that it is true for a given j. Then

$$
\begin{aligned}
R &= S[R,F]R^p \\
&= S[SR_{(j)}, F](SR_{(j)})^p \\[2mm]
&= S[S,F][R_{(j)}, F]S^p (R_{(j)})^p \\
&= S[R_{(j)}, F](R_{(j)})^p \\
&= SR_{(j+1)}.
\end{aligned}
$$

In the first equality, we are using the claim proved above. In the second, the inductive hypothesis is used. In the third equality, we are using the fact that $[AB,C] = [A,C][B,C]$, for normal subgroups A, B and C of a group. $\square$

3. Finding homology in covering spaces

The aim of this section is to prove Theorem 1.6, which provides lower bounds on $b_1(K; \mathbb{F}_p)$, for certain subgroups K of a group G. We work with a presentation $\langle X_1, X_2, X_3 \mid R_1, R_2, R_3 \rangle$ for G, satisfying conditions (i) - (vii) of Section 2. This determines a 2-complex L, in the usual way. It has a single 0-cell, which we take to be its basepoint, an oriented 1-cell for each generator, and a 2-cell for each relation. Then $\pi_1(L)$ is isomorphic to G. We are considering a normal subgroup K of G, such that G/K is an elementary abelian p-group of rank n, for some prime p. Associated with K, there is a finite-sheeted cover $\tilde{L}$ of L. We fix a basepoint for $\tilde{L}$

that maps to the 0-cell of L. The plan is to find a lower bound on $b_1(\tilde{L}; \mathbb{F}_p)$. Since $b_1(\tilde{L}; \mathbb{F}_p)$ equals $b_1(K; \mathbb{F}_p)$, this will give the required lower bound on $b_1(K; \mathbb{F}_p)$. Now, $b_1(\tilde{L}; \mathbb{F}_p)$ is the dimension of $H^1(\tilde{L}; \mathbb{F}_p)$. Recall that this is the vector space Z^1/B^1. Here, B^1 and Z^1 are subspaces of C^1, the space of all 1-cochains on $\tilde{L}$, with mod p coefficients. The 1-coboundaries B^1 are precisely those 1-cochains such that their evaluation on any closed loop is trivial. The 1-cocycles Z^1 are those 1-cochains with trivial evaluation on the boundary of any 2-cell. Thus, Z^1 is a subspace of C^1, obtained by imposing $p^n|R_1 \cup R_2 \cup R_3|$ linear constraints, since $p^n|R_1 \cup R_2 \cup R_3|$ is the number of 2-cells of $\tilde{L}$ (which may be infinite). In our approach to the proof of Theorem 1.6, we work with certain subspaces of C^1, which we denote by U^1_ℓ, for integers ℓ between 0 and n. (The integer ℓ is the same as that in the statement of Theorem 1.6, and is known as the 'level' of the subspace U^1_ℓ.) These subspaces form a hierarchy

$$C^1 \supseteq U^1_n \supseteq U^1_{n-1} \supseteq \ldots \supseteq U^1_0.$$

Each has the following dimension:

$$\dim(U^1_\ell) = b_1(G; \mathbb{F}_p) \sum_{i=0}^{\ell} \binom{n}{i}.$$

These subspaces have the following nice property. Roughly speaking, to determine whether or not certain cochains in U^1_ℓ are in fact cocycles, one does not need to verify that their evaluation around every 2-cell is zero. Instead, it suffices to check a certain set of 2-cells, with cardinality at most

$$b_2(G; \mathbb{F}_p) \sum_{i=0}^{\ell-1} \binom{n}{i}.$$

More precisely, if an element of U^1_ℓ has zero evaluation around these 2-cells, then a cocycle may be constructed from it. Of course, some of these cocycles may be coboundaries, but in fact this is true only for a relatively small subspace.

For any $j \in X_1 \cup X_2$, let c_j be the 1-cochain on L which sends the edge of L labelled j to $1 \in \mathbb{F}_p$, and maps the remaining edges to 0. These are cocycles because $X_1 \cup X_2$ forms a basis for $H_1(G; \mathbb{F}_p)$.

We lift the orientations on the 1-cells of L to the 1-cells of $\tilde{L}$. For each 1-cell e of $\tilde{L}$, let $i(e)$ denote its initial vertex. The map $\tilde{L} \to L$ sends e to a 1-cell of L, and so e has a well-defined evaluation under each c_j. We denote this by $c_j(e)$.

Every vertex v of $\tilde{L}$ also has a well-defined evaluation under c_j, for each $j \in X_1$, defined as follows. Pick a path from the basepoint of $\tilde{L}$ to v. This projects to a loop α in L. Define $c_j(v)$ to be $c_j(\alpha)$. This is well-defined, because if α' is another path from the basepoint to v, then

$$c_j(\alpha') = c_j(\alpha'.\alpha^{-1}) + c_j(\alpha) = c_j(\alpha).$$

The final equality holds because $\alpha'.\alpha^{-1}$ is a loop in $\tilde{L}$ and so $\alpha'.\alpha^{-1} \in K$, which implies that $c_j(\alpha'.\alpha^{-1}) = 0$.

We can now define the subspace U_ℓ^1 of C^1, for each integer ℓ between 0 and n. We do this by specifying a spanning set. For each subset A of X_1 with size at most ℓ, and for each element $y \in X_1 \cup X_2$, define $c(A, y)$ to be the following 1-cochain. On an edge e of $\tilde{L}$, let

$$c(A, y)(e) = \left(\prod_{j \in A} c_j(i(e)) \right) c_y(e).$$

When $A = \emptyset$, we take this to mean that $c(A, y)(e) = c_y(e)$, by convention. Then U_ℓ^1 is defined to be the subspace of C^1 spanned by these cochains.

Example. Let G be the free group on 3 generators, let $K = [G, G]G^2$ and let $p = 2$. Then L is the wedge of 3 circles, and $\tilde{L}$ is the cube-shaped graph shown twice in Figure 1. Note that the dotted edges in each figure join up with each other. Then the support of the cochains $c(\{1, 2\}, 3)$ and $c(\{1\}, 1)$ is shown in bold.
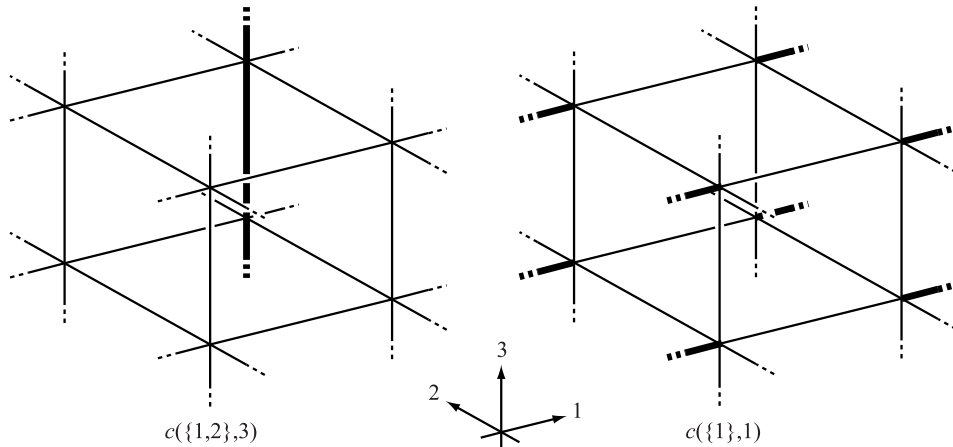

$c(\{1,2\},3)$ $c(\{1\},1)$

Figure 1.

It is possible to prove that the cochains $c(A, y)$ are linearly independent and so form a basis for U^1_ℓ. This is a reassuring result. But since we will not use this fact, we omit its proof.

Given any element $g \in F$ and a 1-cochain c on $\tilde{L}$, we define $c(g)$ as follows. Pick a word representing g, which specifies a path in $\tilde{L}$ starting at the basepoint. Define $c(g)$ to be the evaluation of c on this path. This is clearly independent of the choice of word representing g.

Pick a total ordering on X_1. For a subset E of X_1, with elements $i_1 < i_2 < \ldots < i_{|E|}$, let w_E be the word $i_1 \ldots i_{|E|}$ representing an element of the free group F. When $E = \emptyset$, then w_E is the identity element of F.

Our key technical result is the following.

Proposition 3.1. *Let z be an element of U^1_ℓ. Let k be an element of R. Suppose that $z(w_E k w_E^{-1}) = 0$, for all subsets E of X_1 with size at most $\ell - 1$. Then $z(gkg^{-1}) = 0$ for all $g \in F$.*

We prove this using a series of lemmas. Throughout, A is a subset of X_1 with size at most ℓ, and y is an element of $X_1 \cup X_2$. We define KR to be the subgroup of F that maps to K under the quotient homomorphism $F \to G$. Note that KR is the set of elements of F that form closed loops in $\tilde{L}$.

Lemma 3.2. *For $k \in KR$ and $g \in F$,*

$$c(A, y)([g, k]) = \sum_{\substack{B \subseteq A \\ B \neq \emptyset}} \left(c(A - B, y)(k) \prod_{j \in B} c_j(g) \right).$$

Proof. Represent k by the word $x_1 \ldots x_s$ in the generators of F. Because this is a closed loop in $\tilde{L}$, the parts of $[g, k]$ in g and g^{-1} run along the same edges in reverse and hence cancel. Thus, we need only consider evaluation of $c(A, y)$ on the parts of $[g, k]$ in k and k^{-1}. Let $e_1, \ldots, e_s$ be the edges of k. Then the k part of $[g, k]$ runs along $ge_1, \ldots, ge_s$, where each ge_r denotes the copy of e_r translated by the covering transformation of $\tilde{L}$ corresponding to g. The k^{-1} part of $[g, k]$ runs along $e_1, \ldots, e_s$ in reverse. Let β_r be ± 1, according to whether k runs forwards or

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backwards along the edge e_r. So,

$$c(A, y)([g, k])$$

$$= \sum_{r=1}^{s} \beta_r c(A, y)(ge_r) - \sum_{r=1}^{s} \beta_r c(A, y)(e_r)$$

$$= \sum_{r=1}^{s} \left(\left(\prod_{j \in A} c_j(i(ge_r)) \right) \beta_r c_y(e_r) - \left(\prod_{j \in A} c_j(i(e_r)) \right) \beta_r c_y(e_r) \right)$$

$$= \sum_{r=1}^{s} \left(\prod_{j \in A} (c_j(g) + c_j(i(e_r))) - \prod_{j \in A} c_j(i(e_r)) \right) \beta_r c_y(e_r)$$

$$= \sum_{r=1}^{s} \left(\sum_{B \subseteq A} \left(\prod_{j \in B} c_j(g) \prod_{j \in A-B} c_j(i(e_r)) \right) - \prod_{j \in A} c_j(i(e_r)) \right) \beta_r c_y(e_r)$$

$$= \sum_{\substack{B \subseteq A \\ B \neq \emptyset}} \left(\prod_{j \in B} c_j(g) \right) \left(\sum_{r=1}^{s} \left(\prod_{j \in A-B} c_j(i(e_r)) \right) \beta_r c_y(e_r) \right)$$

$$= \sum_{\substack{B \subseteq A \\ B \neq \emptyset}} \left(c(A - B, y)(k) \prod_{j \in B} c_j(g) \right),$$

as required. $\square$

Corollary 3.3. *For any $k \in KR$ and $j \in X_1 \cup X_2 \cup X_3$,*

$$c(A, y)([j, k]) = \begin{cases} c(A - \{j\}, y)(k) & \text{if } j \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.4. *For any $E \subseteq X_1$, and any $k \in KR$,*

$$c(A, y)(w_E k w_E^{-1}) = \sum_{B \subseteq A \cap E} c(A - B, y)(k).$$

Proof. We prove this by induction on $|E|$. For $E = \emptyset$, it is trivial. For the inductive step, let j be the first element of E, and let D be $E - \{j\}$. Then

$$c(A, y)(w_E k w_E^{-1}) = c(A, y)(j w_D k w_D^{-1} j^{-1}).$$

If j is not in A, then, by Corollary 3.3, this equals

$$c(A, y)(w_D k w_D^{-1}) = \sum_{B \subseteq A \cap D} c(A - B, y)(k) = \sum_{B \subseteq A \cap E} c(A - B, y)(k),$$

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as required. On the other hand, if j is in A, then it equals

$$c(A, y)(w_D k w_D^{-1}) + c(A, y)([j, w_D k w_D^{-1}])$$

$$= c(A, y)(w_D k w_D^{-1}) + c(A - \{j\}, y)(w_D k w_D^{-1})$$

$$= \sum_{B \subseteq A \cap D} c(A - B, y)(k) + \sum_{B \subseteq (A - \{j\}) \cap D} c(A - \{j\} - B, y)(k)$$

$$= \sum_{B \subseteq A \cap E} c(A - B, y)(k).$$

The first equality above relies on Corollary 3.3. The second equality uses the inductive hypothesis. The induction is established. ☐

Lemma 3.5. *Let z be an element of U_ℓ^1. Thus,*

$$z = \sum_{A, y} \lambda_{A, y} c(A, y),$$

where A ranges over all subsets of X_1 with size at most ℓ and $y \in X_1 \cup X_2$, and where $\lambda_{A, y}$ are coefficients in $\mathbb{F}_p$. Let k be an element of R. Suppose that $z(w_E k w_E^{-1}) = 0$ for all subsets E of X_1 with size at most $\ell - 1$. Then, for any subset E of X_1,

$$\sum_{\{A : E \subseteq A, \ |A| \leq \ell\} \atop y} \lambda_{A, y} c(A - E, y)(k) = 0.$$

Proof. We prove the lemma by induction on $|E|$. For $|E| = 0$, it is trivial. The inductive step, when $|E| \leq \ell - 1$, follows from the fact that

$$0 = z(w_E k w_E^{-1})$$

$$= \sum_{A, y} \lambda_{A, y} c(A, y)(w_E k w_E^{-1})$$

$$= \sum_{A, y} \lambda_{A, y} \sum_{B \subseteq A \cap E} c(A - B, y)(k)$$

$$= \sum_{B \subseteq E} \sum_{\{A : B \subseteq A, \ |A| \leq \ell\} \atop y} \lambda_{A, y} c(A - B, y)(k).$$

The third equality is an application of Lemma 3.4. By induction

$$\sum_{\{A : B \subseteq A, \ |A| \leq \ell\} \atop y} \lambda_{A, y} c(A - B, y)(k)$$

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is zero, when B is strictly contained in E. Hence,

$$\sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)(k)$$

is also zero, as required. For $|E| \geq \ell$, note that if $E \subseteq A$, then $E = A$. Thus, in this case, the formula we must prove is $\sum_y \lambda_{E,y} c(\emptyset, y)(k) = 0$. But, $c(\emptyset, y)(k) = c_y(k)$, which is zero because we are assuming that $k \in R$. $\square$

Lemma 3.6. *Let z and k be as in Lemma 3.5. Thus,*

$$z = \sum_{A,y} \lambda_{A,y} c(A,y),$$

where A ranges over all subsets of X_1 with size at most ℓ and $y \in X_1 \cup X_2$, and where $\lambda_{A,y}$ are coefficients in $\mathbb{F}_p$. Then for any $g \in F$ and any $E \subseteq X_1$,

$$\sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)(gkg^{-1}) = 0.$$

Proof. We represent g by a word in the generators. We prove the lemma by induction on the length of this word. We already know it to be the case when g is the identity, by Lemma 3.5. For the inductive step, applied to some word $g = ju$:

$$\sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)(gkg^{-1})$$

$$= \sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)(juku^{-1}j^{-1})$$

$$= \sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)(uku^{-1})$$

$$+ \sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)([j, uku^{-1}])$$

$$= \sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell\} \\ y}} \lambda_{A,y} c(A-E,y)(uku^{-1})$$

$$+ \sum_{\substack{\{A:E\subseteq A,\ |A|\leq\ell,\ j\in A-E\} \\ y}} \lambda_{A,y} c((A-E)-\{j\})(uku^{-1}),$$

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by Corollary 3.3. The first sum is zero by induction, as is the second, since it equals

$$\sum_{\{A : E \cup \{j\} \underset{y}{\subsetneq} A, \; |A| \leq \ell\}} \lambda_{A,y} c(A - (E \cup \{j\}))(uku^{-1}).$$

$\square$

Proof of Proposition 3.1. Set $E = \emptyset$ in Lemma 3.6. We obtain

$$z(gkg^{-1}) = \sum_{A,y} \lambda_{A,y} c(A, y)(gkg^{-1}) = 0$$

for all $g \in F$. $\square$

So far, we have focused on cochains supported on edges labelled by generators in $X_1 \cup X_2$. Let U^1 be the space of all such cochains. We now show how a cochain in U^1 has a natural modification, which has the same values on the edges labelled by $X_1 \cup X_2$, but so that its support might also include edges labelled by X_3. This modification depends on the presentation for G that we fixed in Section 2. We will define a linear map $\psi \colon U^1 \to C^1$. The modification of a cochain z will be $\psi(z)$. We define $\psi(z)$ to agree with z on the edges of $\tilde{L}$ labelled by $X_1 \cup X_2$. Each remaining edge e is labelled by an element $x_3 \in X_3$. For this x_3, there is a relation of the form $x_3 = f(x_3)$ in R_3. Note that e is a loop based at the vertex $i(e)$. Let g be a word in X_1 which specifies a path from the basepoint of $\tilde{L}$ to $i(e)$. We define $(\psi(z))(e)$ to be $z(gf(x_3)g^{-1})$. This is clearly independent of the choice of g, since the g and g^{-1} parts of the loop $gf(x_3)g^{-1}$ traverse the same edges in the opposite direction. Note that ψ is an injection.

Lemma 3.7. *The restriction of ψ to B^1 is the identity. Hence, $\psi(B^1) = B^1$.*

Proof. Note first that each element of B^1 is supported on edges labelled by $X_1 \cup X_2$, and so B^1 lies in U^1, which is the domain of ψ. Thus, it makes sense to speak of the restriction of ψ to B^1.

Let z be an element of B^1. On each edge e labelled by $x_3 \in X_3$, $(\psi(z))(e)$ is defined to be the evaluation of z on $gf(x_3)g^{-1}$, for suitable $g \in F$. This is a closed loop, and so its evaluation under the coboundary z is trivial. Thus, $\psi(z)$ is trivial on all edges labelled by X_3. On the remaining edges, $\psi(z)$ and z agree. Thus, $\psi(z) = z$, as required. $\square$

We now focus on a subspace of U_ℓ^1. This will arise as the kernel of a linear map $\phi_\ell \colon U_\ell^1 \to V_\ell$, where V_ℓ is the product of

$$b_2(G; \mathbb{F}_p) \sum_{i=0}^{\ell-1} \binom{n}{i}$$

copies of $\mathbb{F}_p$. We parametrise the co-ordinates of each element of V_ℓ by pairs (r_1, E), where $r_1 \in R_1$ and $E \subset X_1$, with $|E| \le \ell - 1$. The map ϕ_ℓ is defined as follows. If $z \in U_\ell^1$, then for each pair (r_1, E), the (r_1, E) co-ordinate of $\phi_\ell(z)$ is $z(w_E r_1 w_E^{-1})$.

We now define the subspace C_ℓ^1 of C^1 to be $\psi(\ker(\phi_\ell))$. Thus, to construct an element of C_ℓ^1, start with a linear combination z of the cochains $c(A, y)$. Evaluate z on the words $w_E r_1 w_E^{-1}$. Restrict attention to those z that have zero evaluation on these words. Given such a z, modify it to $\psi(z)$, which assigns certain values to edges labelled by X_3. Each such $\psi(z)$ is an element of C_ℓ^1, and conversely each element of C_ℓ^1 is constructed in this way.

Theorem 3.8. *Each element of C_ℓ^1 is a cocycle.*

Before we prove this, we need a lemma. Recall that m is the integer from Lemma 2.1.

Lemma 3.9. *For all elements $g \in \gamma_m(F)$ and all cochains c in C^1, $c(g) = 0$.*

Proof. This is simply a restatement of Lemma 2.1 in topological language. $\square$

Proof of Theorem 3.8. Let z be an element of $\ker(\phi_\ell)$. We want to prove that $\psi(z)$ is a cocycle. Lemma 2.2 gives that $R = SR_{(m)}$, where $S = \langle\!\langle R_1, R_3 \rangle\!\rangle$. Hence, to show that $(\psi(z))(r) = 0$ for all $r \in R$, it suffices to check that $(\psi(z))(r) = 0$ for all $r \in R_{(m)} \cup S$. But, $R_{(m)}$ lies in $\gamma_m(F)$, by the definition of $R_{(m)}$. Lemma 3.9 implies that the evaluation of any 1-cochain in $\tilde{L}$ on an element of $\gamma_m(F)$ is trivial. Thus, $(\psi(z))(r) = 0$ for all $r \in R_{(m)}$. We therefore only need to prove that $(\psi(z))(r) = 0$ for all $r \in \langle\!\langle R_1 \rangle\!\rangle \cup \langle\!\langle R_3 \rangle\!\rangle$.

Claim 1. Let e be an edge labelled by $x_3 \in X_3$, and let g be as in the definition of $(\psi(z))(e)$. Then, $(\psi(z))(g f(x_3) g^{-1}) = z(g f(x_3) g^{-1})$.

Now, $f(x_3)$ is the product of an element of $\gamma_2(F_-)$ and an element of $\gamma_m(F)$. The evaluation of the latter under any 1-cochain in $\tilde{L}$ is trivial, by Lemma 3.9.

Thus, $(\psi(z))(gy_3g^{-1})$ is equal to the evaluation under $\psi(z)$ of the word in $\gamma_2(F_-)$, conjugated by g. This lies in F_-, and hence the corresponding loop is supported on the edges labelled by $X_1 \cup X_2$. But z and $\psi(z)$ agree on these edges. This proves the claim.

The claim implies that, for each 2-cell of $\tilde{L}$ labelled by an element of R_3, the evaluation of $\psi(z)$ around its boundary is zero. Hence, $(\psi(z))(r) = 0$ for all $r \in \langle\!\langle R_3 \rangle\!\rangle$.

Claim 2. For each 2-cell of $\tilde{L}$ labelled by an element in R_1, the evaluation of $\psi(z)$ around its boundary is zero.

By property (vii) of Section 2, each such element lies in $\gamma_2(F_-)\gamma_m(F)$. Thus, by the argument of Claim 1, its evaluation under $\psi(z)$ equals its evaluation under z. Now, z lies in $\ker(\phi_\ell)$ and so its evaluation on each word $w_E r_1 w_E^{-1}$ is zero (where $r_1 \in R_1$ and $E \subseteq X_1$ with $|E| \leq \ell - 1$). Proposition 3.1 then implies that its evaluation on gr_1g^{-1} for any $r_1 \in R_1$ and $g \in F$ is zero. This proves the claim and the theorem. $\square$

Theorem 3.8 establishes that the cochains in C_ℓ^1 are cocycles. But in order to prove Theorem 1.6, we need to know how many of these cocycles are coboundaries. We start by examining which elements of U_ℓ^1 are coboundaries. There will, in general, be some non-zero coboundaries, as the following example demonstrates.
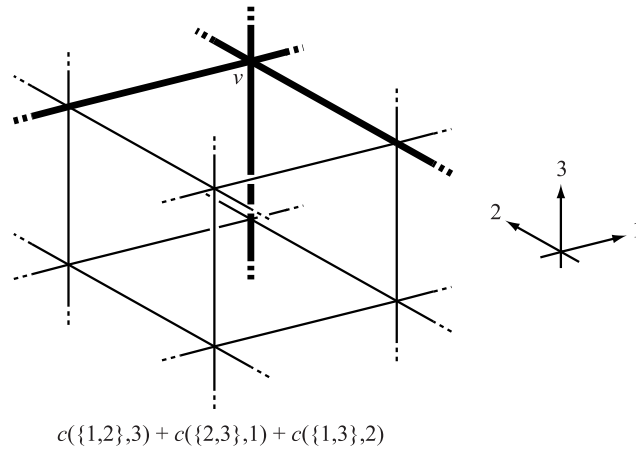

$c(\{1,2\},3) + c(\{2,3\},1) + c(\{1,3\},2)$

Figure 2.

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Example. Let G be the free group on 3 generators, let K be $[G,G]G^2$, and let $p = 2$. Then, on $\tilde{L}$,

$$c(\{1,2\},3) + c(\{2,3\},1) + c(\{1,3\},2)$$

is a coboundary (see Figure 2). It is the coboundary of the function supported at the vertex v where $c_1(v) = c_2(v) = c_3(v) = 1$.

In fact, it is not hard to show that, more generally, if A is any non-empty subset of X_1, then

$$\sum_{y \in A} c(A - \{y\}, y)$$

is a coboundary, although we will not need this fact.

Proposition 3.10. *The dimension of $U_\ell^1 / (B^1 \cap U_\ell^1)$ is at least*

$$\sum_{r=2}^{\ell+1} \binom{n}{r} (r-1) + (b_1(G; \mathbb{F}_p) - n) \sum_{r=0}^{\ell} \binom{n}{r}$$

if p is odd, and at least

$$b_1(G; \mathbb{F}_2) \sum_{r=0}^{\ell} \binom{n}{r} - \sum_{r=1}^{\ell+1} \binom{n}{r}$$

if $p = 2$.

We prove this by evaluating elements of U_ℓ^1 on a certain set T of 'test' loops in $\tilde{L}$. The number of such loops will be equal to the quantities given in Proposition 3.10. Let $U_\ell^1 \to \mathbb{F}_p^T$ be the map that sends an element of U_ℓ^1 to its evaluation under the test loops T. We will show that this map has rank $|T|$. Since this map factors through $U_\ell^1 / (B^1 \cap U_\ell^1)$, this will prove the proposition.

Let A be a non-empty subset of X_1 with cardinality at most ℓ. Let y be an element of $(X_1 \cup X_2) - A$. Let y_1 be the smallest element of A. (Recall that X_1 comes with a total ordering.) We insist that if y lies in X_1, then it is larger than y_1. Define $t(A, y)$ to be the loop $[y, y_1^{-1}]$ which starts at the vertex v such that

$$c_j(v) = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{if } j \notin A. \end{cases}$$

When $y \in X_2$, define $t(\emptyset, y)$ to be the loop y which starts and ends at the basepoint of $\tilde{L}$. When p is odd, the set T of test loops will be all such $t(A, y)$. When $p = 2$,

the test loops will be all these loops, together with the following. Let A be as above, but now let y be an element of A. Define $t(A, y)$ to be the loop y^2 which starts at the vertex v defined above. We include all such $t(A, y)$ as test loops.

Lemma 3.11. *The number of test loops T is given by the quantities in Proposition 3.10.*

Proof. Let us first consider the case where p is odd. We wish to count the number of pairs (A, y) satisfying the above conditions. If $y \in X_1$, then we simply count the possibilities for $A \cup y$. For each such set $A \cup y$, with cardinality r between 2 and $\ell + 1$, there are $r - 1$ choices for y, since y cannot be the smallest element of $A \cup y$. Thus, the number of such pairs (A, y) where $y \in X_1$ is

$$\sum_{r=2}^{\ell+1} \binom{n}{r} (r - 1).$$

The number of pairs (A, y) where $y \in X_2$ is clearly

$$(b_1(G; \mathbb{F}_p) - n) \sum_{r=0}^{\ell} \binom{n}{r}$$

since $|X_2| = (b_1(G; \mathbb{F}_p) - n)$.

Let us now examine the case where $p = 2$. Here, we count all pairs (A, y) where $A \subset X_1$, $|A| \leq \ell$ and $y \in X_1 \cup X_2$. We then subtract off the number that do not satisfy the given condition. The first count gives

$$b_1(G; \mathbb{F}_2) \sum_{r=0}^{\ell} \binom{n}{r}.$$

If (A, y) does not satisfy the condition required to define a test loop, then y lies in X_1 and is strictly smaller than every element of A. Thus, given $A \cup y$, it is possible to determine y. Thus, we need only count the number of possibilities for $A \cup y$:

$$\sum_{r=1}^{\ell+1} \binom{n}{r}.$$

The required formula follows immediately. $\square$

Lemma 3.12. *Let A' be a subset of X_1 with size at most ℓ, and let y' be an element of $X_1 \cup X_2$. Let A be a subset of X_1 with size at most $|A'|$, and let y be an element of $(X_1 \cup X_2) - A$. Suppose that, if y lies in X_1, then A is non-empty and y is larger than the smallest element y_1 of A. Similarly, suppose that if y' lies in X_1 and A' is non-empty, then y' is at least as large as the smallest element of A'. Then*

$$c(A', y')(t(A, y)) = \begin{cases} 1 & \text{if } A = A' \text{ and } y = y' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us first consider the case where $A = \emptyset$. Then $y \in X_2$, by assumption, and $t(A, y)$ is a single edge e labelled y based at the basepoint of $\tilde{L}$. The evaluation of $c(A', y')$ on e is

$$c(A', y')(e) = \left(\prod_{j \in A'} c_j(i(e)) \right) c_{y'}(e).$$

For this to be non-zero, we must have $y = y'$ and $A' = \emptyset$. In this case, the evaluation is 1, as required.

Let us now suppose that $A \neq \emptyset$ and hence that $A' \neq \emptyset$. The loop $t(A, y)$ then consists of two edges labelled y and two edges labelled y_1. Now, the evaluation of $c(A', y')$ on each such edge e is again

$$c(A', y')(e) = \left(\prod_{j \in A'} c_j(i(e)) \right) c_{y'}(e).$$

This is zero if $c_{y'}(e) = 0$. Thus, the evaluation of $t(A, y)$ is zero unless $y' = y$ or $y' = y_1$. If neither of these equalities holds, the lemma is true.

Case 1. $y' = y$.

Then, only the edges labelled y contribute to the evaluation of $t(A, y)$. Their initial vertices are v and vy_1^{-1}. Thus, the total evaluation is

$$\prod_{j \in A'} c_j(v) - \prod_{j \in A'} c_j(vy_1^{-1}).$$

The first term is 1 if and only if $A' \subseteq A$. Since we are assuming $|A'| \geq |A|$, this happens if and only if $A = A'$. Otherwise, the first term is zero. The latter term is

always zero, since to be non-zero, it would have to be the case that $A' \subseteq A - \{y_1\}$. This proves the lemma in the first case.

Case 2. $y' = y_1$.

In this case, only the edges labelled y_1 contribute to the evaluation of $t(A, y)$. Their initial vertices are vy_1^{-1} and vyy_1^{-1}. Thus, the total evaluation is

$$\prod_{j \in A'} c_j(vy_1^{-1}) - \prod_{j \in A'} c_j(vyy_1^{-1}).$$

The first term is zero since $A' \not\subseteq A - \{y_1\}$. If the second term is non-zero, then $A' \subseteq A \cup \{y\} - \{y_1\}$. But comparing the sizes of these sets, we see that this must be an equality. Hence, the smallest element of A' is strictly bigger than y_1, which equals y'. (Recall that y_1 is the smallest element of A, and y is larger than y_1.) We therefore deduce that y' is strictly smaller than every element of A', which is contrary to hypothesis. Thus, in Case 2, the evaluation of $t(A, y)$ is zero. $\square$

Lemma 3.13. *Let $p = 2$. Let A' be a subset of X_1 with size at most ℓ, and let y' be an element of $X_1 \cup X_2$. Let A be a subset of X_1 with size at most $|A'|$, and let y be an element of A. Then*

$$c(A', y')(t(A, y)) = \begin{cases} 1 & \text{if } A = A' \text{ and } y = y' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The test loop $t(A, y)$ has two edges labelled y, with initial vertices v and vy. So, its evaluation under $c(A', y')$ is

$$\left(\prod_{j \in A'} c_j(v) + \prod_{j \in A'} c_j(vy) \right) c_{y'}(y).$$

This is zero unless $y' = y$. The first term in the brackets is zero unless $A' \subseteq A$, which happens if and only if $A' = A$. The second term is always zero, since $A' \not\subseteq A - \{y\}$. $\square$

The reason why Theorem 1.6 treats odd primes in a different way from the prime 2 arises in the above lemma. For odd p, one may also define test loops $t(A, y)$ when $y \in A$: this is a loop based at a suitable vertex v (depending on A) running along the edges y^p. However, the evaluation of $c(A, y)$ on this $t(A, y)$ is $\sum_{r=0}^{p-1} r$, which is zero modulo p, when p is odd.

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Proof of Proposition 3.10. We pick a total order on the test loops T, subject to the condition that if $|A| < |A'|$, then $t(A, y) < t(A', y')$, whenever these loops are defined. For each test loop $t' = t(A', y')$, let $z_{t'}$ be the cochain $c(A', y')$ in U_ℓ^1. Then, by Lemmas 3.12 and 3.13, $z_{t'}(t') = 1$ and $z_{t'}(t) = 0$ for all $t < t'$. This proves the proposition. $\square$

Proposition 3.14. *The dimension of $C_\ell^1/(B^1 \cap C_\ell^1)$ is at least*

$$\sum_{r=2}^{\ell+1} \binom{n}{r} (r-1) + (b_1(G; \mathbb{F}_p) - n) \sum_{r=0}^{\ell} \binom{n}{r} - b_2(G; \mathbb{F}_p) \sum_{r=0}^{\ell-1} \binom{n}{r}$$

if p is odd, and at least

$$b_1(G; \mathbb{F}_2) \sum_{r=0}^{\ell} \binom{n}{r} - \sum_{r=1}^{\ell+1} \binom{n}{r} - b_2(G; \mathbb{F}_2) \sum_{r=0}^{\ell-1} \binom{n}{r}$$

if $p = 2$.

Proof. We have the isomorphisms

$$\frac{C_\ell^1}{C_\ell^1 \cap B^1} = \frac{C_\ell^1}{C_\ell^1 \cap \psi(B^1)} = \frac{\psi(\ker(\phi_\ell))}{\psi(\ker(\phi_\ell)) \cap \psi(B^1)} \cong \frac{\ker(\phi_\ell)}{\ker(\phi_\ell) \cap B^1}.$$

The first equality is a consequence of Lemma 3.7. The second is just the definition of C_ℓ^1. The final isomorphism is a consequence of the fact that ψ is injective. Thus,

$$\dim(C_\ell^1/(C_\ell^1 \cap B^1)) \geq \dim(\ker(\phi_\ell)) - \dim(U_\ell^1 \cap B^1)$$

$$\geq \dim(U_\ell^1) - \dim(U_\ell^1 \cap B^1) - \dim(V_\ell).$$

The proposition now follows from Proposition 3.10, which gives a lower bound on the dimension of $U_\ell^1/(U_\ell^1 \cap B^1)$, and the formula for the dimension of V_ℓ. $\square$

Theorem 1.6 immediately follows from this proposition and Theorem 3.8, since $C_\ell^1/(B^1 \cap C_\ell^1)$ is a subspace of $H^1(\tilde{L}; \mathbb{F}_p)$.

4. Relationship with the lower central p-series

The proof of Theorem 1.6 was fairly formal. In this section, we aim to explain it in terms that are possibly more familiar.

It is instructive to consider the case where $\ell = 1$, $p = 2$ and $n = b_1(G; \mathbb{F}_2)$ in Theorem 1.6. This forces K to be $\gamma_2(G)$, the second term in the lower-central

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2-series for G. Theorem 1.6 gives the inequality

$$b_1(\gamma_2(G); \mathbb{F}_2) \geq \binom{b_1(G; \mathbb{F}_2)}{2} + b_1(G; \mathbb{F}_2) - b_2(G; \mathbb{F}_2).$$

This lower bound on $b_1(\gamma_2(G); \mathbb{F}_2)$ was already known. Indeed, the following result was proved by Shalen and Wagreich (see Lemma 1.3 of [6]).

Theorem 4.1. *Let G be a group and let p be a prime. Suppose that $b_1(G; \mathbb{F}_p)$ and $b_2(G; \mathbb{F}_p)$ are finite. Then*

$$b_1(\gamma_2(G); \mathbb{F}_p) \geq \binom{b_1(G; \mathbb{F}_p)}{2} + b_1(G; \mathbb{F}_p) - b_2(G; \mathbb{F}_p).$$

They proved this using the following exact sequence of Stallings [7]:

$$H_2(G; \mathbb{F}_p) \to H_2(G/\gamma_2(G); \mathbb{F}_p) \to \gamma_2(G)/\gamma_3(G) \to 0.$$

Now, $G/\gamma_2(G)$ is an elementary abelian p-group with rank $b_1(G; \mathbb{F}_p)$. Its second homology is well-known to have rank

$$\binom{b_1(G; \mathbb{F}_p)}{2} + b_1(G; \mathbb{F}_p),$$

via the Künneth formula. Thus, exactness of the sequence gives that $\gamma_2(G)/\gamma_3(G)$ has dimension at least

$$\binom{b_1(G; \mathbb{F}_p)}{2} + b_1(G; \mathbb{F}_p) - b_2(G; \mathbb{F}_p).$$

Since $\gamma_2(G)/\gamma_3(G)$ is a quotient of $H_1(\gamma_2(G); \mathbb{F}_p)$, we deduce the required lower bound on $b_1(\gamma_2(G); \mathbb{F}_p)$.

Now, $\gamma_2(G)/\gamma_3(G)$ is an elementary abelian p-group, and hence it is isomorphic to $(\gamma_2(G)/\gamma_3(G))^*$, which is $\mathrm{Hom}(\gamma_2(G)/\gamma_3(G); \mathbb{F}_p)$. It is often useful to work with this latter group. Any homomorphism $\gamma_2(G)/\gamma_3(G) \to \mathbb{F}_p$ arises from a homomorphism $\gamma_2(G) \to \mathbb{F}_p$ that is trivial on $\gamma_3(G)$. Conversely, any such homomorphism gives an element of $(\gamma_2(G)/\gamma_3(G))^*$. Thus, one can consider $(\gamma_2(G)/\gamma_3(G))^*$ to be a subgroup of the set of all homomorphisms from $\gamma_2(G)$ to $\mathbb{F}_p$. This is just $H^1(\gamma_2(G); \mathbb{F}_p)$. Now, $H^1(\gamma_2(G); \mathbb{F}_p)$ is isomorphic to $H^1(\tilde{L}; \mathbb{F}_p)$, where $\tilde{L}$ is the 2-complex from Section 3. The Stallings exact sequence gives a

lower bound on the dimension of $(\gamma_2(G)/\gamma_3(G))^*$. Each element in here gives an element of $H^1(\tilde{L}; \mathbb{F}_p)$, which is represented by a 1-cocycle on $\tilde{L}$. What are these cocycles? When $p = 2$, they are precisely C_ℓ^1 for $\ell = 1$.

Thus, the level $\ell = 1$ is the topological analogue of $\gamma_2(G)/\gamma_3(G)$. As one might expect, higher values of ℓ do indeed correspond to sections further down the lower central p-series of G. Specifically, one can consider the covering space $\tilde{L}_{\ell+1}$ corresponding to the subgroup $\gamma_{\ell+1}(G)$ of G. It is possible to construct explicit 1-cocycles on $\tilde{L}_{\ell+1}$ representing certain elements of $(\gamma_{\ell+1}(G)/\gamma_{\ell+2}(G))^*$. These cocycles turn out to be invariant under the action of the covering group $\gamma_2(G)/\gamma_{\ell+1}(G)$ and so descend to cocycles on $\tilde{L}$. These lie in C_ℓ^1.

Thus, Theorem 1.6 arose from an attempt to understand the Stallings exact sequence topologically, and to explore its possible analogues further down the lower central p-series. So the appearance at various stages in Sections 2 and 3 of $\gamma_m(F)$ was more than just a technical device. The lower central p-series has a crucial rôle in the interpretation of Theorem 1.6.

5. NORMAL SUBGROUPS WITH LARGE HOMOLOGY

Most of the theorems in this paper make the hypothesis that the group G contains finite index subgroups G_1 where $b_1(G_1; \mathbb{F}_p)$ is arbitrarily large. In this section, we show that we may assume that these subgroups G_1 are, in addition, normal. We will need to make a hypothesis about G that is much weaker than the $b_2 - b_1$ and b_2/b_1 conditions.

This section is not in fact required for most of the results in this paper. It is necessary only to prove that the subgroups G_i in Theorem 1.7 with fast homology growth are normal in G. If one is content with the weaker conclusion that they are just normal in G_1, then this section could be omitted entirely. From this, there is an easy argument which gives that we may take each G_i to be subnormal in G. This would be sufficient to deduce Theorem 1.1. However, we prefer to pursue the strongest possible conclusion for Theorem 1.7: that each G_i is normal in G. For this, it appears that more technology is required: we need some results about p-adic analytic pro-p groups.

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Our main result in this section is the following.

Theorem 5.1. *Let G be a finitely generated group and let p be a prime. Suppose that, for some finite index subgroup G_1 of G, $b_1(G_1; \mathbb{F}_p) > 1$ and $b_2(G_1; \mathbb{F}_p) < b_1(G_1; \mathbb{F}_p)^2/4$. Then*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index normal subgroup of } G\} = \infty.$$

We will need to quote two facts about p-adic analytic pro-p groups. For the following, see Interlude D in [1].

Theorem 5.2. *Let G be a finitely generated group and let p be a prime. Suppose that $b_1(G; \mathbb{F}_p) > 1$ and $b_2(G; \mathbb{F}_p) < b_1(G; \mathbb{F}_p)^2/4$. Then the pro-$p$ completion of G, denoted $\hat{G}_{(p)}$, is not p-adic analytic.*

A proof of the following can be found in [4].

Theorem 5.3. *Let G be a finitely generated group and let p be a prime. Then the following are equivalent:*

1. *$\hat{G}_{(p)}$ is p-adic analytic;*

2. *the supremum of $b_1(G_i; \mathbb{F}_p)$, as G_i ranges over all characteristic subgroups of G with index a power of p, is finite;*

3. *the supremum of $b_1(G_i; \mathbb{F}_p)$, as G_i ranges over all normal subgroups of G with index a power of p, is finite.*

Proof of Theorem 5.1. By hypothesis, there is a finite index subgroup G_1 of G such that $b_1(G_1; \mathbb{F}_p) > 1$ and $b_2(G_1; \mathbb{F}_p) < b_1(G_1; \mathbb{F}_p)^2/4$. So, by Theorem 5.2, the pro-p completion of G_1 is not p-adic analytic. Therefore, by Theorem 5.3, G_1 contains a sequence of characteristic subgroups G_i, each with index a power of p, such that $b_1(G_i; \mathbb{F}_p)$ tends to infinity.

Let K be a finite index normal subgroup of G that lies in G_1. Let K_i be the intersection of K and G_i. Then,

$$b_1(K_i; \mathbb{F}_p) \geq b_1(G_i; \mathbb{F}_p) - b_1(G_i/K_i; \mathbb{F}_p).$$

But G_i/K_i is isomorphic to G_iK/K, which is a subgroup of G_1/K. Since this finite group has only finitely many subgroups, $b_1(G_i/K_i; \mathbb{F}_p)$ is uniformly bounded

above. Hence, $b_1(K_i; \mathbb{F}_p)$ tends to infinity. Now, G_i is normal in G_1 and has index a power of p. So, K_i is normal in K and has index a power of p. Hence, by Theorem 5.3, K has a sequence of finite index characteristic subgroups L_i such that $b_1(L_i; \mathbb{F}_p)$ tends to infinity. Since these are characteristic in K, which is normal in G, they are therefore normal in G. These are the required subgroups of G. $\square$

6. Homology growth and subgroup growth

In this section, we use the homological lower bounds of Theorem 1.6 to deduce Theorem 1.7. We are assuming that G is a finitely generated group having the $b_2 - b_1$ property with respect to the prime 2 and that

$$\sup\{b_1(G_i; \mathbb{F}_2) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Pick G_1 where $b_1(G_1; \mathbb{F}_2)$ is bigger than 1 and large enough so that $b_2(G_1; \mathbb{F}_p) < b_1(G_1; \mathbb{F}_p)^2/4$. Theorem 5.1 states that

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index normal subgroup of } G\} = \infty.$$

Thus, we may assume not only that $b_1(G_1; \mathbb{F}_p)$ is large but also that G_1 is normal in G.

Define subgroups G_i of G recursively, by setting $G_{i+1} = [G_i, G_i](G_i)^2$. Thus, G_i is just the derived 2-series for G_1. Let $x_i = b_1(G_i; \mathbb{F}_2)$. According to Theorem 1.6, setting $\ell = \lfloor x_i/2 \rfloor$ and $n = x_i$, we have

$$x_{i+1} \geq x_i \sum_{r=0}^{\lfloor x_i/2 \rfloor} \binom{x_i}{r} - \sum_{r=1}^{\lfloor x_i/2 \rfloor+1} \binom{x_i}{r} - b_2(G_i; \mathbb{F}_2) \sum_{r=0}^{\lfloor x_i/2 \rfloor-1} \binom{x_i}{r}.$$

Now, the middle summation is bounded below by -2^{x_i}. The third summation can be compared with all but the highest term in the first summation. Thus,

$$x_{i+1} \geq x_i \binom{x_i}{\lfloor x_i/2 \rfloor} - 2^{x_i} \max\{1, 1 + b_2(G_i; \mathbb{F}_2) - x_i\}. \tag{1}$$

Claim 1. Let λ be any positive real number less than $\sqrt{2/\pi}$. Then, provided x_1 is sufficiently big,

$$x_{i+1} \geq \lambda 2^{x_i} \sqrt{x_i},$$

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for all $i \geq 1$.

According to Stirling's formula,

$$x! \sim \sqrt{2\pi x} \; x^x e^{-x},$$

as $x \to \infty$. So,

$$\binom{x}{\lfloor x/2 \rfloor} \sim \frac{\sqrt{2\pi x} x^x e^{-x}}{2\pi (x/2)^{x+1} e^{-x}} = \sqrt{\frac{2}{\pi}} \frac{2^x}{\sqrt{x}}.$$

Thus, the first term in (1) is at least

$$\lambda 2^{x_i} \sqrt{x_i},$$

when x_i is sufficiently large. Now, by the $b_2 - b_1$ condition, there is a universal upper bound on $b_2(G_i; \mathbb{F}_2) - x_i$. Thus, the first term of (1) dominates, and the claim is proved. In fact, to prove this claim, one does not need the full strength of the $b_2 - b_1$ condition. It suffices to assume that, when x_i is sufficiently large, $b_2(G_i; \mathbb{F}_2) - x_i$ is negative or small compared with $\sqrt{x_i}$.

Note that, by the claim, if we pick $b_1(G_1; \mathbb{F}_2)$ to be sufficiently big, then x_i is a strictly increasing function.

Set

$$\sigma_i = \sum_{j=1}^{i} x_j.$$

Claim 2. Provided x_1 is sufficiently big, then for all $i \geq 1$,

$$2^{\sigma_i} \leq \lambda 2^{x_i} x_i (\log x_i)^{2/3}.$$

We prove this by induction on i. It is clear for $i = 1$. For the inductive step, note that

$$
\begin{aligned}
2^{\sigma_{i+1}} &= 2^{x_{i+1}} 2^{\sigma_i} \\
&\leq 2^{x_{i+1}} \lambda 2^{x_i} x_i (\log x_i)^{2/3} \\
&\leq 2^{x_{i+1}} x_{i+1} x_i^{1/2} (\log x_i)^{2/3} \\
&\leq 2^{x_{i+1}} x_{i+1} (\log x_{i+1})^{2/3},
\end{aligned}
$$

where the second inequality is consequence of Claim 1 and the final step follows from the fact that

$$(\log x_{i+1})^{2/3} \geq (x_i \log 2 + \log \sqrt{x_i} + \log \lambda)^{2/3} \geq x_i^{1/2} \log x_i,$$

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provided x_i is sufficiently big. This certainly holds if x_1 is sufficiently large.

Claim 3. As $i \to \infty$,

$$x_i = \Omega \left(\frac{2^{\sigma_{i-1}}}{\sigma_{i-1}^{1/2} \log \sigma_{i-1}} \right).$$

This follows from

$$x_i \geq \lambda 2^{x_{i-1}} x_{i-1}^{1/2} \geq \frac{2^{\sigma_{i-1}}}{x_{i-1}^{1/2} (\log x_{i-1})^{2/3}} \geq \frac{2^{\sigma_{i-1}}}{\sigma_{i-1}^{1/2} (\log \sigma_{i-1})^{2/3}} = \Omega \left(\frac{2^{\sigma_{i-1}}}{\sigma_{i-1}^{1/2} \log \sigma_{i-1}} \right).$$

The second inequality is an application of Claim 2, and the penultimate inequality comes from the fact that $\sigma_{i-1} \geq x_{i-1}$.

The proof of Theorem 1.7 follows quickly from this claim. This is because $x_i = b_1(G_i; \mathbb{F}_2)$, by definition, and

$$[G_1 : G_i] = \prod_{j=1}^{i-1} [G_j : G_{j+1}] = 2^{\sum_{j=1}^{i-1} x_j} = 2^{\sigma_{i-1}}.$$

This completes the proof of Theorem 1.7. We saw in the introduction that Theorem 1.1 is a rapid consequence of Theorem 1.7.

It should now be apparent why Theorems 1.1 and 1.7 work only when $p = 2$. This is *not* due to the two different lower bounds in Theorem 1.6. Instead, it is a consequence of the behaviour of binomial coefficients. If one wants to establish fast homology growth for the sequence of subgroups G_i, one needs to know that the ratio $b_1(G_{i+1}; \mathbb{F}_p)/b_1(G_i; \mathbb{F}_p)$ is comparable with the index $[G_i : G_{i+1}]$. Now, the latter is

$$p^{b_1(G_i; \mathbb{F}_p)}.$$

A lower bound on the former comes from Theorem 1.6. When $b_2(G_i; \mathbb{F}_p) = b_1(G_i; \mathbb{F}_p)$, say, the formula in Theorem 1.6 is at most

$$b_1(G_i; \mathbb{F}_p) \binom{b_1(G_i; \mathbb{F}_p)}{\ell}.$$

So, no matter what value of ℓ is chosen, the lower bound on $b_1(G_{i+1}; \mathbb{F}_p)/b_1(G_i; \mathbb{F}_p)$ that we obtain is at most

$$2^{b_1(G_i; \mathbb{F}_p)}.$$

Thus, the only situation in which we can prove that the growth in homology is comparable to the growth in the subgroups' index is when $p = 2$. If one wanted

to prove similar results when p is odd, a significantly stronger version of Theorem 1.6 would be required.

It also clear that, in practice, one does need the full hypotheses of Theorem 1.1 to deduce its conclusion. Firstly, as noted in the proof of Claim 1, one may weaken the $b_2 - b_1$ condition to the hypothesis that, for any sequence of finite index subgroups G_i of G such that $b_1(G_i; \mathbb{F}_2) \to \infty$,

$$\limsup_i \frac{b_2(G_i; \mathbb{F}_2) - b_1(G_i; \mathbb{F}_2)}{\sqrt{b_1(G_i; \mathbb{F}_2)}} \leq 0.$$

Secondly, one does not need to *assume* that

$$\sup\{b_1(G_i; \mathbb{F}_2) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

This is in fact a consequence of the $b_2 - b_1$ condition, together with the fact that $b_1(G_1; \mathbb{F}_2)$ is 'sufficiently large', for some finite index subgroup G_1 of G. For, suppose that one has an upper bound (k, say) on $b_2(G_i; \mathbb{F}_2) - b_1(G_i; \mathbb{F}_2)$ for all finite index subgroups G_i. One can then apply Theorem 1.6 with $\ell = 1$ (or Theorem 4.1) to a finite index subgroup G_1 of G to give

$$b_1(\gamma_2(G_1); \mathbb{F}_2) \geq \binom{b_1(G_1; \mathbb{F}_2)}{2} + b_1(G_1; \mathbb{F}_2) - b_2(G_1; \mathbb{F}_2) \geq \binom{b_1(G_1; \mathbb{F}_2)}{2} - k.$$

Suppose $b_1(G_1; \mathbb{F}_2)$ is big enough (depending on k) to ensure that $b_1(\gamma_2(G_1); \mathbb{F}_2) > b_1(G_1; \mathbb{F}_2)$. We may then repeat to find a lower bound on $b_1(\gamma_2(\gamma_2(G_1)); \mathbb{F}_2)$, and so on. Thus, we obtain a sequence of finite index subgroups $\{G_i\}$ such that $b_1(G_i; \mathbb{F}_2)$ tends to infinity. Moreover, this is the derived 2-series of some finite index subgroup G_1 of G. We will see this approach put into practice in Section 8 with the proofs of Theorems 1.4 and 1.5.

This then leads to a third way that the hypotheses of Theorem 1.1 may be weakened. One does not need to assume a uniform upper bound on $b_2(G_i; \mathbb{F}_2) - b_1(G_i; \mathbb{F}_2)$ over all finite index subgroups G_i. Once one has a finite index subgroup G_1 of G such that $b_1(G_i; \mathbb{F}_2)$ tends to infinity for the derived 2-series $\{G_i\}$ for G_1, then one only needs to assume an upper bound on $b_2(G_i; \mathbb{F}_2) - b_1(G_i; \mathbb{F}_2)$ for this collection of finite index subgroups.

7. THE b_2/b_1 CONDITION

Let G be a group satisfying the b_2/b_1 condition with respect to the prime p. Let m be the supremum of

$$\left\lceil \frac{b_2(G_i; \mathbb{F}_p)}{b_1(G_i; \mathbb{F}_p)} \right\rceil$$

as G_i ranges over all finite index subgroups such that $b_1(G_i; \mathbb{F}_p) > 0$. Then for all such G_i, $b_2(G_i; \mathbb{F}_p) \leq m \, b_1(G_i; \mathbb{F}_p)$. Suppose also that

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

We will prove that there is a sequence of finite index subgroups $G \triangleright G_1 \triangleright G_2 \triangleright \ldots$ with the following property. For all $\epsilon > 0$,

$$b_1(G_i; \mathbb{F}_p) \geq [G : G_i]^{(1/(m+p-1))-\epsilon}$$

for all sufficiently large i. This will prove Theorems 1.8 and 1.2. It also provides a lower bound on the constant k in these theorems.

By Theorem 5.1, there are finite index normal subgroups G_1 of G where $b_1(G_1; \mathbb{F}_p)$ is arbitrarily large. Set $n = m + p - 1$ and set $\ell = 1$. Define a sequence of subgroups $\{G_i\}$ of G, as follows. For $i \geq 1$, let G_{i+1} be a normal subgroup of G_i such that G_i/G_{i+1} is an elementary abelian p-group of rank n. Applying Theorem 1.6, obtain the inequality

$$b_1(G_{i+1}; \mathbb{F}_p) \geq \binom{n}{2} + (b_1(G_i; \mathbb{F}_p) - n)(n+1) - b_2(G_i; \mathbb{F}_p)$$

$$\geq b_1(G_i; \mathbb{F}_p)p - n^2/2 - 3n/2.$$

Now, when $b_1(G_i; \mathbb{F}_p)$ is sufficiently large, the $n^2/2$ and $3n/2$ terms are negligible compared with the first term. In particular, $b_1(G_{i+1}; \mathbb{F}_p)$ is strictly greater than $b_1(G_i; \mathbb{F}_p)$. Thus, we may repeat this indefinitely, obtaining a sequence of finite index subgroups $G_1 \triangleright G_2 \triangleright \ldots$ such that $b_1(G_i; \mathbb{F}_p) \to \infty$. Thus,

$$\liminf_i \frac{b_1(G_{i+1}; \mathbb{F}_p)}{b_1(G_i; \mathbb{F}_p)} \geq p.$$

But, $[G_i : G_{i+1}] = p^{(m+p-1)}$. Thus,

$$\liminf_i \frac{(\log_p b_1(G_{i+1}; \mathbb{F}_p) - \log_p b_1(G_i; \mathbb{F}_p))}{\log_p[G_i : G_{i+1}]} \geq \frac{1}{m+p-1}.$$

Thus,

$$\liminf_i \frac{\log_p b_1(G_i; \mathbb{F}_p)}{\log_p [G : G_i]} \geq \liminf_i \frac{\sum_{j=1}^{i-1} (\log_p b_1(G_{j+1}; \mathbb{F}_p) - \log_p b_1(G_j; \mathbb{F}_p))}{\log_p [G : G_1] + \sum_{j=1}^{i-1} \log_p [G_j : G_{j+1}]}$$

$$\geq \frac{1}{m + p - 1}.$$

This implies that, for all $\epsilon > 0$,

$$b_1(G_i; \mathbb{F}_p) \geq [G : G_i]^{(1/(m+p-1))-\epsilon},$$

for all sufficiently large i. $\square$

8. Examples

In this section, we show that the $b_2 - b_1$ and b_2/b_1 conditions hold for several different classes of groups. The aim is to establish Theorems 1.4 and 1.5.

Proposition 8.1. *Let G be the fundamental group of a compact 3-manifold M that is either closed or orientable, and let p be a prime. Then $b_2(G_i; \mathbb{F}_p) \leq b_1(G_i; \mathbb{F}_p)$ for any finite index subgroup G_i of G.*

Proof. Any finite index subgroup G_i of G is the fundamental group of a compact 3-manifold $\tilde{M}$ that is either closed or orientable. By attaching 3-balls to $\tilde{M}$ if necessary, we may assume that it has no 2-sphere boundary components. Thus, its Euler characteristic is non-positive, by Poincaré duality. So,

$$0 \geq \chi(\tilde{M}) = b_0(\tilde{M}; \mathbb{F}_p) - b_1(\tilde{M}; \mathbb{F}_p) + b_2(\tilde{M}; \mathbb{F}_p) - b_3(\tilde{M}; \mathbb{F}_p)$$

$$\geq -b_1(\tilde{M}; \mathbb{F}_p) + b_2(\tilde{M}; \mathbb{F}_p).$$

The proposition is therefore a consequence of the following easy fact. $\square$

Lemma 8.2. *Let X be a topological space with fundamental group G. Then*

$$b_1(G; \mathbb{F}_p) = b_1(X; \mathbb{F}_p)$$

$$b_2(G; \mathbb{F}_p) \leq b_2(X; \mathbb{F}_p).$$

Proof. An Eilenberg-Maclane space $K(G, 1)$ can be constructed from X by attaching cells with dimensions at least 3. This does not affect b_1 and it does not increase b_2. $\square$

Any lattice in $\mathrm{PSL}(2, \mathbb{C})$ has a finite index subgroup that is the fundamental group of a compact orientable 3-manifold. So, Proposition 8.1, Theorem 1.3 and Theorem 1.1 deal with case (1) of Theorem 1.4. The following deals with case (2).

Proposition 8.3. *Let G and p be as in Proposition 8.1. Suppose that $b_1(G_i; \mathbb{F}_p) > 3$ for some finite index subgroup G_i of G. Then*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Proof. By Theorem 4.1 and Proposition 8.1,

$$b_1(\gamma_2(G_i); \mathbb{F}_p) \geq \binom{b_1(G_i; \mathbb{F}_p)}{2} + b_1(G_i; \mathbb{F}_p) - b_2(G_i; \mathbb{F}_p) \geq \binom{b_1(G_i; \mathbb{F}_p)}{2}.$$

This is strictly greater than $b_1(G_i; \mathbb{F}_p)$, since $b_1(G_i; \mathbb{F}_p) > 3$. Repeating for $\gamma_2(\gamma_2(G_i))$, and so on, we obtain a sequence $\{G_j\}$ of finite index subgroups of G, such that $b_1(G_j; \mathbb{F}_p)$ tends to infinity. $\square$

Note that the hypothesis that $b_1(G_i; \mathbb{F}_p) > 3$ is necessary here. For example, when G is the fundamental group of the 3-torus, then any finite index subgroup of G is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Note also that here, G has polynomial subgroup growth.

We now consider groups with deficiency at least 1. By definition, these are groups with a finite presentation $\langle X | R \rangle$ where $|X| - |R| = 1$. Note that, by the Reidemeister-Schreier process, any finite index subgroup of a group with deficiency at least 1 also has deficiency at least 1.

Proposition 8.4. *Let G be a group with deficiency at least 1. Then*

$$b_2(G; \mathbb{F}_p) - b_1(G; \mathbb{F}_p) \leq -1.$$

Proof. Let L be the 2-complex arising from the presentation $\langle X | R \rangle$ for G where $|X| - |R| = 1$. Then

$$b_2(L; \mathbb{F}_p) - b_1(L; \mathbb{F}_p) + 1 = \chi(L) = |R| - |X| + 1 = 0.$$

Now apply Lemma 8.2. $\square$

Corollary 8.5. *Any group with deficiency at least 1 satisfies the $b_2 - b_1$ condition.*

Proposition 8.6. *Let G be a group with deficiency at least 1. Suppose that $b_1(G_i; \mathbb{F}_p) > 2$ for some finite index subgroup G_i of G. Then*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Proof. This is essentially the same as the proof of Proposition 8.3, but we use Proposition 8.4 rather than Proposition 8.1. □

This deals with case (3) of Theorem 1.4. For case (4), note that any (finitely generated free)-by-cyclic group $F \rtimes \mathbb{Z}$ has deficiency at least 1. Thus, by Corollary 8.5, it satisfies the $b_2 - b_1$ condition. We also have the following, which completes the analysis of case (4) of Theorem 1.4.

Proposition 8.7. *Let G be a (finitely generated free non-abelian)-by-cyclic group. Then, for any prime p,*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Proof. The group G is a semi-direct product $F \rtimes_\phi \mathbb{Z}$, determined by an automorphism ϕ of the finitely generated free non-abelian group F. Let K be any finite index characteristic subgroup of F. This is preserved by ϕ. It is clear that $K \rtimes_{\phi|K} \mathbb{Z}$ is a finite index subgroup of $F \rtimes_\phi \mathbb{Z}$. In this way, we may assume that F has arbitrarily large rank. Now, ϕ induces an automorphism $\phi_*: H_1(F; \mathbb{F}_p) \to H_1(F; \mathbb{F}_p)$. Since $H_1(F; \mathbb{F}_p)$ has finite order, ϕ_*^n is the identity for some positive integer n. The kernel of the map $F \rtimes_\phi \mathbb{Z} \to \mathbb{Z} \to (\mathbb{Z}/n\mathbb{Z})$ is isomorphic to $F \rtimes_{\phi^n} \mathbb{Z}$. Note that $b_1(F \rtimes_{\phi^n} \mathbb{Z}; \mathbb{F}_p) = b_1(F; \mathbb{F}_p) + 1$. Since $b_1(F; \mathbb{F}_p)$ can be assumed to be arbitrarily large, the proposition is proved. □

We end with a large class of examples of b_2/b_1 groups. These give case (3) of Theorem 1.5.

Proposition 8.8. *Let G be the fundamental group of a closed 4-manifold M with non-positive Euler characteristic, and let p be a prime. Suppose, in addition, that if p is odd, then M is orientable. Then G satisfies the b_2/b_1 condition.*

Proof. Corresponding to any finite index subgroup G_i of G, there is a finite-sheeted covering space $\tilde{M}$ of M. This also has non-positive Euler characteristic. When M is orientable, so is $\tilde{M}$. Now, Poincaré duality applied to $\tilde{M}$ gives that

$$0 \geq \chi(\tilde{M}) = b_0(\tilde{M}; \mathbb{F}_p) - b_1(\tilde{M}; \mathbb{F}_p) + b_2(\tilde{M}; \mathbb{F}_p) - b_3(\tilde{M}; \mathbb{F}_p) + b_4(\tilde{M}; \mathbb{F}_p)$$
$$= 2 - 2b_1(\tilde{M}; \mathbb{F}_p) + b_2(\tilde{M}; \mathbb{F}_p).$$

Thus, by Lemma 8.2,

$$b_2(G_i; \mathbb{F}_p) \leq b_2(\tilde{M}; \mathbb{F}_p) \leq 2b_1(G_i; \mathbb{F}_p) - 2.$$

□

Proposition 8.9. *Let G and p be as in Proposition 8.8. Suppose that $b_1(G_i; \mathbb{F}_p) > 4$ for some finite index subgroup G_i of G. Then*

$$\sup\{b_1(G_i; \mathbb{F}_p) : G_i \text{ is a finite index subgroup of } G\} = \infty.$$

Proof. This follows the same lines as the proof of Proposition 8.3, using the inequality

$$b_2(G_i; \mathbb{F}_p) \leq 2b_1(G_i; \mathbb{F}_p) - 2$$

that was established in the proof of Proposition 8.8. □

Case (3) of Theorem 1.5 is proved by applying Propositions 8.8 and 8.9 and Theorem 1.2. When M is orientable or $p = 2$, the result follows immediately. When M is non-orientable and $p > 2$, we must first pass to the orientable double cover $\tilde{M}$ of M. Let K_i be $\pi_1(\tilde{M}) \cap G_i$. Then

$$b_1(K_i; \mathbb{F}_p) \geq b_1(G_i; \mathbb{F}_p) - b_1(G_i/K_i; \mathbb{F}_p) = b_1(G_i; \mathbb{F}_p) > 4.$$

Thus, $\pi_1(\tilde{M})$ satisfies the hypotheses of Propositions 8.8 and 8.9. Hence, Propositions 8.8 and 8.9 and Theorem 1.2 establish the theorem in this case.

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