# Surface subgroups of arithmetic Kleinian groups

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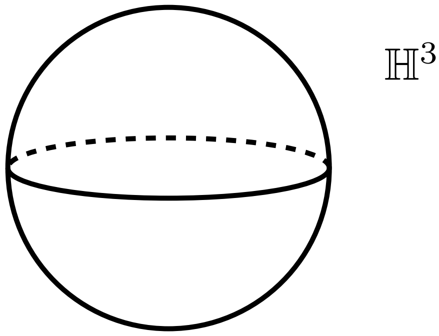
## Kleinian groups

A Kleinian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

$\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$

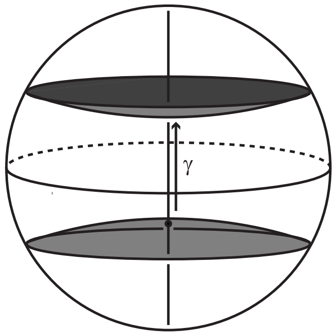



$\Gamma$ is a lattice if $\mathbb{H}^3/\Gamma$ has finite volume.

Given a Kleinian group $\Gamma$ acting freely on $\mathbb{H}^3$, the quotient space $\mathbb{H}^3/\Gamma$ is hyperbolic 3-manifold.

## Example

Suppose that $\Gamma = \langle \gamma \rangle$, where $\gamma$ is a loxodromic element, translating along a geodesic.



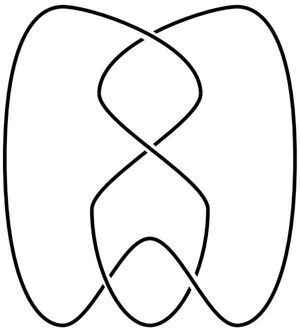

Then $\mathbb{H}^3/\Gamma$ is an open solid torus ($\cong \mathbb{R}^2 \times S^1$).

# Another example

Let $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[\omega])$, where $\omega = e^{2\pi i/3}$.

This is an example of an arithmetic Kleinian group.

In fact, $\Gamma$ has an index 12 subgroup $\tilde{\Gamma}$ s.t. $\mathbb{H}^3/\tilde{\Gamma}$ is homeomorphic to the complement of the figure-eight knot.



A Bianchi group is $\mathrm{PSL}(2, R)$, where $R$ is the ring of integers in $\mathbb{Q}(\sqrt{-d})$.

# Geometrisation

An orientable 3-manifold $M$ has a hyperbolic structure if its interior is $\mathbb{H}^3/\Gamma$ for some Kleinian group $\Gamma$ acting freely on $\mathbb{H}^3$.

Theorem: [Thurston, Perelman] A compact orientable 3-manifold $M$ admits a hyperbolic structure iff

- it is irreducible ie any 2-sphere bounds a 3-ball;
- it is atoroidal ie any embedded torus in $M$ lies in a 3-ball, bounds a solid torus or is parallel to a component of $\partial M$;
- it is not Seifert fibred ie it does not admit a foliation by circles.

So, ‘almost every’ 3-manifold is hyperbolic.

## Surfaces in 3-manifolds

Throughout: all surfaces are closed orientable and have positive genus.

Let $S$ be a surface.

Let $M$ be a closed orientable 3-manifold.

A map $i\colon S \to M$ is $\pi_1$-injective if $i_*\colon \pi_1(S) \to \pi_1(M)$ is an injection.

$M$ is Haken if it is irreducible and contains an embedded $\pi_1$-injective surface.

Theorem: [Waldhausen] If two closed Haken 3-manifolds have isomorphic $\pi_1$, they are homeomorphic.

Unfortunately, many 3-manifolds are non-Haken.

# Surface subgroups

Surface subgroup conjecture: Every finitely generated Kleinian group is either finite, virtually free or contains the fundamental group of a surface.

Equivalently: Every closed hyperbolic 3-manifold contains an immersed $\pi_1$-injective surface.

Stronger conjectures:

1. [Waldhausen, Thurston] Every closed hyperbolic 3-manifold is finitely covered by a Haken 3-manifold.
2. [Waldhausen, Thurston] Every closed hyperbolic 3-manifold is finitely covered by a 3-manifold with positive first Betti number.
3. [Gromov] Every infinite word-hyperbolic group is either virtually free or contains a surface subgroup.

## Progress on these problems

[Grunewald-Schwermer] The Bianchi groups have finite index subgroups with arbitrarily large cuspidal cohomology (and hence a closed embedded $\pi_1$-injective surface).

[Labesse-Schwermer] Any arithmetic Kleinian group for which the trace field has a subfield of index 2 has finite index subgroups with arbitrarily large first Betti number.

There have been many other contributors, particularly Clozel and Lubotzky.

But we still don't know whether every arithmetic 3-manifold satisfies Waldhausen's and Thurston's conjectures!

Theorem: [L] Every arithmetic hyperbolic 3-manifold contains an immersed $\pi_1$-injective surface.

## Main theorem

Theorem: [L] Every arithmetic hyperbolic 3-manifold contains an immersed $\pi_1$-injective surface.

Ingredients of the proof:

- 3-orbifolds
- Golod-Shafarevich inequality
- Perelman's solution to the geometrisation conjecture
- Cheeger constants
- The first eigenvalue of the Laplacian
- The critical exponent of Kleinian groups
- Some classical 3-manifold theory
- A little arithmetic machinery

## Contributors

M. Lackenby, *Heegaard splittings, the virtually Haken conjecture and Property $(\tau)$*

M. Lackenby, *Covering spaces of 3-orbifolds*

M. Lackenby, D. Long, A. Reid, *Covering spaces of arithmetic 3-orbifolds*

L. Bowen, *Free groups in lattices*

M. Lackenby, D. Long, A. Reid, *LERF and the Lubotzky-Sarnak conjecture*

M. Lackenby, *Surface subgroups of Kleinian groups with torsion*

# Hyperbolic 3-orbifolds

If $\Gamma$ is a discrete subgroup of $\text{Isom}^+(\mathbb{H}^3)$, not necessarily acting freely, then $\mathbb{H}^3/\Gamma$ is an orientable hyperbolic 3-orbifold $O$.

One keeps track not just of the underlying space $|O|$ but also the isotropy data.

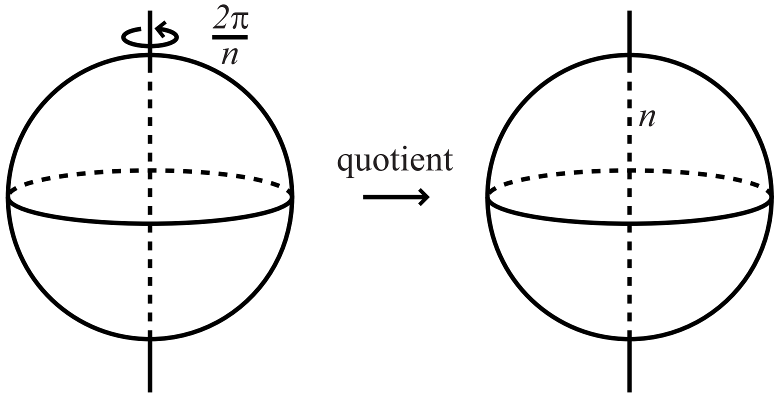
ie, for $x \in O$, consider $\tilde{x} \in$ inverse image of $x$, and define the local group of $x$ to be $\text{Stab}_\Gamma(\tilde{x})$.

The singular locus is the set of points in $O$ with non-trivial local group.

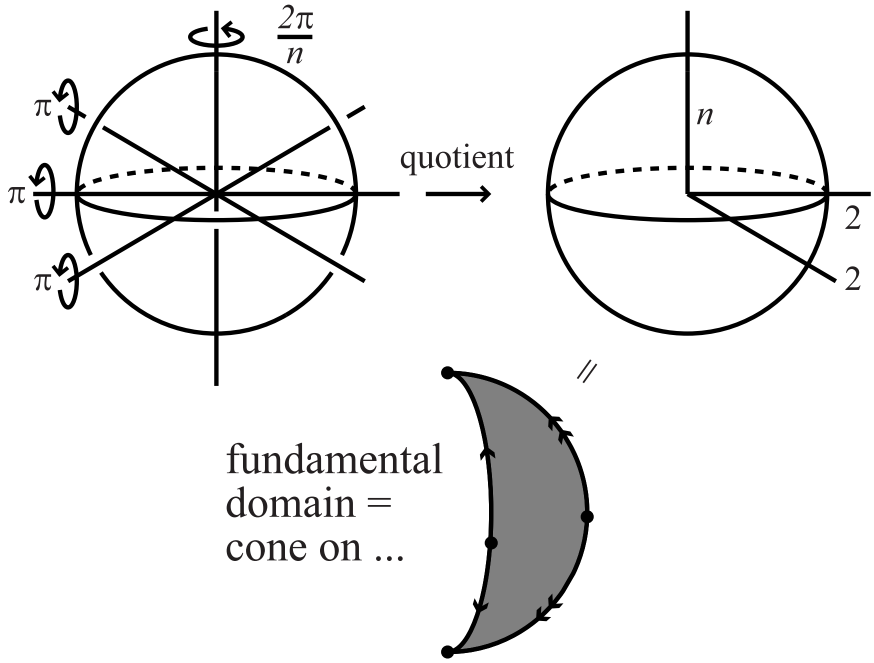
The local group of any $x$ is a finite subgroup of $SO(3)$ ie:

- cyclic,
- dihedral (including $\mathbb{Z}_2 \times \mathbb{Z}_2$)
- $A_4$, $S_4$, $A_5$.

## Cyclic local group:



Dihedral local group:




In fact, $|O|$ is always a 3-manifold and $\text{sing}(O)$ is always a collection of simple closed curves and trivalent graphs.

## Surface subgroups again

Theorem: [L] Any finitely generated Kleinian group $\Gamma$ containing a finite non-cyclic subgroup is either finite, virtually free or contains a surface subgroup.

The main case is when $\Gamma$ is co-compact.

Equivalently in this case: Any closed hyperbolic 3-orbifold that contains a singular vertex admits an immersed $\pi_1$-injective surface.

# Commensurable groups

Two groups $\Gamma_1$ and $\Gamma_2$ are commensurable if there are finite index subgroups $\Gamma_1' \leq \Gamma_1$ and $\Gamma_2' \leq \Gamma_2$ such that $\Gamma_1' \cong \Gamma_2'$.

$\Gamma_1$ contains a surface subgroup iff $\Gamma_2$ does.

Theorem: [L-Long-Reid] Any arithmetic Kleinian group is commensurable with one that contains $\mathbb{Z}_2 \times \mathbb{Z}_2$.

## Fundamental group and covering spaces of orbifolds

Let $O = \mathbb{H}^3/\Gamma$.

Its fundamental group $\pi_1(O)$ is $\Gamma$.

This is not the same as $\pi_1(|O|)$.

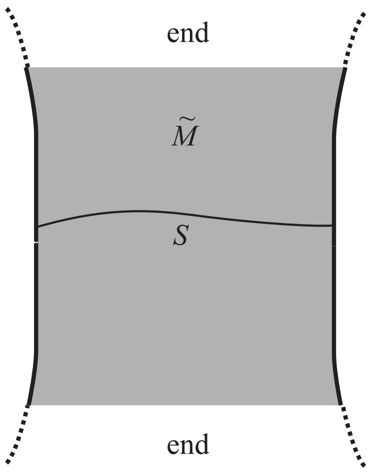
If $\Gamma'$ is a subgroup of $\Gamma$, then

$$\mathbb{H}^3/\Gamma' \to \mathbb{H}^3/\Gamma$$

is a covering space.

# End of the proof

Aim: To find a manifold cover $\tilde{M}$ of $O$ with at least two ends.



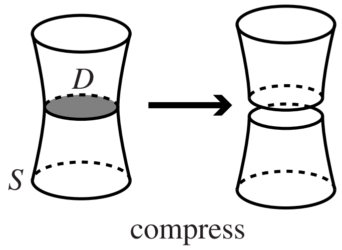

Then apply:

Lemma: Let $M$ be an orientable hyperbolic 3-manifold with at least 2 ends. Then $\pi_1(M)$ contains a surface subgroup.

Proof:

Let $S$ be a closed orientable surface separating two ends of $M$.

Compress $S$ as much as possible to $\overline{S}$.




Some component of $\overline{S}$ still separates two ends of $M$.

So, it's not a sphere, because $M$ is irreducible.

By the loop theorem, it's $\pi_1$-injective.

## Start of the proof

Let's suppose that $\Gamma$ contains $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $O = \mathbb{H}^3/\Gamma$.

Theorem: $O$ has a finite cover $\tilde{O}$ such that

(i) every arc and simple closed curve of $\mathrm{sing}(\tilde{O})$ has order 2;

(ii) $\tilde{O}$ has at least one singular vertex;

(iii) $\pi_1(|\tilde{O}|)$ is infinite.

The proof uses the Golod-Shafarevich inequality:

If $G$ is a group with finite presentation $\langle X|R\rangle$ and

$$\frac{d_p(G)^2}{4} > d_p(G) - |X| + |R|,$$

where $p$ is a prime and $d_p(G) = \dim H_1(G;\mathbb{F}_p)$, then $G$ is infinite.

## Hyperbolic underlying space

Let $M = |\tilde{O}|$.

Well known theorem: If $M$ is a closed orientable 3-manifold with infinite $\pi_1$, then either

(i) $M$ has a finite cover with $b_1 > 0$, or

(ii) $M$ is hyperbolic.

The proof uses Perelman's solution to the geometrisation conjecture.

So, wlog $|\tilde{O}|$ is hyperbolic.

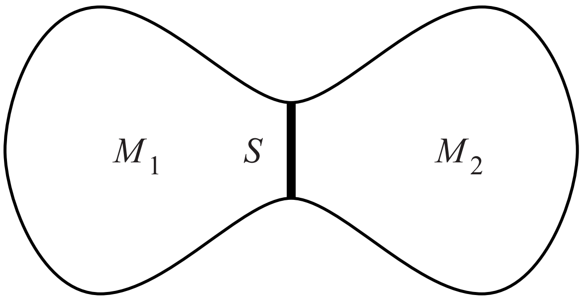
## Cheeger constants

Let $M$ be a complete Riemannian manifold.

If $M$ has finite volume, then its Cheeger constant $h(M)$ is

$$\inf_S \left\{ \frac{\mathrm{Area}(S)}{\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)\}} \right\},$$

as $S$ ranges over all embedded codimension 1 submanifolds that separate $M$ into $M_1$ and $M_2$.



If $M$ has infinite volume, then its Cheeger constant $h(M)$ is

$$\inf_S \left\{ \frac{\mathrm{Area}(S)}{\mathrm{Vol}(M_1)} \right\},$$

as $S$ ranges over all embedded codimension 1 submanifolds that bound a compact submanifold $M_1$.

Theorem: [L-Long-Reid] Let $M$ be a closed hyperbolic 3-manifold. Then $M$ has infinite-sheeted covers $M_i$ such that $h(M_i) \to 0$.

## Cheeger constants

Theorem: [L-Long-Reid] Let $M$ be a closed hyperbolic 3-manifold. Then $M$ has infinite-sheeted covers $M_i$ such that $h(M_i) \to 0$.

This is a consequence of:

Theorem: [Bowen] $\Gamma = \pi_1(M)$ has a sequence of finitely generated free subgroups $\Gamma_i$ such that $\delta(\Gamma_i) \to 2$.

Here $\delta(\Gamma_i)$ = the 'critical exponent' of $\Gamma_i$

Theorem: [Sullivan]

$$\lambda_1(\Gamma_i \backslash \mathbb{H}^3) = \begin{cases} \delta(\Gamma_i)(2 - \delta(\Gamma_i)) & \text{if } \delta(\Gamma_i) \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Here $\lambda(\Gamma_i \backslash \mathbb{H}^3)$ = the first eigenvalue of the Laplacian of $\Gamma_i \backslash \mathbb{H}^3$.

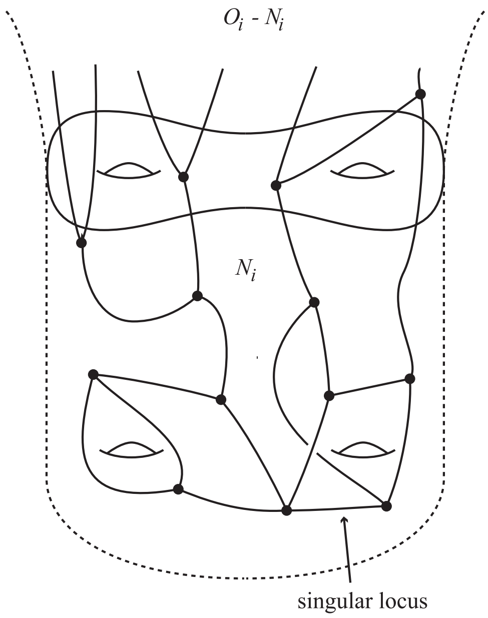
Theorem: [Cheeger] For any complete Riemannian manifold $M_i$, $\lambda_1(M_i) \geq h(M_i)^2/4$.

Apply this theorem to $M = |\tilde{O}|$.

We get a sequence of covering spaces $M_i$.

There are induced covers $O_i \to \tilde{O}$, where $|O_i| = M_i$.

Let $|N_i|$ be a compact submanifold of $|O_i|$ such that $\frac{\text{Area}(|\partial N_i|)}{\text{Vol}(|N_i|)} \to 0$.



# The homology of orbifolds

Given any orbifold $O$ and prime $p$, one can define its first homology $H_1(O;\mathbb{F}_p) = H_1(\pi_1(O);\mathbb{F}_p)$.

Lemma: If $N$ is a compact orientable 3-orbifold, and each arc and circle of $\mathrm{sing}(N)$ has order 2, then

$$\dim\, H_1(N;\mathbb{F}_2) \geq b_1(\mathrm{sing}(N)).$$

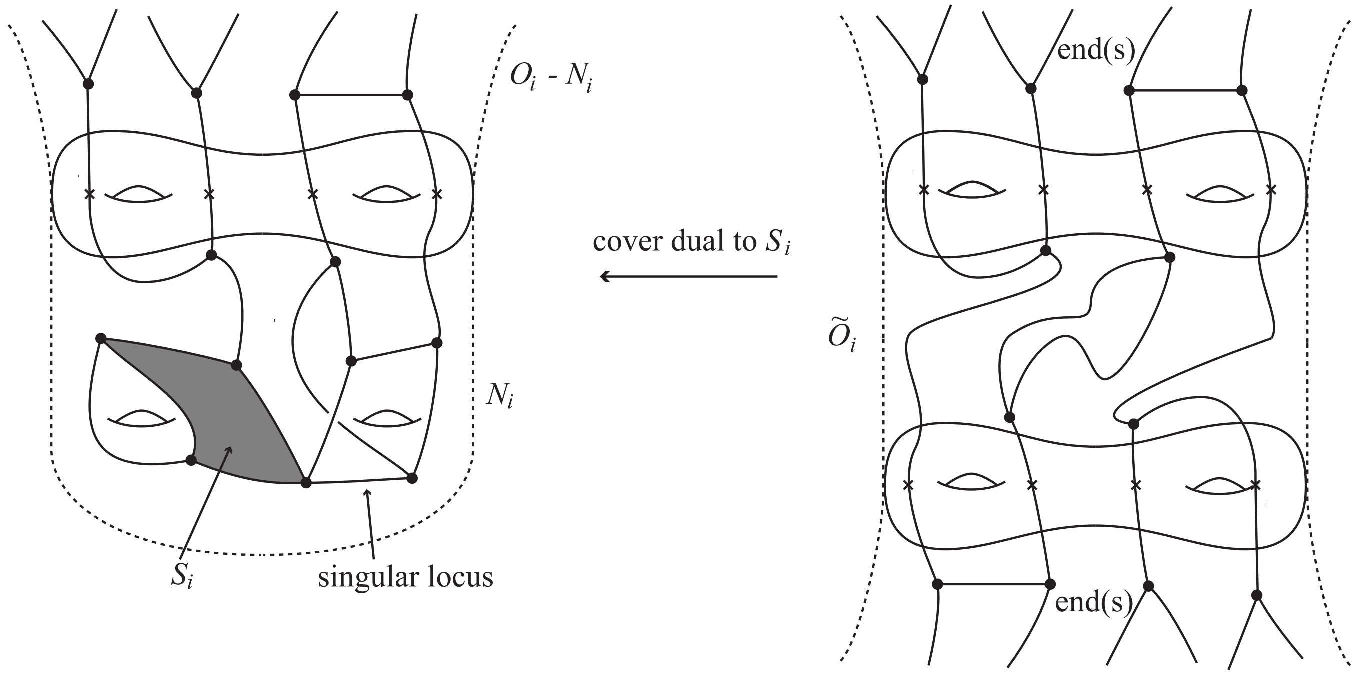
So, for $i >> 0$,

$$\dim\, H_1(N_i;\mathbb{F}_2) > \dim\, H_1(\partial N_i;\mathbb{F}_2).$$

So, $\ker\, H^1(N_i;\mathbb{F}_2) \to H^1(\partial N_i;\mathbb{F}_2)$ is non-trivial.

Let $S_i$ be a surface properly embedded in $N_i - \mathrm{sing}(N_i)$ dual to a non-trivial element of this kernel, and that is disjoint from $\partial N_i$.

Let $\tilde{O}_i$ be the 2-fold cover of $O$ dual to $S_i$.




This has at least two ends.