Certifying the hyperbolicity of knots and links

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Hyperbolic structures

The Geometrisation Conjecture was very difficult to prove.

But in practice, it is remarkably easy to find a hyperbolic structure on a 3-manifold.

Why?!

Finding hyperbolic structures

Question: What is the computational complexity of determining whether a compact 3-manifold is hyperbolic and, if it is hyperbolic, how hard is it to find the hyperbolic structure?

Previous work

[Manning, Casson] One can determine whether a closed 3-manifold M is hyperbolic and find its hyperbolic structure, as long as $\pi_1(M)$ has solvable word problem.

[Kuperberg] Gave an algorithm to determine whether M is hyperbolic and to find its hyperbolic structure that runs in time that is elementary recursive ie at most

$$2^{\underbrace{2^{2^{\cdots^{t}}}}_{\text{fixed height}}}$$

where t is the number of tetrahedra in a given triangulation of M.

[Scull] The running time is at most

$$2^{2^{t^{O(t)}}}$$

Ruling out hyperbolic structures

If a closed orientable 3-manifold is not hyperbolic, then it is one of:

- Seifert fibred
- reducible
- toroidal.

Theorem: [Ivanov, Schleimer] S^3 recognition is in NP.

Theorem: [Lackenby-Schleimer] Recognition of elliptic 3-manifolds is in NP.

Theorem: [Jackson] Recognition of Seifert fibre spaces with non-empty boundary is in NP.

Closed Seifert fibre spaces remain a problem, particularly the small ones.

Hyperbolic structures on link complements

Problem: (LINK HYPERBOLICITY) Given a diagram of a link L with c crossings, is L hyperbolic?

Theorem: [Haraway-Hoffman, Badwin-Sivek] This problem is in co-NP.

Theorem: [Baroni, Lackenby] This problem is in NP.

Showing that a link is not hyperbolic

Theorem: [Thurston] Let L be a link in the 3-sphere. Then one of the following holds:

- L is the unknot;
- L is split;
- there is an essential torus in $S^3 - L$;
- there is an essential annulus in $S^3 - L$;
- L is hyperbolic.

Haraway-Hoffman used the following fact:

Theorem: [Lackenby] Deciding whether a compact orientable 3-manifold has incompressible boundary is in NP.

Dividing into two cases

From now onwards, we'll focus on:

Theorem: [Baroni, Lackenby] Link hyperbolicity is in NP.

Given a hyperbolic link L, the proof of its hyperbolicity divides into two cases:

- L is a fibred knot [Baroni]
- L is not fibred or has more than one component [Lackenby].

We will start by examining the fibred case.

The Nielsen-Thurston type of a surface automorphism

Let S be an orientable surface of finite type and $\chi(S) < 0$, and let $\phi \colon S \to S$ be a homeomorphism. Then exactly one of the following holds:

1. ϕ is periodic;
2. ϕ is reducible;
3. ϕ is pseudo-anosov ($\Leftrightarrow$ $(S \times I)/\phi$ is hyperbolic)

Suppose that we are given ϕ as a word w in 'standard generators' in the mapping class group of S.

Theorem: [Bell-Webb] For a fixed surface S (with at least one puncture), there is an algorithm to determine the Nielsen-Thurston type of ϕ that runs in polynomial time in the length of w.

Theorem: [Baroni] There is an algorithm to determine the Nielsen-Thurston type of ϕ that runs in polynomial time in the length of w and in $|\chi(S)|$.

Computing distance in the curve complex

This relies on:

Theorem: [Bell-Webb] For a fixed compact orientable triangulated surface S (with non-empty boundary), there is an algorithm to determine distance in the curve complex between two curves C_1 and C_2. This runs in polynomial time as a function of the $\log(\text{weight}(C_1))$ and $\log(\text{weight}(C_2))$. Indeed, the algorithm provides a tight geodesic between C_1 and C_2.

Theorem: [Baroni] There is an algorithm to determine the distance in the curve complex between two curves C_1 and C_2 in a compact orientable surface S with a triangulation $\mathcal{T}$, up to a bounded $(\text{poly}(\chi(S)))$ additive and multiplicative error. This runs in polynomial time as a function of the number of triangles of $\mathcal{T}$, $\log(\text{weight}(C_1))$ and $\log(\text{weight}(C_2))$. Indeed, the algorithm provides a quasi-geodesic between C_1 and C_2.

Deciding whether a mapping class is pseudo-anosov

The stable translation length $\ell(\phi)$ of $\phi\colon S \to S$ is $\lim_{N\to\infty} d(C, \phi^N(C))/N$ for any essential curve C. This is positive iff ϕ is pseudo-anosov.

To decide whether ϕ is pseudo-anosov:

- Bell and Webb pick an essential curve C that is short with respect to the given triangulation of S.
- They compute $\phi^N(C)$ for some 'large' N.
- They find a geodesic in the curve complex of S joining C and $\phi^N(C)$. Let C' be its midpoint.
- Then the stable translation length of ϕ is $d(C', \phi^N(C'))/N$ rounded to a suitable fraction.

Baroni uses the same argument, but with coarse distances and a quasi-geodesic.

Certifying fibred hyperbolic knot

We are given a diagram D of a knot L.

From this we build a triangulation for the exterior of L with $O(c(D))$ tetrahedra.

The fibre surface S can be arranged to be a fundamental normal surface, and hence have weight at most $2^{O(c(D))}$.

This is part of the certificate.

Then we can also certify that the exterior of S is a copy of $S \times [0, 1]$.

It seems hard to 'write down' the monodromy ϕ.

But one can find a 'short' curve C in S and then compute $\phi^N(C)$ for $N = \text{poly}(c(D))$.

This is enough to determine whether $\ell(\phi) > 0$.

The non-fibred case

This uses hierarchies.

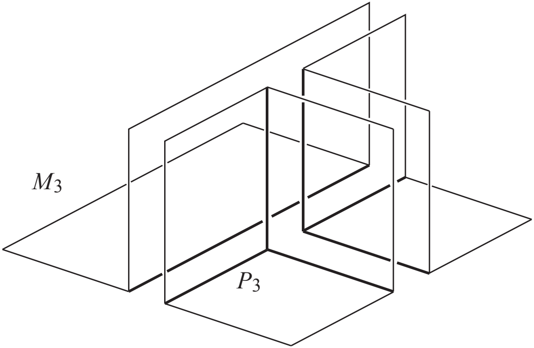
A hierarchy is a sequence of compact orientable 3-manifolds $M = M_1, \dots, M_{\ell+1}$ and orientable surfaces $S_1, \dots, S_\ell$ such that:

- each S_i is properly embedded in M_i;
- each $M_{i+1} = M_i \backslash\backslash S_i$;
- $M_{\ell+1}$ is a collection of 3-balls.

Boundary patterns

A boundary pattern for a 3-manifold M is subset P of ∂M that is a disjoint union of simple closed curves and trivalent graphs.

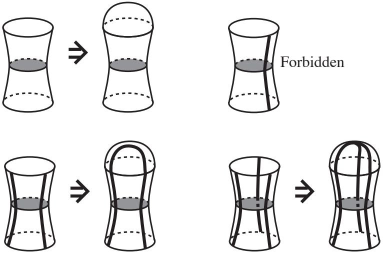
The manifolds in a hierarchy naturally inherit a boundary pattern.



Essential boundary patterns

A boundary pattern P is essential if, for any properly embedded disc D that intersects P at most three times, ∂D bounds a disc D' in ∂M that intersects P in one of the following:

- the empty set,
- an arc,
- a tripod.



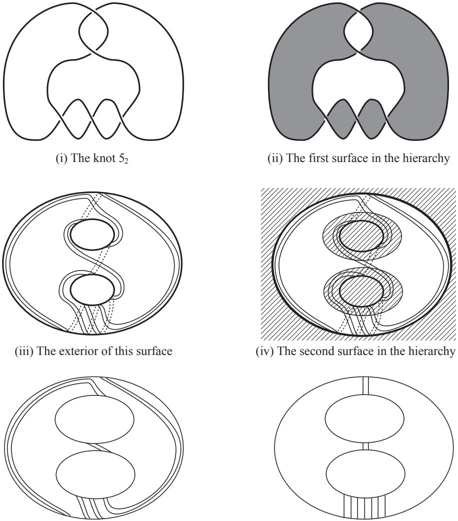
Essential hierarchies

A hierarchy $M = M_1, \ldots, M_{\ell+1}$ is essential if the final manifold $M_{\ell+1}$ inherits an essential boundary pattern.

Theorem: [Waldhausen, Johansson] Let M be a compact orientable 3-manifold with non-empty boundary and empty boundary pattern. Then the following are equivalent:

- ∂M is incompressible and M is irreducible;
- M has an essential hierarchy.

An example: the knot 5_2



(i) The knot 5_2

(ii) The first surface in the hierarchy

(iii) The exterior of this surface

(iv) The second surface in the hierarchy

(v) The pattern of one of the balls

(vi) A simplified copy of the pattern

So, 5_2 is non-trivial.

Low genus hierarchies

Theorem: Let M be a compact orientable irreducible 3-manifold with an essential boundary pattern P, and let $\mathcal{H}$ be a handle structure for (M, P). Then (M, P) admits a hierarchy

$$(M, P) = (M_1, P_1) \xrightarrow{S_1} (M_2, P_2) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_{n+1}, P_{n+1})$$

and each (M_i, P_i) has a handle structure $\mathcal{H}_i$ such that the following hold:

1. each S_i is normal and fundamental in $\mathcal{H}_i$;
2. complexity$(\mathcal{H}_{i+2}) <$ complexity$(\mathcal{H}_i)$.

Moreover: When M is the exterior of a link L and $P = \emptyset$, and $\mathcal{H}$ is the handle structure arising from a diagram D, then $|\chi(S_i)|$ and $|S_i \cap P_i|$ are both $O(c(D)^2)$, where $c(D)$ is the crossing number of D.

Determining the gluing maps

Let

$$(M_i, P_i) \xrightarrow{S_i} (M_{i+1}, P_{i+1})$$

be a decomposition in this hierarchy.

There are two copies of S_i in ∂M_i, denoted S_i^- and S_i^+.

Let $\phi\colon S_i^- \to S_i^+$ be the map that glues them up.

When S_i is not a fibre, there is an algorithm to 'write down' ϕ.

This expresses ϕ as a composition of Pachner moves between certain triangulations of S_i^- and S_i^+.

The JSJ surfaces for a manifold with pattern

There is an analogue of the JSJ decomposition for a manifold M with essential pattern P.

The decomposing surfaces are tori, annuli (disjoint from P) and squares (ie discs intersecting P four times).

Theorem: The JSJ surfaces decompose (M, P) into the following pieces:

- 'simple' manifolds (ie they contain no essential tori, annuli disjoint from P, or squares);
- Seifert fibre manifolds,
- patterned I-bundles.

The I-bundles $\mathcal{B}$ determine a transfer map $\tau \colon \partial_h \mathcal{B} \to \partial_h \mathcal{B}$.

Determining the JSJ by induction

Let $(M, P) \xrightarrow{S} (M', P')$ be a decomposition with gluing map ϕ.

Any essential torus/annulus/square in (M, P) is cut up by S into a torus/annuli/squares in (M', P').

So, the I-bundle pieces of the JSJ for (M, P) are obtained by patching together the I-bundle pieces $\mathcal{B}'$ of the JSJ for (M', P').

But $\partial_h \mathcal{B}'$ might not patch together precisely under ϕ.

So one shrinks $\mathcal{B}'$, forming a smaller I-bundle $\mathcal{B}'_2$.

Again, this might not patch together correctly under ϕ.

So one shrinks $\mathcal{B}'_2$, forming a smaller I-bundle $\mathcal{B}'_3$, etc.

This process stabilises after $O(|\chi(S)|)$ steps.

So in this way we end with the JSJ for (M, P).

(In fact, one keeps track of just its transfer map τ.)

Note that this does not work when S is a fibre.

Completing the proof

We are given a diagram D for the non-fibred hyperbolic link L.

The hierarchy for the exterior of L is given to us as part of the certificate.

This shows that L is not the unknot and not split.

We compute the JSJ of the manifolds, working backwards along the hierarchy.

So if L is hyperbolic and non-fibred, we may certify that $S^3 - L$ has no JSJ tori and is not Seifert fibred.

In other words, we have a polynomial time certificate that L is hyperbolic.

Further questions

- What about general Haken 3-manifolds?
- What about general 3-manifolds?
- Finding the hyperbolic structure: is this in FNP?
- Do hyperbolic quantities (eg volume) relate to the combinatorics of an essential hierarchy?