Certifying the hyperbolicity of knots and links

Marc Lackenby

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Why?!

Finding hyperbolic structures

Question: What is the computational complexity of determining whether a compact 3-manifold is hyperbolic and, if it is hyperbolic, how hard is it to find the hyperbolic structure?

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[Scull] The running time is at most

$$2^{2^{t^{O(t)}}}$$

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Closed Seifert fibre spaces remain a problem, particularly the small ones.

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Theorem: [Baroni, Lackenby] This problem is in NP.

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Theorem: [Lackenby] Deciding whether a compact orientable 3-manifold has incompressible boundary is in NP.

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We will start by examining the fibred case.

The Nielsen-Thurston type of a surface automorphism

Let S be an orientable surface of finite type and $\chi(S) < 0$, and let $\phi\colon S \to S$ be a homeomorphism. Then exactly one of the following holds:

1. ϕ is periodic;
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Theorem: [Baroni] There is an algorithm to determine the Nielsen-Thurston type of ϕ that runs in polynomial time in the length of w and in $|\chi(S)|$.

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Theorem: [Baroni] There is an algorithm to determine the distance in the curve complex between two curves C_1 and C_2 in a compact orientable surface S with a triangulation $\mathcal{T}$, up to a bounded $(\text{poly}(\chi(S)))$ additive and multiplicative error. This runs in polynomial time as a function of the number of triangles of $\mathcal{T}$, $\log(\text{weight}(C_1))$ and $\log(\text{weight}(C_2))$. Indeed, the algorithm provides a quasi-geodesic between C_1 and C_2.

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Baroni uses the same argument, but with coarse distances and a quasi-geodesic.

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This is enough to determine whether $\ell(\phi) > 0$.

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A hierarchy is a sequence of compact orientable 3-manifolds $M = M_1, \dots, M_{\ell+1}$ and orientable surfaces $S_1, \dots, S_\ell$ such that:

- each S_i is properly embedded in M_i;
- each $M_{i+1} = M_i \backslash\backslash S_i$;
- $M_{\ell+1}$ is a collection of 3-balls.

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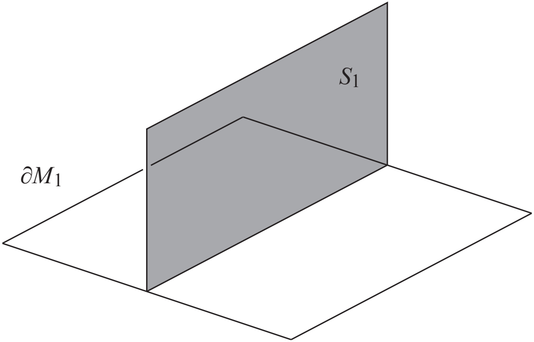

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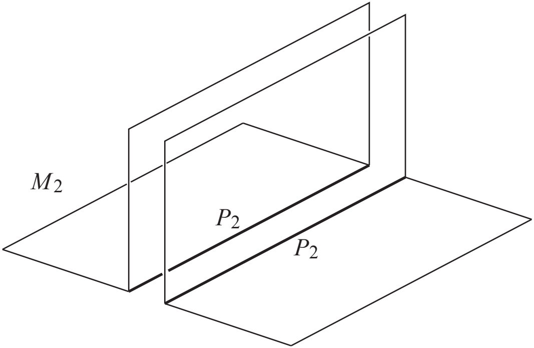

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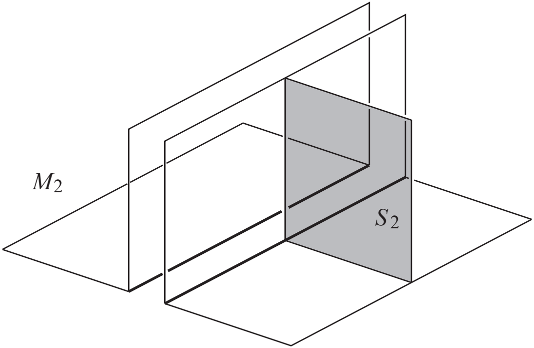

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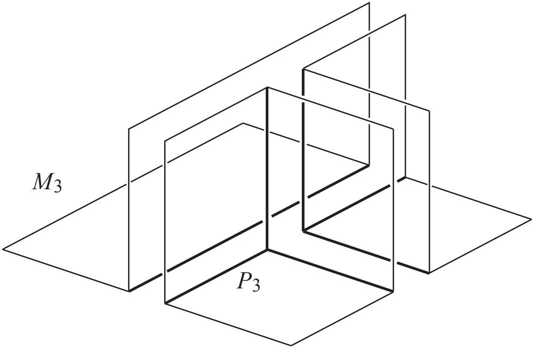
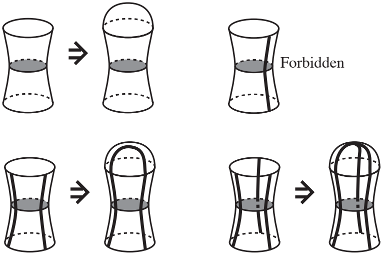


Essential boundary patterns

A boundary pattern P is essential if, for any properly embedded disc D that intersects P at most three times, ∂D bounds a disc D' in ∂M that intersects P in one of the following:

- the empty set,
- an arc,
- a tripod.



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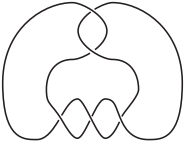
- M has an essential hierarchy.

An example: the knot 5_2



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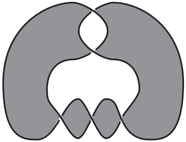
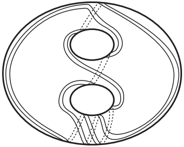
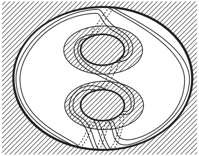


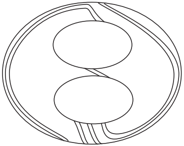
(ii) The first surface in the hierarchy



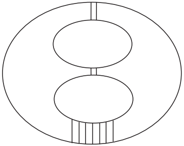
(iii) The exterior of this surface



(iv) The second surface in the hierarchy



(v) The pattern of one of the balls



(vi) A simplified copy of the pattern

So, 5_2 is non-trivial.

Low genus hierarchies

Theorem: Let M be a compact orientable irreducible 3-manifold with an essential boundary pattern P, and let $\mathcal{H}$ be a handle structure for (M, P). Then (M, P) admits a hierarchy

$$(M, P) = (M_1, P_1) \xrightarrow{S_1} (M_2, P_2) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_{n+1}, P_{n+1})$$

and each (M_i, P_i) has a handle structure $\mathcal{H}_i$ such that the following hold:

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2. complexity$(\mathcal{H}_{i+2}) <$ complexity$(\mathcal{H}_i)$.

Moreover: When M is the exterior of a link L and $P = \emptyset$, and $\mathcal{H}$ is the handle structure arising from a diagram D, then $|\chi(S_i)|$ and $|S_i \cap P_i|$ are both $O(c(D)^2)$, where $c(D)$ is the crossing number of D.

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This expresses ϕ as a composition of Pachner moves between certain triangulations of S_i^- and S_i^+.

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Theorem: The JSJ surfaces decompose (M, P) into the following pieces:

- 'simple' manifolds (ie they contain no essential tori, annuli disjoint from P, or squares);
- Seifert fibre manifolds,
- patterned I-bundles.

The JSJ surfaces for a manifold with pattern

There is an analogue of the JSJ decomposition for a manifold M with essential pattern P.

The decomposing surfaces are tori, annuli (disjoint from P) and squares (ie discs intersecting P four times).

Theorem: The JSJ surfaces decompose (M, P) into the following pieces:

- 'simple' manifolds (ie they contain no essential tori, annuli disjoint from P, or squares);
- Seifert fibre manifolds,
- patterned I-bundles.

The I-bundles $\mathcal{B}$ determine a transfer map $\tau\colon \partial_h \mathcal{B} \to \partial_h \mathcal{B}$.

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Note that this does not work when S is a fibre.

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So if L is hyperbolic and non-fibred, we may certify that $S^3 - L$ has no JSJ tori and is not Seifert fibred.

In other words, we have a polynomial time certificate that L is hyperbolic.

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- Do hyperbolic quantities (eg volume) relate to the combinatorics of an essential hierarchy?