Topology & Groups

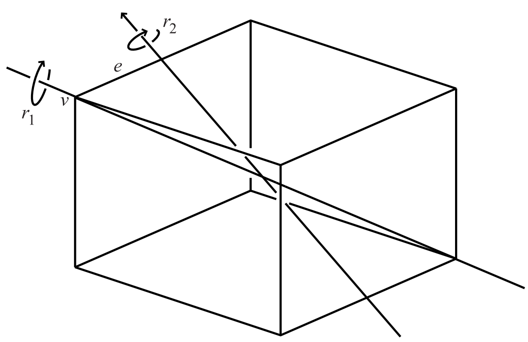
Michaelmas 2008

Question Sheet 1

1. For each of the following groups $G$ and generating sets $S$, draw the resulting Cayley graph:

   (i) $G = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $S = \{(1,0),(0,1)\}$;

   (ii) $G =$ the group of rotational symmetries of the cube, $S = \{r_1, r_2\}$, where $r_1$ is a rotation of order 3 about about an axis through a vertex $v$, and $r_2$ is a rotation of order 2 about an axis through an edge $e$ which has $v$ as an endpoint.




2. Let $\Gamma$ be the Cayley graph of a group $G$ with respect to a generating set $S$.

   (i) Show that the following is a metric on $G$: $d(g_1, g_2) =$ the shortest number of edges in a path in $\Gamma$ joining the vertex labelled $g_1$ to the vertex labelled $g_2$.

   (ii) Show that $d(g_1, g_2)$ equals the smallest non-negative integer $n$ such that

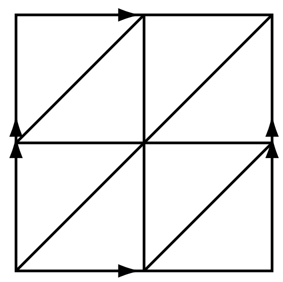
$$g_2 = g_1 s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n}$$

   where each $s_i \in S$ and $\epsilon_i \in \{-1, 1\}$.

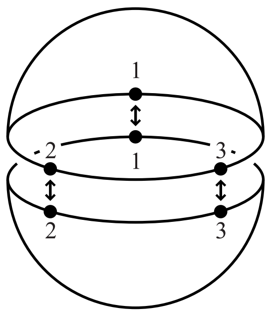
   (iii) Let $\mathrm{Isom}(G)$ be the group of isometries of $G$ with this metric. Prove that $G$ can be realised as a subgroup of $\mathrm{Isom}(G)$.

   (iv) Find an example where $G \subsetneqq \mathrm{Isom}(G)$.

3. (i) Explain why the following diagram is *not* a triangulation of the torus:



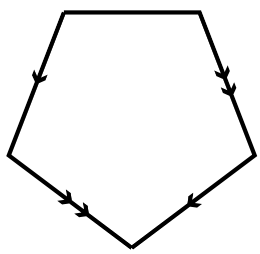
(ii) Explain why the following is *not* a triangulation of the 2-sphere:




4. Show that any finite simplicial complex may be realised a subspace of $\mathbb{R}^n$, for some $n$.

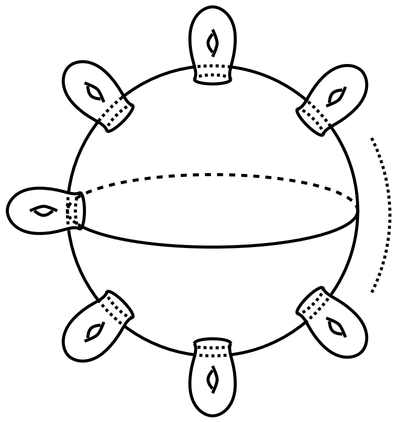
5. [*In this question, you are required to show that certain spaces are homeomorphic. Precise formulae are not required here. Instead, describe these homeomorphisms using pictures.*]

(i). Let $X$ be the torus with the interior of a small disc removed. Prove that $X$ can be constructed by identifying the sides of a pentagon as shown below.

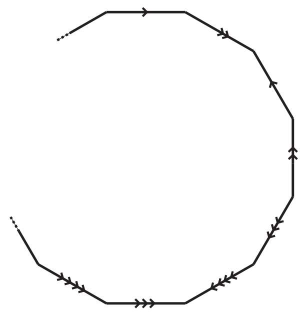


Deduce that $X$ can be given the structure of a cell complex, with a single 0-cell, three 1-cells, and one 2-cell.

(ii) Let $S$ be the closed surface with $g$ handles, for $g \in \mathbb{N}$, as shown below.



Show that $S$ can be constructed as follows. Start with a $4g$-sided polygon, and identify its sides in pairs, according the following recipe:



[Hint: divide the polygon up into $g$ pentagons and a $g$-sided polygon.]

Deduce that $S$ can be given the structure of a cell complex, with a single 0-cell, $2g$ 1-cells and a single 2-cell.