TOPOLOGY & GROUPS

MICHAELMAS 2008

QUESTION SHEET 2

Questions with an asterisk $*$ beside them are primarily aimed at 4th year students.

1. Let $\alpha\colon S^n \to S^n$ be the antipodal map (defined by $\alpha(x) = -x$). Prove that α is homotopic to the identity if n is odd.

2. For any two maps $f, g\colon X \to S^n$ such that $f(x) \neq -g(x)$ for all $x \in X$, show that $f \simeq g$.

3. Let X be a contractible space and let Y be any space. Show that

   (i) X is path-connected;

   (ii) $X \times Y$ is homotopy equivalent to Y;

   (iii) any two maps from Y to X are homotopic;

   (iv) if Y is path-connected, any two maps from X to Y are homotopic.

The *wedge* $X \vee Y$ of two spaces X and Y, containing basepoints x and y, is the space obtained from the disjoint union of X and Y by identifying x and y. Often, the resulting space is independent of the choice of basepoints, in which case there is no need to specify them. (See Definition V.26.)

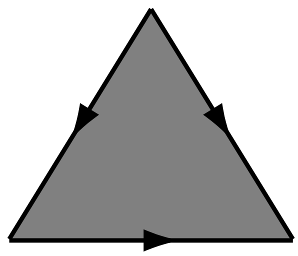
4. Prove that the following spaces are homotopy equivalent:

   (i) $S^1 \vee S^1$,

   (ii) the torus with one point removed,

   (iii) $\mathbb{R}^2$ minus two points.

$*$ 5. *(Harder)* For maps $f, g\colon S^{n-1} \to X$, let $X \cup_f D^n$ and $X \cup_g D^n$ be the spaces obtained by attaching n-cells to X along f and g respectively. Show that if f and g are homotopic maps $S^{n-1} \to X$, then $X \cup_f D^n$ and $X \cup_g D^n$ are homotopy equivalent. Deduce that the space (known as the 'dunce cap') obtained by identifying the three sides of a triangle, as shown overleaf, is contractible.



6. Let K and L be finite simplicial complexes. Prove that there are only countably many homotopy classes of maps $|K| \to |L|$.

7. Prove that any two maps $S^m \to S^n$, where $m < n$, are homotopic. [Hint: use the Simplicial Approximation Theorem.]