

TOPOLOGY & GROUPS
MICHAELMAS 2008
QUESTION SHEET 3

Questions with an asterisk * beside them are primarily aimed at 4th year students.

You may assume throughout this sheet that $\pi_1(S^1) \cong \mathbb{Z}$, and that a generator for $\pi_1(S^1)$ is represented by the loop $t \mapsto e^{2\pi it}$.

1. Show that for a space X , the following three conditions are equivalent:

- (i) Every map $S^1 \rightarrow X$ is homotopic to a constant map.
- (ii) Every map $S^1 \rightarrow X$ extends to a map $D^2 \rightarrow X$.
- (iii) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \rightarrow X$ are homotopic. [In this problem, ‘homotopic’ means ‘homotopic without regard to basepoints’.]

2. Let X and Y be spaces with basepoints x_0 and y_0 . Show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Deduce that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

3. A *retraction* of a space X onto a subspace A is a map $r: X \rightarrow A$ such that $ri = \text{id}_A$, where $i: A \rightarrow X$ is the inclusion map.

- (i) Prove that there is no retraction map $r: D^2 \rightarrow S^1$.
- (ii) Our aim here is to show that any map $f: D^2 \rightarrow D^2$ has a fixed point. Suppose that, on the contrary, f has no fixed point; in other words $f(x) \neq x$ for all $x \in D^2$. Use the pairs $(x, f(x))$ to construct a retraction $D^2 \rightarrow S^1$. Thus, we deduce that any map $D^2 \rightarrow D^2$ must have a fixed point.

This fact has many applications outside of topology. For example, it can be used to show that certain differential equations always have a solution.

4. For $n > 2$, prove that no two of \mathbb{R} , \mathbb{R}^2 and \mathbb{R}^n are homeomorphic.

* 5. Show that there is no retraction of a Möbius band onto its boundary.