Topology & Groups

Michaelmas 2008

Question Sheet 5

1. If $G_1 = \langle X_1 \mid R_1 \rangle$ and $G_2 = \langle X_2 \mid R_2 \rangle$, supply (with proof) a presentation for $G_1 \times G_2$. Deduce that if G_1 and G_2 are finitely presented, then so is $G_1 \times G_2$.

2. Show that $\langle a, b \mid aba^{-1}b^{-1}, a^5b^2, a^2b \rangle$ is the trivial group. [Hint: don't try to use Tietze transformations.]

3. Show that the group of symmetries of a regular n-sided polygon is $\langle a, b \mid a^n, b^2, abab \rangle$. [Hint: you will find it useful to show that $\langle a, b \mid a^n, b^2, abab \rangle$ has at most $2n$ elements.]

4. Show that $\langle a, b \mid abab^{-1} \rangle \cong \langle c, d \mid c^2d^2 \rangle$, by setting up an explicit isomorphism between them. [Hint: Note that in the first group, $(ab)(ab) = (aba)b = b^2$.]

5. Prove that the push-out of

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 2} & & \\ \mathbb{Z} & & \end{array}$$

is isomorphic to $\mathbb{Z}$.

6. Show that the group $\langle x, y \mid xyx = yxy \rangle$ is isomorphic to the push-out of

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ \big\downarrow{\scriptstyle \times 3} & & \\ \mathbb{Z} & & \end{array}$$

[*Hint: consider the elements xy and yxy.*] Is this an amalgamated free product?

7. A group-theoretic property P is known as *semi-decidable* if there is an algorithm that starts with a finite presentation of a group G and *either* terminates with a 'yes' answer if G has property P, *or* does not terminate if G does not have property P. Prove that the following properties of a group are semi-decidable:

(i) being abelian;

(ii) being free;

(iii) being a specific finite group (which is given by its multiplication table);

(iv) being finite.