Questions with an asterisk * beside them are optional.

1. For each of the following subgroups of \( \langle x, y \rangle = \pi_1(S^1 \vee S^1) \), construct a based covering map \( p: (\tilde{X}, \tilde{b}) \to (S^1 \vee S^1, b) \) such that \( p_\ast \pi_1(\tilde{X}, \tilde{b}) \) is that subgroup:

   (i) \( \langle x \rangle \)

   (ii) \( \{x^{m_1}y^{m_2}x^{n_2}y^{m_2} \ldots y^{m_k} : \sum m_i \text{ is even} \} \).

   (iii) the kernel of the homomorphism \( \langle x, y \rangle \to \mathbb{Z} \times \mathbb{Z} \) that sends \( x \) to \((1, 0)\), and \( y \) to \((0, 1)\).

2. Recall that the Klein bottle \( K \) is defined to be a square with the following side identifications.

   Construct a covering map from \( \mathbb{R}^2 \) to \( K \) and use it to show that \( \pi_1(K) \) is isomorphic to the group whose elements are pairs \((m, n)\) of integers, with the non-abelian group operation given by

   \[(m, n) \ast (x, y) = (m + (-1)^n x, n + y).\]

3. Suppose that a group \( G \) has a left action on a path-connected space \( Y \). This means that there is a homomorphism \( \phi: G \to \text{Homeo}(Y) \), where \( \text{Homeo}(Y) \) is the group of homeomorphisms of \( Y \). This is known as a covering space action if each \( y \in Y \) has an open neighbourhood \( U \) such that \( \phi(g)(U) \cap U = \emptyset \) for each \( g \in G - \{e\} \). Let \( Y/G \) denote the quotient space that identifies two points \( y_1 \) and \( y_2 \) in \( Y \) if and only if \( \phi(g)(y_1) = y_2 \) for some \( g \in G \).

   (i) Prove that the quotient map \( Y \to Y/G \) is a covering map.

   (ii) When \( Y \) is simply-connected, prove that \( \pi_1(Y/G) \cong G \).

   (iii) Find a covering space action of \( \mathbb{Z} \times \mathbb{Z} \) on \( \mathbb{R}^2 \), and use this to provide another proof that the fundamental group of the torus is \( \mathbb{Z} \times \mathbb{Z} \).
4. Construct Cayley graphs for each of the following groups, with respect to the given generators:

(i) \((\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^2, b^2 \rangle\), with respect to the generators \(a\) and \(b\);

(ii) \((\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^3, b^2 \rangle\), with respect to the generators \(a\) and \(b\).

[You should find Question 2 on Problem Sheet 6 useful.]

5. (i) Using covering spaces, prove that for each integer \(n \geq 2\), \(F_n\) is a finite index subgroup of \(F_2\), where \(F_n\) is the free group on \(n\) generators.

(ii) Prove that if \(F_m\) is subgroup of \(F_n\) with index \(i\), then \(m = ni - i + 1\). Deduce that \(m \geq n\).

(iii) Prove that \(F_2\) is a subgroup of \(F_3\) (but necessarily its index is infinite). Construct a covering space of \(S^1 \vee S^1 \vee S^1\) corresponding to this subgroup.

6. Prove that there are 13 index 3 subgroups of \(F_2\), of which 4 are normal.

7. Let \(G\) be the group \(\langle x, y \mid x^3y^3 \rangle\). Let \(\phi: G \rightarrow \mathbb{Z}/3\mathbb{Z}\) be the homomorphism that sends \(x\) and \(y\) to the generator of \(\mathbb{Z}/3\mathbb{Z}\). Find a finite presentation for the kernel of this homomorphism. [We know from Theorem VI.37 that a finite index subgroup of a finitely presented group is again finitely presented. To answer this question, you should use the proof of this theorem.]