Topology & Groups

Michaelmas 2016

Question Sheet 7

Questions with an asterisk $*$ beside them are optional.

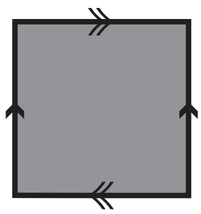
1. For each of the following subgroups of $\langle x, y\rangle = \pi_1(S^1 \vee S^1)$, construct a based covering map $p\colon (\tilde{X}, \tilde{b}) \to (S^1 \vee S^1, b)$ such that $p_* \pi_1(\tilde{X}, \tilde{b})$ is that subgroup:

   (i) $\langle x\rangle$

   (ii) $\{x^{n_1} y^{m_1} x^{n_2} y^{m_2} \dots y^{m_k} : \sum m_i \text{ is even}\}$.

   (iii) the kernel of the homomorphism $\langle x, y\rangle \to \mathbb{Z} \times \mathbb{Z}$ that sends x to $(1, 0)$, and y to $(0, 1)$.

2. Recall that the Klein bottle K is defined to be a square with the following side identifications.



   Construct a covering map from $\mathbb{R}^2$ to K and use it show that $\pi_1(K)$ is isomorphic to the group whose elements are pairs (m, n) of integers, with the non-abelian group operation given by

$$(m, n) \star (x, y) = (m + (-1)^n x, n + y).$$

3. Suppose that a group G has a left action on a path-connected space Y. This means that there is a homomorphism $\phi\colon G \to \mathrm{Homeo}(Y)$, where $\mathrm{Homeo}(Y)$ is the group of homeomorphisms of Y. This is known as a *covering space action* if each $y \in Y$ has an open neighbourhood U such that $\phi(g)(U) \cap U = \emptyset$ for each $g \in G - \{e\}$. Let Y/G denote the quotient space that identifies two points y_1 and y_2 in Y if and only if $\phi(g)(y_1) = y_2$ for some $g \in G$.

   (i) Prove that the quotient map $Y \to Y/G$ is a covering map.

   (ii) When Y is simply-connected, prove that $\pi_1(Y/G) \cong G$.

   (iii) Find a covering space action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$, and use this to provide another proof that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

4. Construct Cayley graphs for each of the following groups, with respect to the given generators:

   (i) $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^2, b^2 \rangle$, with respect to the generators a and b;

   (ii) $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^3, b^2 \rangle$, with respect to the generators a and b.

   [*You should find Question 2 on Problem Sheet 6 useful.*]

5. (i) Using covering spaces, prove that for each integer $n \geq 2$, F_n is a finite index subgroup of F_2, where F_n is the free group on n generators.

   (ii) Prove that if F_m is subgroup of F_n with index i, then $m = ni - i + 1$. Deduce that $m \geq n$.

   (iii) Prove that F_2 is a subgroup of F_3 (but necessarily its index is infinite). Construct a covering space of $S^1 \vee S^1 \vee S^1$ corresponding to this subgroup.

$*$ 6. Prove that there are 13 index 3 subgroups of F_2, of which 4 are normal.

7. Let G be the group $\langle x, y \mid x^3 y^3 \rangle$. Let $\phi \colon G \to \mathbb{Z}/3\mathbb{Z}$ be the homomorphism that sends x and y to the generator of $\mathbb{Z}/3\mathbb{Z}$. Find a finite presentation for the kernel of this homomorphism. *[We know from Theorem VI.37 that a finite index subgroup of a finitely presented group is again finitely presented. To answer this question, you should use the proof of this theorem.]*