§4. Basic pl topology

We have already had to state without proof of a number of results of the form 'a certain submanifold has a certain neighbourhood'. It is clear that if we are to argue rigourously, we need to develop a greater understanding of pl topology. The results that we state here without proof can be found in Rourke and Sanderson's book 'Introduction to piecewise-linear topology'.

Regular neighbourhoods

Definition. The *barycentric subdivision* $K^{(1)}$ of the simplicial complex K is constructed as follows. It has precisely one vertex in the interior of each simplex of K (including having a vertex at each vertex of K). A collection of vertices of $K^{(1)}$, in the interior of simplices $\sigma_1, \dots, \sigma_r$ of K, span a simplex of $K^{(1)}$ if and only if σ_1 is a face of σ_2, which is a face of σ_3, etc (possibly after re-ordering $\sigma_1, \dots, \sigma_r$).

An example is given in Figure 14. It is also possible to define $K^{(1)}$ inductively on the dimensions of the simplices of K, as follows. Start with all the vertices of K. Then add a vertex in each 1-simplex of K. Join it to the relevant 0-simplices of K. Then add a vertex in each 2-simplex σ of K. Add 1-simplices and 2-simplices inside σ by 'coning' the subdivision of $\partial\sigma$. Continue analogously with the higher-dimensional simplices.

Definition. The r^{th} *barycentric subdivision* of a simplicial complex K for each $r \in \mathbb{N}$ is defined recursively to be $(K^{(r-1)})^{(1)}$, where $K^{(0)} = K$.

Definition. If L is a subcomplex of the simplicial complex K, then the *regular neighbourhood* $\mathcal{N}(L)$ of K is the closure of the set of simplices in $K^{(2)}$ that intersect L. It is a subcomplex of $K^{(2)}$.

The following result asserts that regular neighbourhoods are essentially independent of the choice of triangulation for K.

Theorem 4.1. (Regular neighbourhoods are ambient isotopic) *Suppose that K' is a subdivision of a simplicial complex K. Let L be a subcomplex of K, and let L' be the subdivision $K' \cap L$. Then the regular neighbourhood of L in K is ambient isotopic to the regular neighbourhood of L' in K'.*

Thus, we may speak of regular neighbourhoods without specifying an initial triangulation.

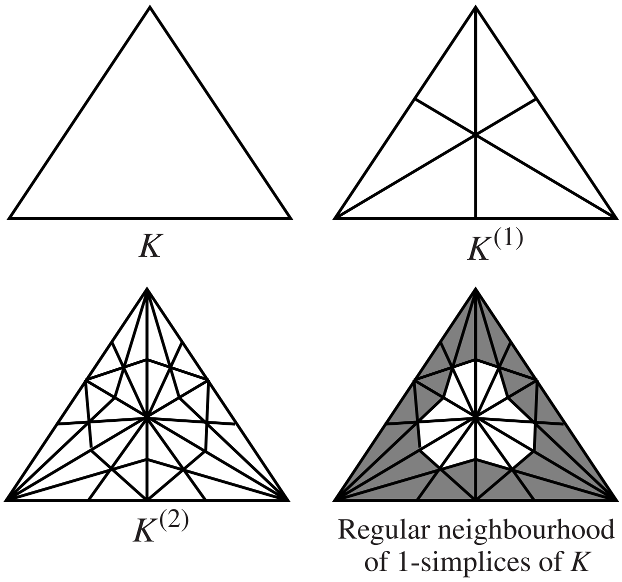

Figure 14.

Handle structures

Definition. A *handle structure* of an n-manifold M is a decomposition of M into $n+1$ sets $\mathcal{H}_0, \dots, \mathcal{H}_n$ having disjoint interiors, such that

- $\mathcal{H}_i$ is a collection of disjoint n-balls, known as *i-handles*, each having a product structure $D^i \times D^{n-i}$,
- for each i-handle $(D^i \times D^{n-i}) \cap (\bigcup_{j=0}^{i-1} \mathcal{H}_j) = \partial D^i \times D^{n-i}$,
- if $H_i = D^i \times D^{n-i}$ (respectively, $H_j = D^j \times D^{n-j}$) is an i-handle (respectively, j-handle) with $j < i$, then $H_i \cap H_j = D^j \times E = F \times D^{n-i}$ for some $(n - j - 1)$-manifold E (respectively, $(i-1)$-manifold F) embedded in ∂D^{n-j} (respectively, ∂D^i).

Here we adopt the convention that D^0 is a single point and $\partial D^0 = \emptyset$.

In words, the third of the above conditions requires that the attaching map of each handle respects the product structures of the handles to which it is attached. For a 3-manifold, this is relevant only for $j = 1$ and $i = 2$.

One should view a handle decomposition as like a CW complex, but with each i-cell thickened to a n-ball.

Theorem 4.2. *Every pl manifold has a handle structure.*

Proof. Pick a triangulation K for the manifold. Let V^i be the vertices of $K^{(1)}$ in the interior of the i-simplices of K. Let $\mathcal{H}^i$ be the closure of the union of the simplices in $K^{(2)}$ touching V^i. These form a handle structure. □

General position

In $\mathbb{R}^n$ it is well-known that two subspaces, of dimensions p and q, intersect in a subspace of dimension at least $p+q-n$, and that if the dimension of their intersection is more than $p+q-n$, then only a small shift of one of them is required to achieve this minimum. Analogous results hold for subcomplexes of a pl manifold. The *dimension* $\dim(P)$ of a simplicial complex P is the maximal dimension of its simplices.

Proposition 4.3. *Suppose that P and Q are subcomplexes of a closed manifold M, with $\dim(P) = p$, $\dim(Q) = q$ and $\dim(M) = M$. Then there is a homeomorphism $h: M \to M$ isotopic to the identity such that $h(P)$ and Q intersect in a simplicial complex of dimension of at most $p+q-m$.*

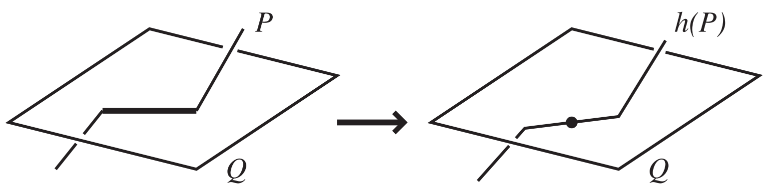

Figure 15.

Then, $h(P)$ and Q are said to be *in general position*. This is one of a number of similar results. They are fairly straightforward, but rather than giving detailed definitions and theorems, we will simply appeal to 'general position' and leave it at that.

Lemma 4.4. *Any pl homeomorphism $\partial D^n \to \partial D^n$ extends to a pl homeomorphism $D^n \to D^n$.*

Proof. See the figure. $\square$

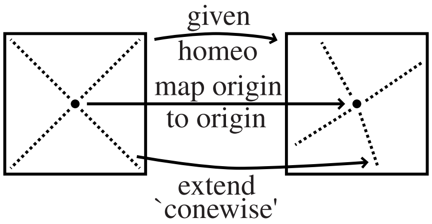

Figure 16.

Remark. The above proof does not extend to the smooth category, and indeed the smooth version is false.

A similar proof gives the following.

Lemma 4.5. *Two homeomorphisms $D^n \to D^n$ which agree on ∂D^n are isotopic.*

Let $r\colon D^n \to D^n$ be the map which changes the sign of the x_n co-ordinate.

Proposition 4.6. *A homeomorphism $D^n \to D^n$ is isotopic either to the identity or to r.*

Proof. By induction on n. First note that there are clearly only two homeomorphisms $\partial D^1 \to \partial D^1$. By Lemma 4.4, these extend to homeomorphisms $D^1 \to D^1$. Now apply Lemma 4.5 to show that any homeomorphism $D^1 \to D^1$ is isotopic to one of these. Now consider a homeomorphism $h\colon \partial D^2 \to \partial D^2$. It takes a 1-simplex σ in ∂D^2 to a 1-simplex in ∂D^2. There are two possibilities up to isotopy for $h|_\sigma$, since σ is a copy of D^1. Note that $\mathrm{cl}(\partial D^2 - \sigma)$ is clearly a copy of a 1-ball. (An explicit homeomorphism is obtained by retracting $\mathrm{cl}(\partial D^2 - \sigma)$ onto one hemisphere of ∂D^2). Hence, each homeomorphism of σ extends to $\partial D^2 - \sigma$, in a way that is unique up to isotopy by Lemmas 4.4 and 4.5. Hence, h is isotopic to $r|_{\partial D^2}$ or $\mathrm{id}|_{\partial D^2}$. Therefore, by Lemma 4.4, any homeomorphism $D^2 \to D^2$ is isotopic to r or id. The inductive step proceeds in all dimensions in this way. $\square$

We end with a couple of further results above spheres and discs that we will use (often implicitly) at a number of points. Their proofs are less trivial than the above results, and are omitted.

Proposition 4.7. *Let $h_1 \colon D^n \to M$ and $h_2 \colon D^n \to M$ be embeddings of the n-ball into an n-manifold. Then there is a homeomorphism $h \colon M \to M$ isotopic to the identity such that $h \circ h_1$ is either h_2 or $h_2 \circ r$.*

Proposition 4.8. *The space obtained by gluing two n-balls along two closed $(n-1)$-balls in their boundaries is homeomorphic to an n-ball.*

§5. Constructing 3-manifolds

The aim now is to give some concrete constructions of 3-manifolds. This will be a useful application of the pl theory outlined in the last section.

Construction 1. Heegaard splittings.

Definition. A *handlebody of genus g* is the 3-manifold with boundary obtained from a 3-ball B^3 by gluing $2g$ disjoint closed 2-discs in ∂B^3 in pairs via orientation-reversing homeomorphisms.

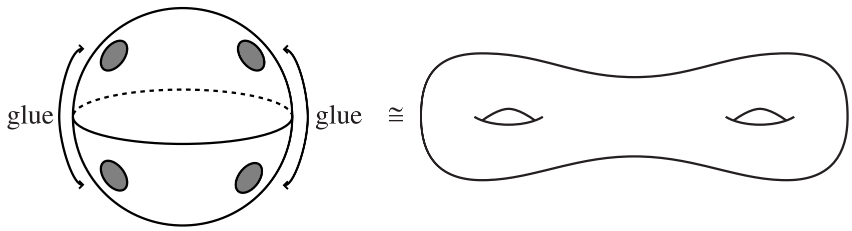

Figure 17.

Lemma 5.1. *Let H be a connected orientable 3-manifold with a handle structure consisting of only 0-handles and 1-handles. Then H is a handlebody.*

Proof. Pick an ordering on the handles of H, and reconstruct H by regluing these balls, one at a time, as specified by this ordering. At each stage, we identify discs, either in distinct components of the 3-manifold, or in the same component of the 3-manifold. Perform all of the former identifications first. The result is a 3-ball. Then perform all of the latter identifications. Each must be orientation-reversing, since H is orientable. Hence, H is a handlebody. □

Let H_1 and H_2 be two genus g handlebodies. Then we can construct a 3-manifold M by gluing H_1 and H_2 via a homeomorphism $h\colon \partial H_1 \to \partial H_2$. This is known as a *Heegaard splitting* of M.

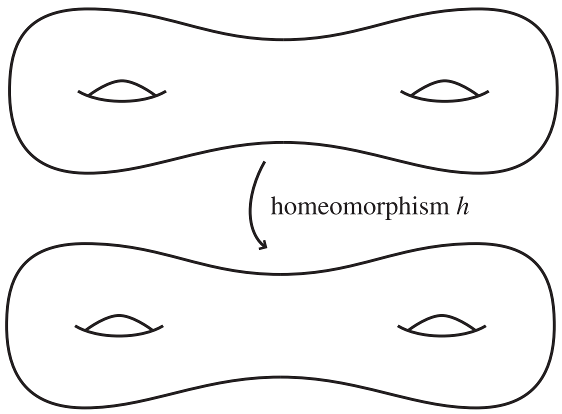

Figure 18.

Exercise. Take two copies of the same genus g handlebody and glue their boundaries via the identity homeomorphism. Show that the resulting space is homeomorphic to the connected sum of g copies of $S^1 \times S^2$.

Exercise. Show that, if H is the genus g handlebody embedded in S^3 in the standard way, then $S^3 - \mathrm{int}(H)$ is also a handlebody. Hence, show that S^3 has Heegaard splittings of all possible genera.

Example. A common example is the case where two solid tori are glued along their boundaries. By the above two exercises, S^3 and $S^2 \times S^1$ have such Heegaard splittings. However, other manifolds can be constructed in this way. A *lens space* is a 3-manifold with a genus 1 Heegaard splitting which is not homeomorphic to S^3 or $S^2 \times S^1$. Note that there are many ways to glue the two solid tori together, because there are many possible homeomorphisms from a torus to itself, constructed as follows. View T^2 as $\mathbb{R}^2/\sim$, where $(x, y) \sim (x+1, y)$ and $(x, y) \sim (x, y+1)$. Then any linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with integer matrix entries and determinant ± 1 descends to a homeomorphism $T^2 \to T^2$.

Theorem 5.2. *Any closed orientable 3-manifold M has a Heegaard splitting.*

Proof. Pick a handle structure for M. The 0-handle and 1-handles form a handlebody. Similarly, the 2-handles and 3-handles form a handlebody. (If one views each i-handle in a handle structure for a closed n-manifold as an $(n-i)$-handle, the result is again a handle structure.) □

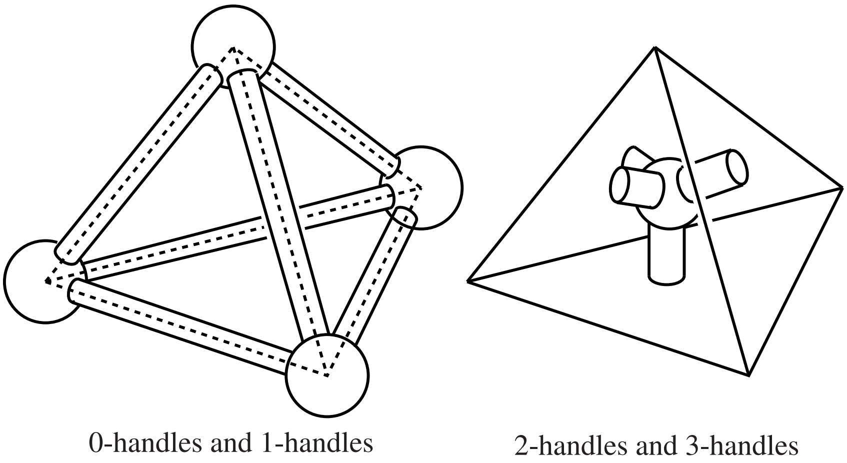

Figure 19.

CONSTRUCTION 2. The mapping cylinder.

Start with a compact orientable surface F. Now glue the two boundary components of $F \times [0,1]$ via an orientation-reversing homeomorphism $h\colon F \times \{0\} \to F \times \{1\}$. The result is a compact orientable 3-manifold $(F \times [0,1])/h$ known as the *mapping cylinder for* h.

Exercise. If two homeomorphisms h_0 and h_1 are isotopic then $(F \times [0,1])/h_0$ and $(F \times [0,1])/h_1$ are homeomorphic.

However, there are many homeomorphisms $F \to F$ not isotopic to the identity.

Definition. Let C be a simple closed curve in the interior of the surface F. Let $\mathcal{N}(C) \cong S^1 \times [-1,1]$ be a regular neighbourhood of C. Then a *Dehn twist* about C is the map $h\colon F \to F$ which is the identity outside $\mathcal{N}(C)$, and inside $\mathcal{N}(C)$ sends (θ, t) to $(\theta + \pi(t+1), t)$.

Note. The choice of identification $\mathcal{N}(C) \cong S^1 \times [-1,1]$ affects the resulting homeomorphism, since it is possible to twist in 'both directions'.

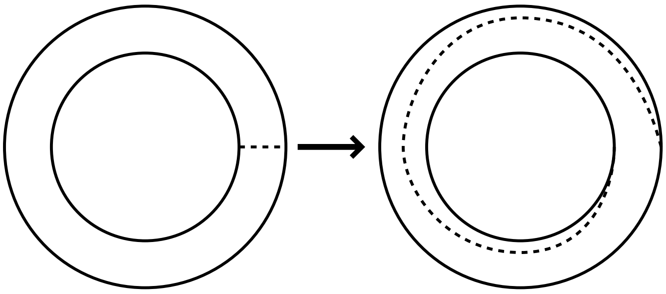

Figure 20.

Exercise. If C bounds a disc in F or is parallel to a boundary component, then a Dehn twist about C is isotopic to the identity. But it is in fact possible to show that if neither of these conditions holds, then a Dehn twist about C is never isotopic to the identity.

Theorem 5.3. [Dehn, Lickorish] *Any orientation preserving homeomorphism of a compact orientable surface to itself is isotopic to the composition of a finite number of Dehn twists.*

CONSTRUCTION 3. Surgery

Let L be a link in S^3 with n components. Then $\mathcal{N}(L)$ is a collection of solid tori. Let M be the 3-manifold obtained from $S^3 - \mathrm{int}(\mathcal{N}(L))$ by gluing in n solid tori $\bigcup_{i=1}^n S^1 \times D^2$, via a homeomorphism $\partial(\bigcup_{i=1}^n S^1 \times D^2) \to \partial\mathcal{N}(L)$. The resulting 3-manifold is *obtained by surgery along L*.

There are many possible ways of gluing in the solid tori, since there are many homeomorphisms from a torus to itself.

Theorem 5.4. [Lickorish, Wallace] *Every closed orientable 3-manifold M is obtained by surgery along some link in S^3.*

Proof. Let $H_1 \cup H_2$ be a Heegaard splitting for M, with gluing homeomorphism $f\colon \partial H_1 \to \partial H_2$. Let $g\colon \partial H_1 \to \partial H_2$ be a gluing homeomorphism for a Heegaard splitting of S^3 of the same genus. Note that H_1 and H_2 inherit orientations from M and S^3, and, with respect to these orientations, f and g are orientation reversing. Then, by Theorem 5.3, $g^{-1} \circ f$ is isotopic to a composition of Dehn twists, $\tau_1, \ldots, \tau_n$ along curves $C_1, \ldots, C_n$, say. Let $k\colon \partial H_1 \times [n, n+1] \to \partial H_1 \times [n, n+1]$ be the isotopy between $\tau_n \circ \ldots \circ \tau_1$ and $g^{-1} \circ f$. A regular neighbourhood

$\mathcal{N}(\partial H_1)$ of ∂H_1 in H_1 is homeomorphic to a product $\partial H_1 \times [0, n+1]$, say, with $\partial H_1 \times \{n+1\} = \partial H_1$. (See Theorem 6.1 in the next section.) For $i = 1, \ldots, n$, let $L_i = \tau_1^{-1} \ldots \tau_{i-1}^{-1} C_i \times \{i - 3/4\} \subset H_1 \subset M$. Define a homeomorphism

$$
\begin{aligned}
M - \bigcup_{i=1}^{n} \mathrm{int}(\mathcal{N}(L_i)) &\rightarrow S^3 - \mathrm{int}(\mathcal{N}(L)) \\
H_1 - (\partial H_1 \times [0, n+1]) &\xrightarrow{\mathrm{id}} H_1 - (\partial H_1 \times [0, n+1]) \\
(\partial H_1 - \mathcal{N}(C_1)) \times [0, 1/2] &\xrightarrow{\mathrm{id}} (\partial H_1 - \mathcal{N}(C_1)) \times [0, 1/2] \\
\partial H_1 \times [1/2, 1] &\xrightarrow{\tau_1} \partial H_1 \times [1/2, 1] \\
(\partial H_1 - \mathcal{N}(\tau_1^{-1} C_2)) \times [1, 3/2] &\xrightarrow{\tau_1} (\partial H_1 - \mathcal{N}(C_2)) \times [1, 3/2] \\
\partial H_1 \times [3/2, 2] &\xrightarrow{\tau_2 \tau_1} \partial H_1 \times [3/2, 2] \\
&\ldots \\
\partial H_1 \times [n - 1/2, n] &\xrightarrow{\tau_n \ldots \tau_1} \partial H_1 \times [n - 1/2, n] \\
\partial H_1 \times [n, n+1] &\xrightarrow{k} \partial H_1 \times [n, n+1] \\
H_2 &\xrightarrow{\mathrm{id}} H_2
\end{aligned}
$$

Here, L is a collection of simple closed curves in $H_1 \subset S^3$. These homeomorphisms all agree, since $\tau_i \ldots \tau_1$ and $\tau_{i-1} \ldots \tau_1$ agree on $\partial H_1 - \tau_1^{-1} \ldots \tau_{i-1}^{-1} \mathcal{N}(C_i)$. Therefore, M is obtained from S^3 by first removing a regular neighbourhood of the link L, and then gluing in n solid tori. □