Three-dimensional manifolds
Michaelas Term 1999
Examples Sheet 1

Surfaces

1. Prove the 2-dimensional pl Schoëflies theorem: any properly embedded simple closed curve in a 2-sphere is ambient isotopic to the standard simple closed curve.

2. Classify (up to ambient isotopy) all the simple closed curves properly embedded in an annulus. What about such curves in a torus?

3. Show that any compact orientable surface with negative Euler characteristic is expressible as union of pairs of pants glued along their boundary curves.

4. Show that, if C is a homotopically trivial simple closed curve properly embedded in an orientable surface, then C bounds an embedded disc. (One approach to this is to use questions 2 and 3).

Surfaces in 3-manifolds

5. Show that if a prime orientable 3-manifold M contains a compressible torus boundary component, then M is the solid torus.

6. Find a compressible torus T properly embedded in some prime orientable 3-manifold M, such that no component of $M - T$ is a solid torus.

7. Let M be a compact 3-manifold. Suppose that we cut this 3-manifold along a sequence of properly embedded incompressible surfaces, and end with a collection of 3-balls. Show that M is prime. Apply this to the 3-manifold given as an example at the end of Lecture 1 (the space obtained by attaching thickened punctured tori to a thickened torus).

Heegaard splittings

8. Show that any closed orientable 3-manifold has Heegaard splittings of arbitrarily high genus.

9. Define the Heegaard genus $h(M)$ of a closed orientable 3-manifold M to be the minimal genus of a Heegaard splitting for M. Show that $h(M_1 \# M_2) \leq h(M_1) + h(M_2)$. (In fact, equality always holds.)

10. Find closed orientable 3-manifolds with arbitrarily large Heegaard genus.

Dehn surgery

11. Let L be a link in S^3. Let M be a 3-manifold obtained by surgery on L. Let C be a collection of simple closed curves, one on each component of $\mathcal{N}(L)$, that each bounds a disc in one of the attached solid tori, but none of which bounds a disc in $\partial\mathcal{N}(L)$. Show that the homeomorphism class of M only depends on the isotopy class of C in $\partial\mathcal{N}(L)$.

These curves C are usually specified by assigning a 'slope' in $\mathbb{Q} \cup \infty$ to each component of L. A slope p/q (where p and q are coprime integers) on a component K of L determines a curve on $\partial\mathcal{N}(K)$, which represents $(p, q) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(\partial\mathcal{N}(K))$. Here, the identification between $\mathbb{Z} \oplus \mathbb{Z}$ and $H_1(\partial\mathcal{N}(K))$ is chosen so that a curve representing $(1, 0)$ bounds a disc in $\partial\mathcal{N}(K)$ and a curve representing $(0, 1)$ is homologically trivial in $H_1(S^3 - K)$.

12. What is the manifold obtained by surgery on the unknot with slope 0? What about $1/q$ surgery, or more generally, p/q surgery on the unknot?

13. Show that any 3-manifold obtained by $1/q$ surgery on a knot in S^3 has the same homology as S^3.

14. Show that any 3-manifold M obtained by surgery on a knot, with slope zero, has $H_1(M) = \mathbb{Z}$. Construct an explicit non-separating orientable surface properly embedded in M.

15. Show that any closed orientable 3-manifold is obtained by surgery on a link in S^3 using only integral surgery slopes.

16. Construct a surgery descriptions of each lens space using only integral surgery slopes. (Express an element of $SL(2, \mathbb{Z})$ as a product of 'standard' matrices.)

17. Using question 12, show that any closed orientable 3-manifold is obtained by surgery on a link in S^3, where each component of the link is unknotted.

18. Is there a way of giving a surgery description of a compact orientable 3-manifold with non-empty boundary?

19. Let M be a 3-manifold obtained by surgery on the trefoil knot (the non-trivial knot with three crossings). Show that M has Heegaard genus at most two. (One of these spaces is the famous Poincaré homology 3-sphere.)

One-sided and two-sided surfaces

20. Show that if an orientable prime 3-manifold M contains a properly embedded $\mathbb{R}P^2$, then M is a copy of $\mathbb{R}P^3$.

21. In the lens space M obtained by $6/1$ surgery on the unknot, construct a properly embedded copy of the non-orientable surface N_3. Show that this is incompressible, but that the map $\pi_1(N_3) \to \pi_1(M)$ induced by inclusion is not injective.