The crossing number of composite knots

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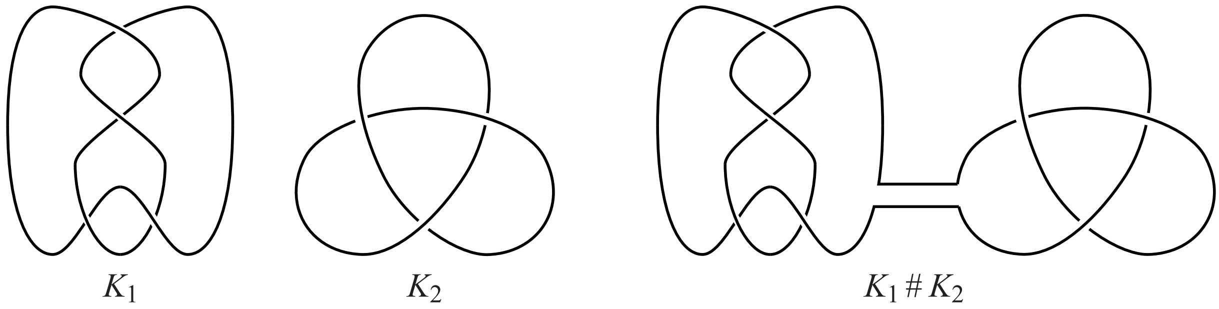
Let K be a knot in S^3.

Its crossing number $c(K)$ is the minimal number of crossings in a diagram for K.

If K_1 and K_2 are oriented knots, their connected sum $K_1 \sharp K_2$ is defined by:



Old conjecture: $c(K_1 \sharp K_2) = c(K_1) + c(K_2)$.

- $c(K_1 \sharp K_2) \leq c(K_1) + c(K_2)$ is trivial.
- True when K_1 and K_2 are alternating [Kauffman], [Murasugi], [Thistlethwaite] - follows from the fact that a reduced alternating diagram has minimal crossing number, which is proved using the Jones polynomial.
- Very little is known in general.

Theorem: [L]

$$\frac{c(K_1) + c(K_2)}{281} \leq c(K_1 \sharp K_2) \leq c(K_1) + c(K_2).$$

Theorem: [L]

$$\frac{c(K_1) + \ldots + c(K_n)}{281} \leq c(K_1 \sharp \ldots \sharp K_n) \leq c(K_1) + \ldots + c(K_n).$$

The advantage of this more general formulation is:

We may assume that each K_i is prime.

Write $K_i = K_{i,1} \sharp \ldots \sharp K_{i,m(i)}$.

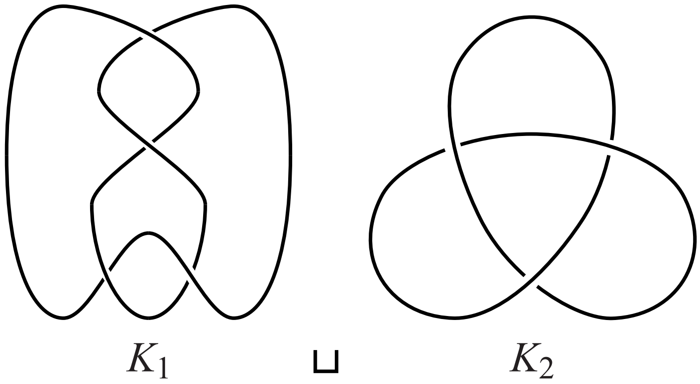
Assuming the theorem for a sum of prime knots:

$$c(K_1 \sharp \ldots \sharp K_n) \geq \frac{\sum_{i=1}^{n} \sum_{j=1}^{m(i)} c(K_{i,j})}{281} \geq \frac{\sum_{i=1}^{n} c(K_i)}{281}.$$

We may also assume that each K_i is non-trivial.

Distant unions

The distant union $K_1 \sqcup \ldots \sqcup K_n$ of knots $K_1, \ldots, K_n$:



Lemma: $c(K_1 \sqcup \ldots \sqcup K_n) = c(K_1) + \ldots + c(K_n)$.

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Proof:

($\leq$): Use minimal crossing number diagrams for $K_1, \ldots, K_n$ to construct a diagram for $K_1 \sqcup \ldots \sqcup K_n$.

($\geq$): Let D be a minimal crossing number diagram of $K_1 \sqcup \ldots \sqcup K_n$.

Use this to construct a diagram D_i of K_i by discarding the remaining components. Then

$$\begin{aligned} c(K_1 \sqcup \ldots \sqcup K_n) &= c(D) \\ &\geq c(D_1) + \ldots + c(D_n) \\ &\geq c(K_1) + \ldots + c(K_n). \end{aligned}$$

Strategy for the main theorem

Let D be a minimal crossing number diagram of $K_1 \sharp \ldots \sharp K_n$.

Use this to construct a diagram D' for $K_1 \sqcup \ldots \sqcup K_n$ such that $c(D') \leq 281\ c(D)$. Then

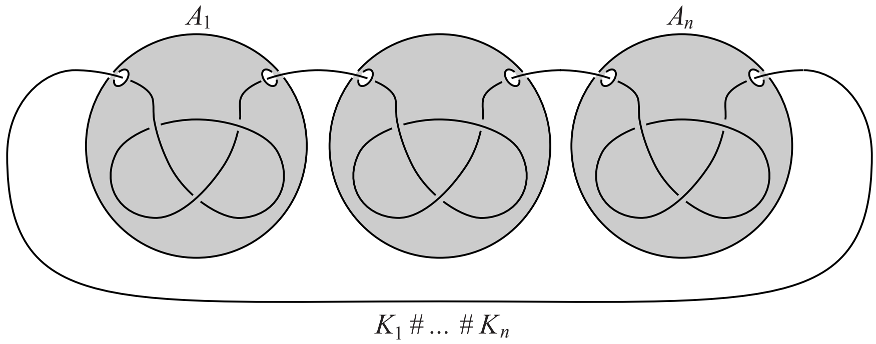
$$\begin{aligned} c(K_1) + \ldots + c(K_n) &= c(K_1 \sqcup \ldots \sqcup K_n) \\ &\leq c(D') \\ &\leq 281\ c(D) \\ &= 281\ c(K_1 \sharp \ldots \sharp K_n). \end{aligned}$$

Creating $K_1 \sqcup \ldots \sqcup K_n$ from $K_1 \sharp \ldots \sharp K_n$

Let $K = K_1 \sharp \ldots \sharp K_n$.

Let X = exterior of K.

Let $A_1, \ldots, A_n$ = the following annuli:



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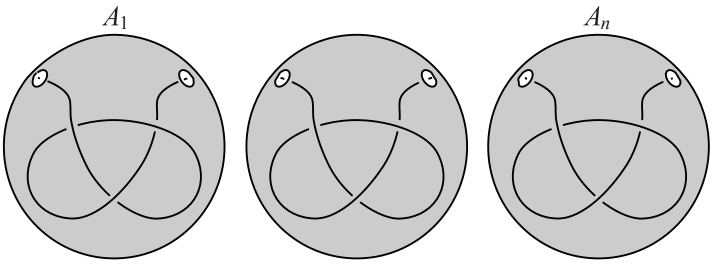

Remove the sub-arcs of K running from A_i to A_{i+1} (mod n).

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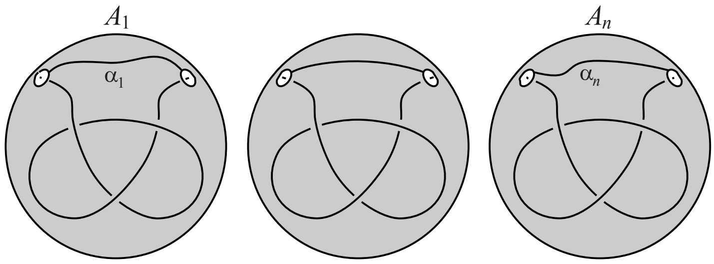
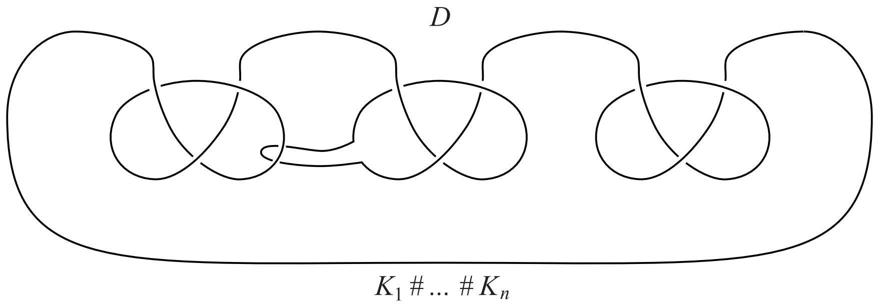


Remove the sub-arcs of K running from A_i to A_{i+1} (mod n).

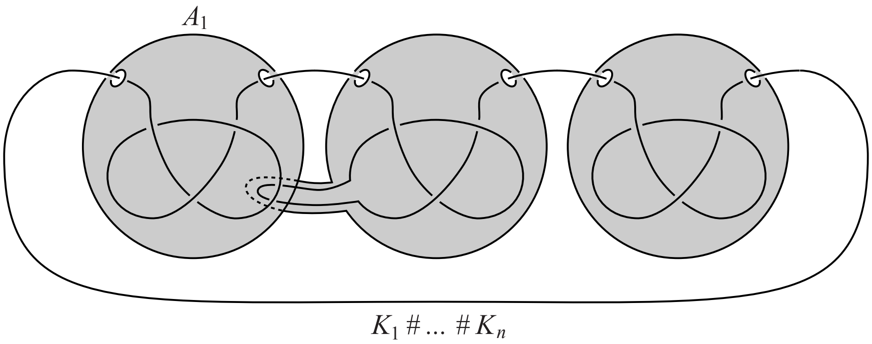
Add an arc α_i on A_i, running between the two boundary components.

(In fact, we do something a bit more complicated than this.)

Creating D' from D

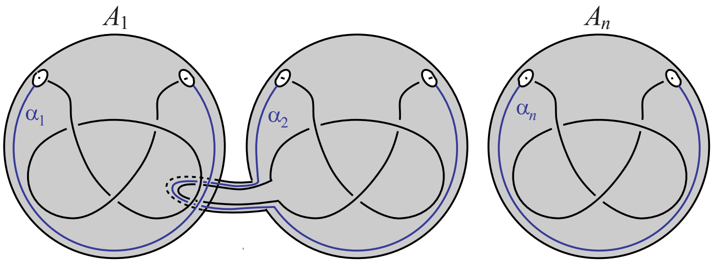


D may be complicated.



D may complicated.

Hence, the annuli $A_1 \cup \ldots \cup A_n$ might be embedded in a 'twisted way'



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Hence, the annuli $A_1 \cup \ldots \cup A_n$ might be embedded in a 'twisted way'

So, when we add the arcs $\alpha_1 \cup \ldots \cup \alpha_n$, we may introduce new crossings.

We need to control the arcs $\alpha_1 \cup \ldots \cup \alpha_n$.

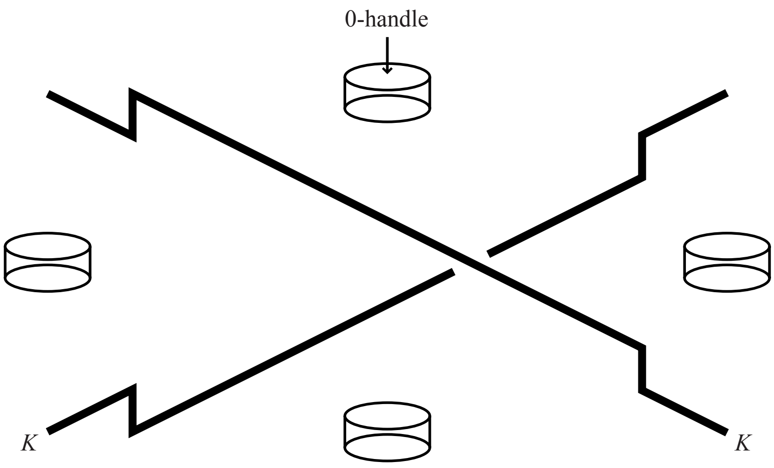
So, we need to control how the annuli A_i are embedded in X (the exterior of K).

For this, we use normal surface theory.

This requires a triangulation of X.

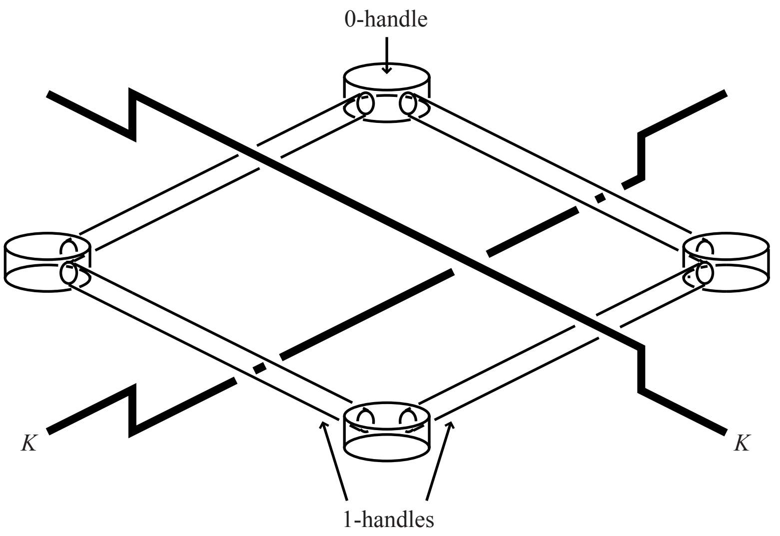
In fact, we'll use a handle structure.

A handle structure on X from D



Place four 0-handles near each crossing.

A HANDLE STRUCTURE FROM D



Place four 0-handles near each crossing.

Add four 1-handles.

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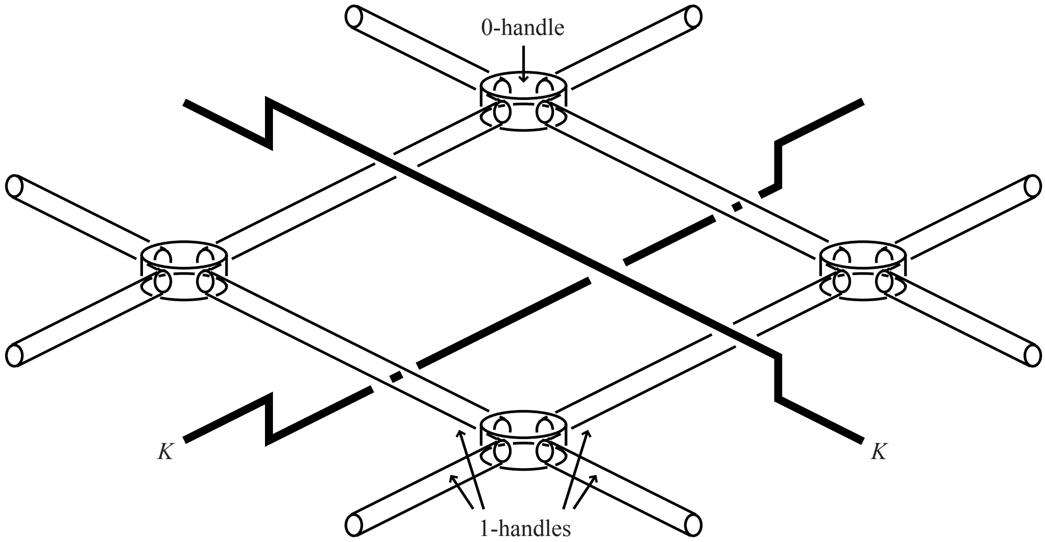
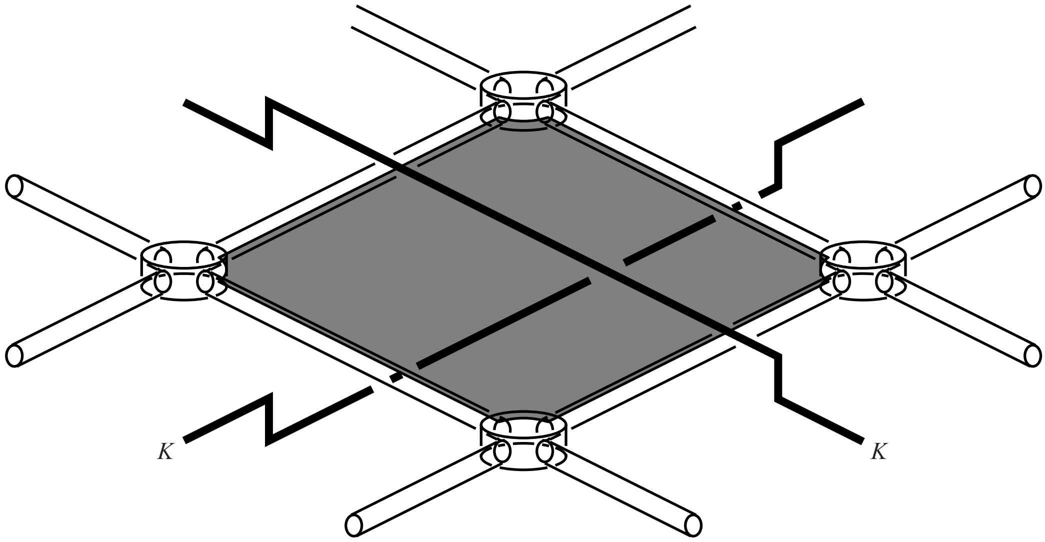


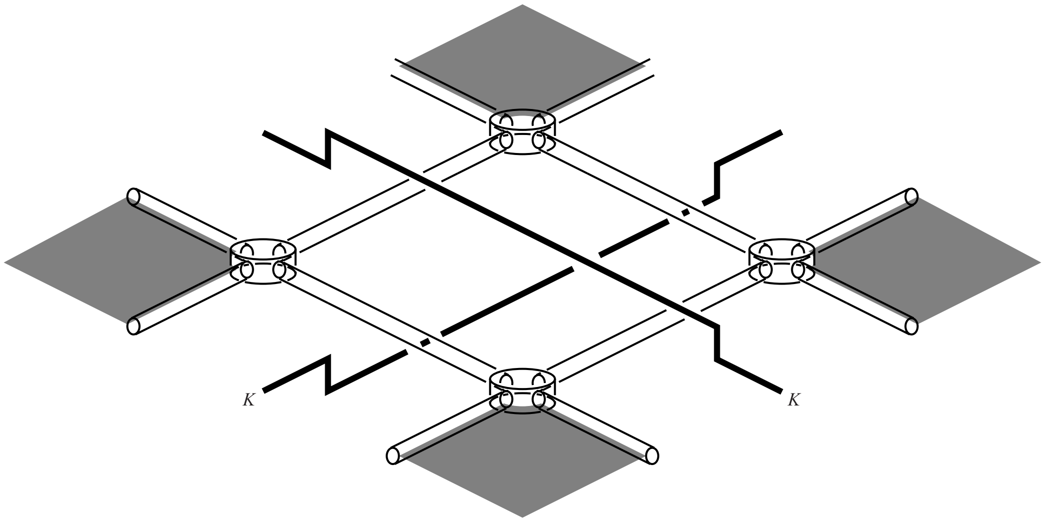
Place four 0-handles near each crossing.

Add four 1-handles.

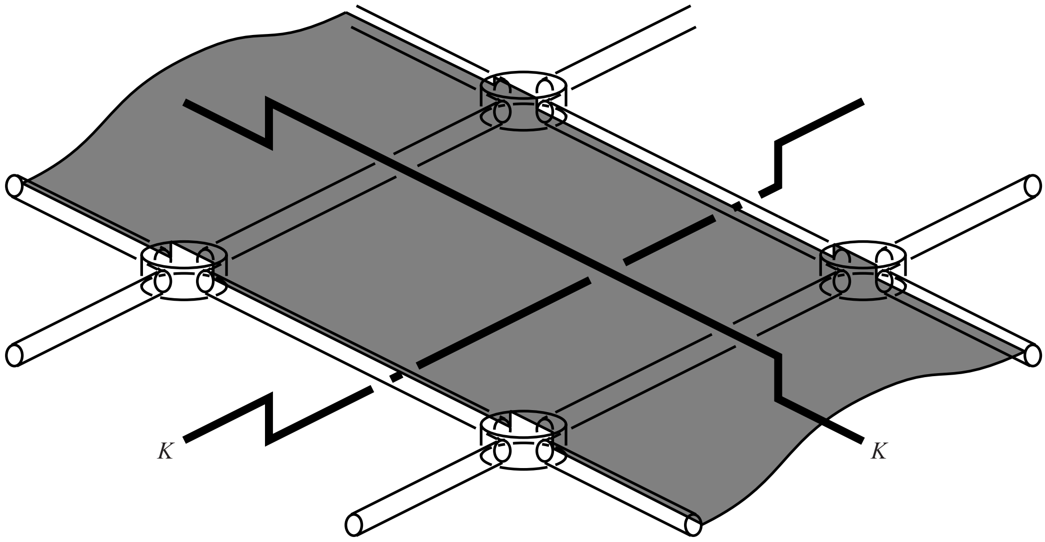
Near each edge of the diagram, add two 1-handles.



Add a 2-handle at each crossing.



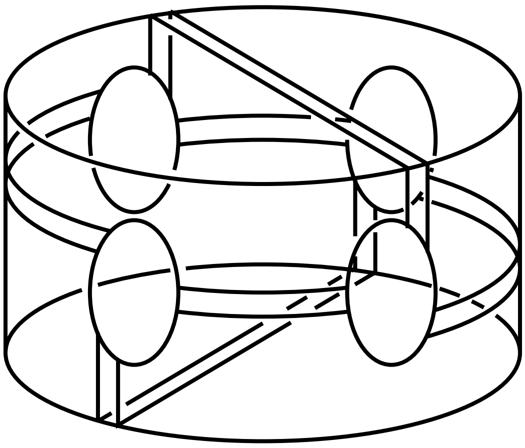
Add a 2-handle in each region.



Add 2-handles along each ‘over-arc’ and ‘under-arc’ of the diagram.

Finally, add 3-handles above and below the diagram.

The local picture near each 0-handle



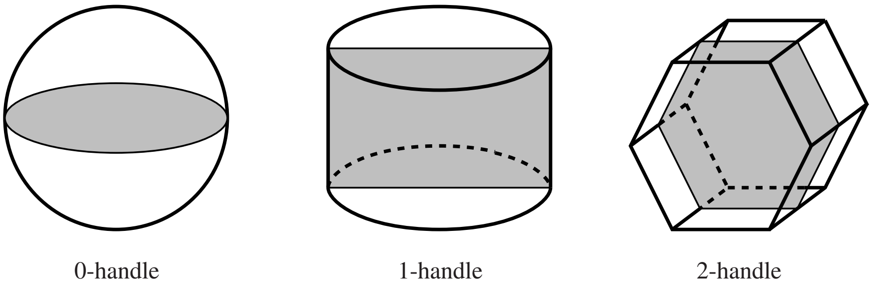
We now have a handle structure on X, the exterior of K.

Let $A = A_1 \cup \ldots \cup A_n$.

We want to ambient isotope A into normal form ...

Because A is incompressible and ∂-incompressible, we may isotope it into normal form.

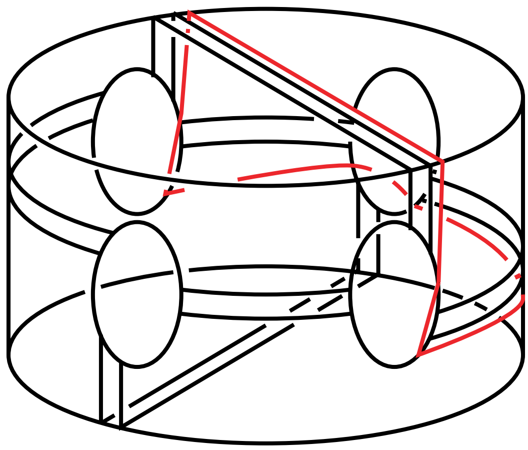
ie. each component of intersection with the handles looks like:



In addition, each component of intersection with the 0-handles satisfies certain conditions.

$\longrightarrow$ only finitely many normal 'disc types'.

An example of a normal disc type:



Recall that we must add an embedded arc α_i on each annulus A_i, running between the two boundary components.

Any such arc will do.

We may arrange that

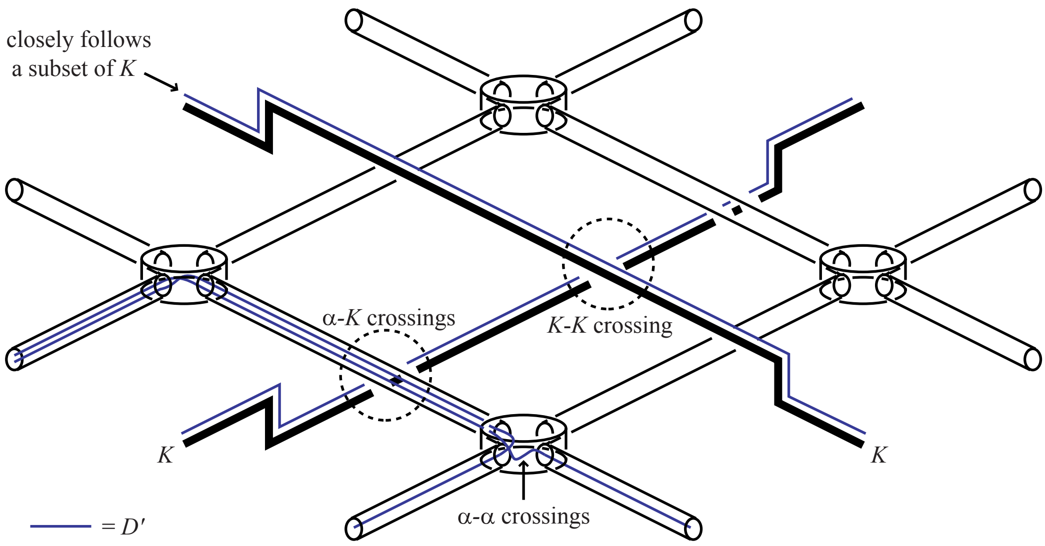
- $\alpha_1 \cup \ldots \cup \alpha_n$ misses the 2-handles
- respects the product structure on the 1-handles.

Pick α_i so that it has minimal length (subject also to an extra condition).

This implies that it intersects each normal disc in at most one arc.

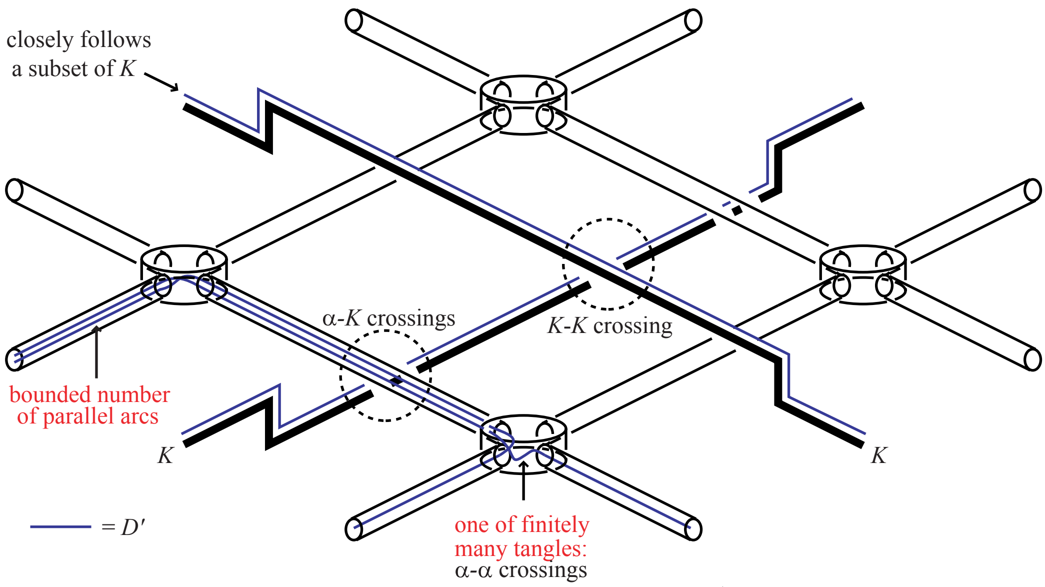
The diagram D'

Inserting the arcs $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ gives a diagram D' for $K_1 \sqcup \ldots \sqcup K_n$:



Its crossings come in 3 types.

Inserting the arcs $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ gives a diagram D' for $K_1 \sqcup \ldots \sqcup K_n$:



D' has 3 types of crossings:

$K - K$ crossings	$\leq c(D)$
$\alpha - K$ crossings	$\leq 4c(D) \times 6$
$\alpha - \alpha$ crossings	$\leq 4c(D) \times 64$
TOTAL	$\leq 281\ c(D)$

Wishful thinking: We'd be done if we could arrange that A intersected each handle in one of finitely many possible configurations.

But this probably isn't possible.

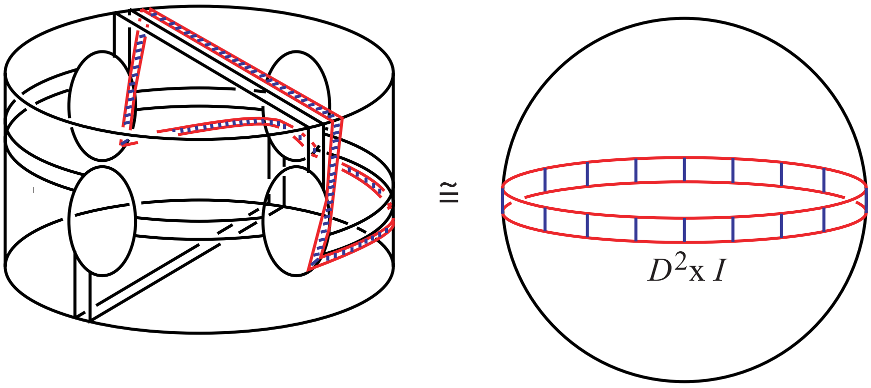
However, there are only finitely many configurations of disc types.

Key Claim 1. α can be chosen to run over at most 2 normal discs of each disc type in each handle.

This $\Rightarrow$ α intersects each handle in one of finitely many possible configurations, and we're done.

What if a handle contains more than one copy of a normal disc? Then, any two such discs are normally parallel.

Between any two normally parallel adjacent discs, there is a copy of $D^2 \times I$.



These patch together to form an I-bundle embedded in the exterior of A, called a 'parallelity bundle' $\mathcal{B}$.

Some terminology

Cut X along the annuli A.

Throw away the component with a copy of A in its boundary.

Let M be the rest.

Then $M = X_1 \sqcup \ldots \sqcup X_n$, where each X_i is the exterior of K_i.

Let S be the copy of A in M.

M inherits a handle structure from the handle structure on X.

The space between two adjacent normal discs of A becomes a 'parallelity handle' of M.

Claim 2. We may pick α so that it misses the parallelity bundle $\mathcal{B}$.

This $\Rightarrow$ Claim 1, because if a normal disc of A has parallel copies on both sides, it lies in a parallelity handle of M.

In fact, it is convenient to enlarge $\mathcal{B}$ to a larger I-bundle $\mathcal{B}'$.

We'll arrange for α to miss $\mathcal{B}'$ and hence $\mathcal{B}$.

A generalised parallelity bundle

is a 3-dimensional submanifold $\mathcal{B}'$ of (M, S) such that

- $\mathcal{B}'$ is an I-bundle over a compact surface F;
- the ∂I-bundle is $\mathcal{B}' \cap S$;
- $\mathcal{B}'$ is a union of handles;
- any handle in $\mathcal{B}'$ that intersects the I-bundle over ∂F is a parallelity handle;
- $\mathrm{cl}(M - \mathcal{B}')$ inherits a handle structure.

The ∂I-bundle is the horizontal boundary of $\mathcal{B}'$.
The I-bundle over ∂F is the vertical boundary.

CLAIM 3: Possibly after modifying its handle structure, (M, S) contains a generalised parallelity bundle $\mathcal{B}'$ such that:

- $\mathcal{B}'$ contains every parallelity handle of M;
- $\mathcal{B}'$ is a collection of I-bundles over discs.

The horizontal boundary is a union of discs in the annuli A.

$\Rightarrow$ it cannot separate the two components of ∂A_i.

$\Rightarrow$ α can be chosen to miss $\mathcal{B}'$

$\Rightarrow$ Claim 2.

How to prove Claim 3

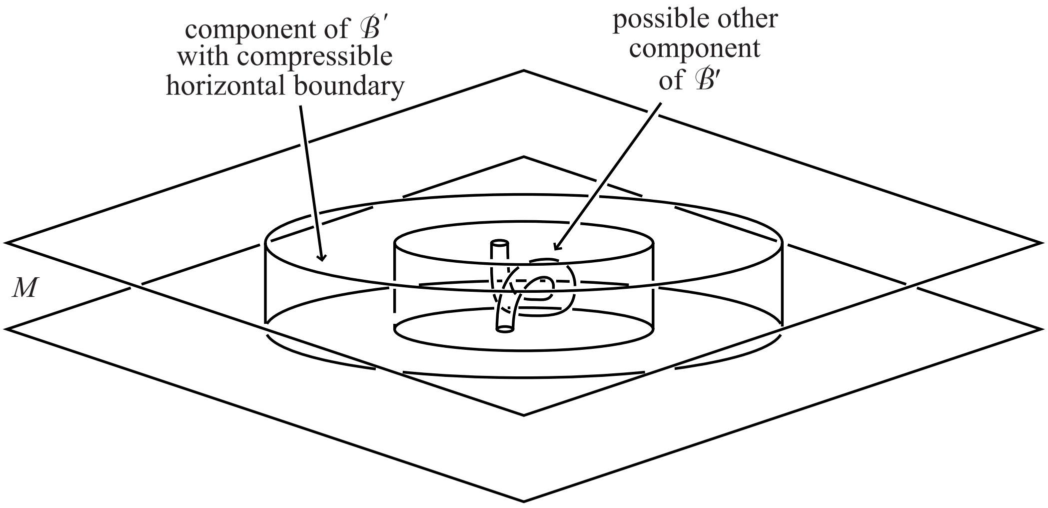
Recall: $\mathcal{B}$ is the parallelity bundle.

Enlarge this to a maximal generalised parallelity bundle $\mathcal{B}'$.

Claim 4: The ∂I-bundle of a maximal generalised parallelity bundle is incompressible.

Main idea of proof:

If the ∂I-bundle were compressible, then we would (probably) get an arrangement like:



$\longrightarrow$ Enlarge $\mathcal{B}'$ over (disc)$\times I$ region.

Contradicts maximality of $\mathcal{B}'$. $\square$ Claim 4.

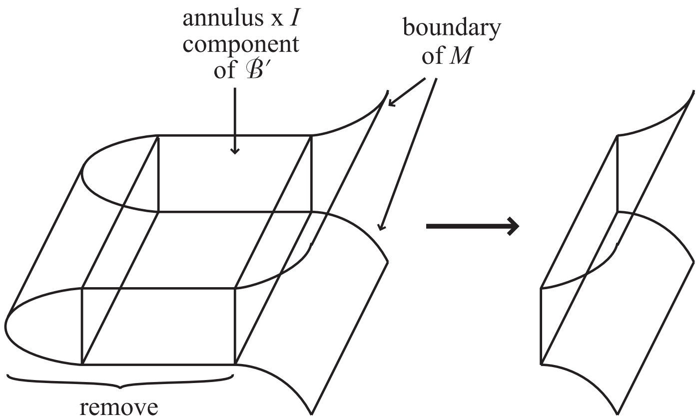
Because the horizontal boundary of $\mathcal{B}'$ is an incompressible subsurface of A, it is a union of discs and annuli parallel to core curves.

How to deal with annular components of the ∂I-bundle

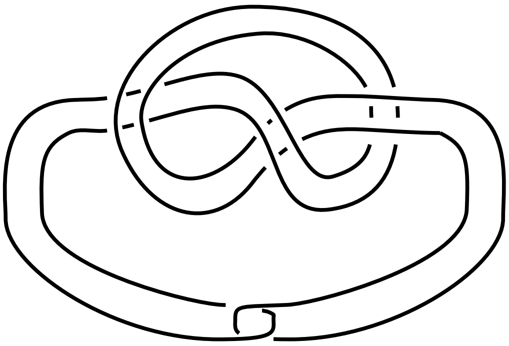
The corresponding component of $\mathcal{B}'$ is an I-bundle over an annulus.

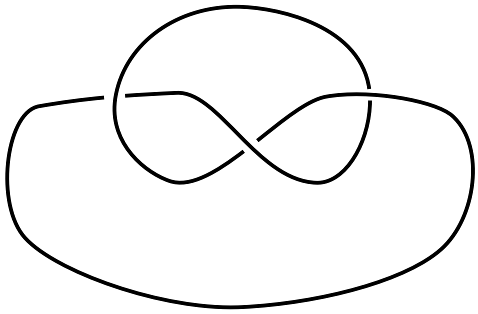
Its vertical boundary is two incompressible annuli in M.

Since each K_i is prime, these are boundary parallel in M.



Keep applying this sort of modification to the handle structure $\longrightarrow$ Claim 3. $\square$ Main Theorem.





Conjecture: $c(K) \geq c(L)$.

Theorem: [L] There is a universal computable constant $N \geq 1$ with following property. If K is a non-trivial satellite knot, with companion knot L, then $c(K) \geq c(L)/N$.