A polynomial upper bound on Reidemeister moves

Marc Lackenby

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Unknot recognition

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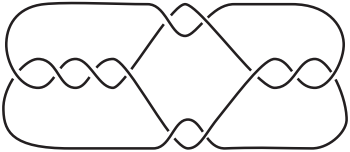

Goeritz's unknot

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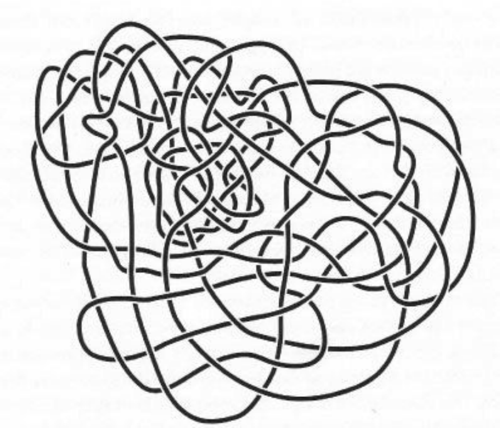

Haken's unknot

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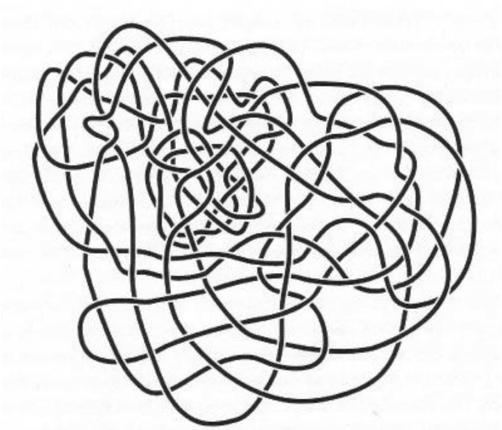
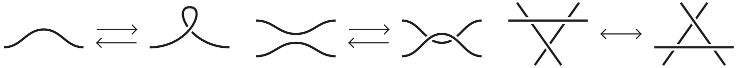


Haken's unknot

There is probably no simple way of doing so.

Reidemeister moves

Any two diagrams of a link differ by a sequence of Reidemeister moves:



If we knew in advance how many moves are required, we would have an algorithm to detect the unknot.

Computable upper bounds

Easy Theorem: The following are equivalent:

- There is an algorithm to decide whether a knot diagram represents the unknot.
- There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that, given an unknot diagram with n crossings, there is a sequence of at most $f(n)$ Reidemeister moves taking it to the trivial diagram.

Upper and lower bounds

Theorem: [Hass-Lagarias, 2001] Given a diagram of the unknot with n crossings, there is a sequence of at most 2^{kn} Reidemeister moves taking it to the trivial diagram, where $k = 10^{11}$.

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Problem: Is there a polynomial upper bound?

[Thurston, 2011] '*a lot of people have thought about this question . . . but this has been a very hard question to resolve*'

A polynomial upper bound

Theorem: [L, 2012] Let D be a diagram of the unknot with n crossings. Then there is a sequence of at most $(231n)^{11}$ Reidemeister moves that transforms D into the trivial diagram.

Haken's algorithm

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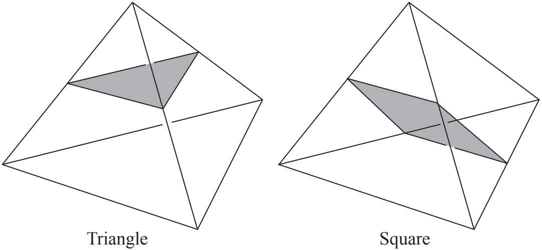
A surface properly embedded in a triangulated 3-manifold is normal if it intersects each tetrahedron in a collection of triangles and squares.



The normal surface equations

Associated to a normal surface S, there is a list of integers which count the number of triangles and squares of each type. This is the vector $[S]$.

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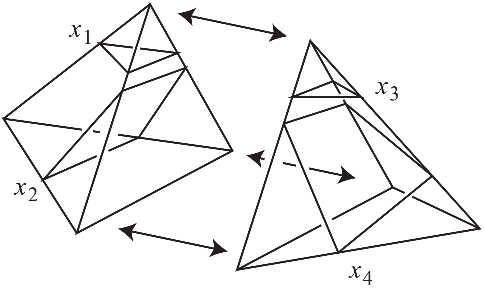
These vectors satisfy a system of equations, called the matching equations.



$$x_1 + x_2 = x_3 + x_4$$

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A normal surface is fundamental if it is not a sum of other normal surfaces.

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Haken's algorithm:

1. Construct a triangulation of the knot exterior.
2. Find all fundamental normal surfaces.
3. Check whether one is a spanning disc.

An exponential upper bound on Reidemeister moves

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Outline of their argument:

1. Construct a triangulation of the knot exterior from the diagram.
2. Find a spanning disc which is fundamental.
3. Slide the unknot over this disc.
4. Each slide across a triangle or square leads to a bounded number of Reidemeister moves.

From exponential to polynomial?

There is no way to improve the estimate on the number of triangles and squares.

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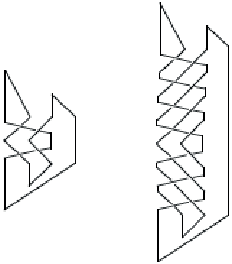
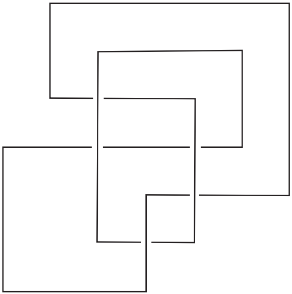
In fact:

Theorem: [Hass-Snoeyink-Thurston, 2001] There exist unknots consisting of $10n + 9$ straight arcs, for which any piecewise linear spanning disc must have at least 2^{n-1} triangular faces.



Rectangular diagrams

A rectangular diagram is a diagram which is a union of horizontal and vertical arcs, such that at each crossing, the over-arc is the vertical one.



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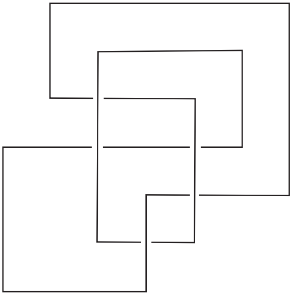
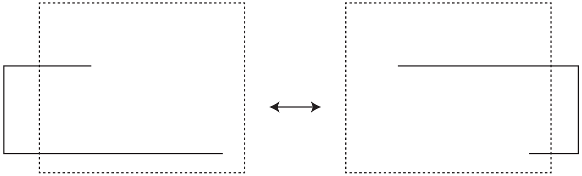


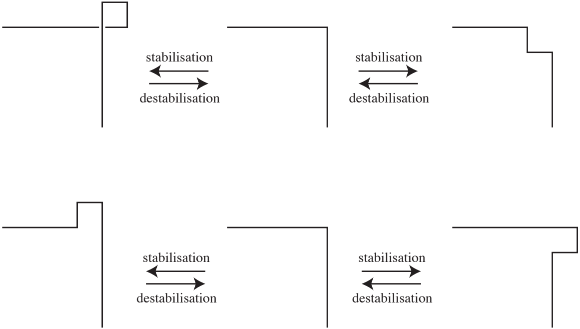
The number of horizontal (or vertical) arcs is the arc index of the diagram.

Moves on rectangular diagrams

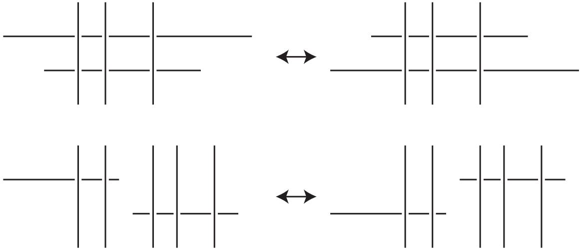


Cyclic permutation of the edges

Moves on rectangular diagrams



Stabilisations and destabilisations



Exchange move:

interchanging parallel edges of the rectangular diagram, as long as they have no edges between them, and their pairs of endpoints do not interleave.

Dynnikov's theorem

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Corollary: [Dynnikov, 2004] Given any diagram of the unknot with n crossings, there is a sequence of Reidemeister moves taking it to the trivial diagram, so that each diagram in this sequence has at most $2(n+1)^2$ crossings.

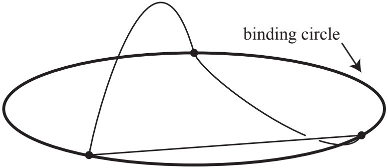
Arc presentations

Let S^1 be the unknot in S^3, called the binding circle.

Foliate the complement of the binding circle by open discs called pages.

A link L is in an arc presentation if

- it intersects the binding circle in finitely many points called vertices;
- it intersects each page in the empty set or a single arc joining distinct vertices.



Arc presentations and rectangular diagrams

There is a one-one correspondence

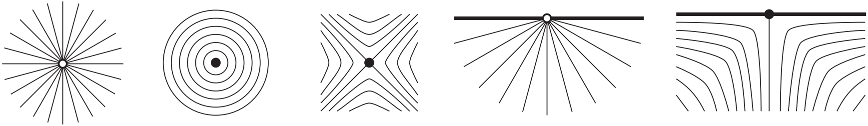
$$\text{arc presentations} \longleftrightarrow \begin{array}{c}\text{rectangular diagrams}\\ \text{up to cyclic permutation}\end{array}$$

Dynnikov's argument

Let S be the spanning disc for the unknot.

Then S inherits a singular foliation from its intersections with the pages.

Local pictures of the singular set:



(a) (b) (c) (d) (e)

(a): vertex of S (where it intersects the binding circle)
(b): local max/min of S (a 'pole')
(c): interior saddle of S
(d): boundary vertex of S
(e): boundary saddle of S

A separatrix is a component of a leaf with an endpoint in a saddle.

Dynnikov's argument

The valence of a vertex of S is the number of separatrices coming out of it.

An Euler characteristic argument implies that there is always one of:

- A pole
- A 2-valent interior vertex
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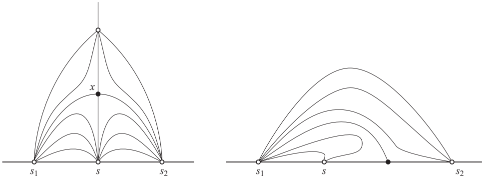
Each modification is achieved using exchange moves, cyclic permutations and possibly a destabilisation.

A 1-valent boundary vertex



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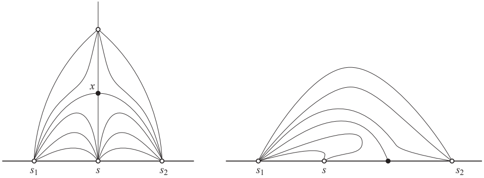
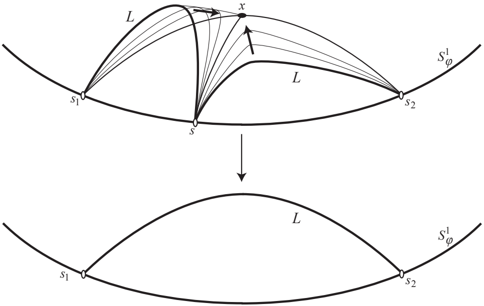
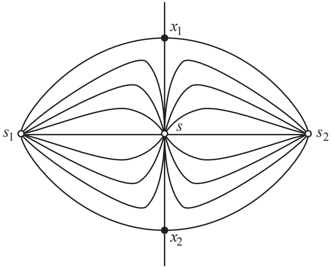




A 2-valent interior vertex



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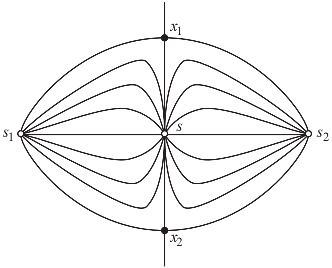
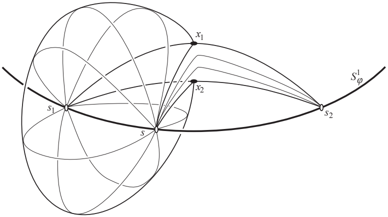




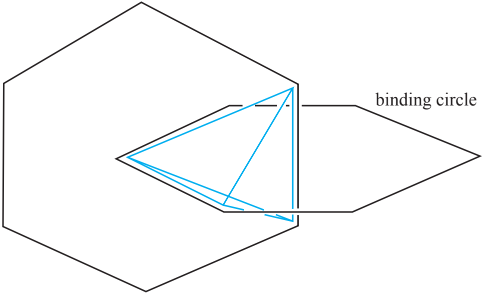
Main idea of proof

- Blend Dynnikov's methods with the use of normal surfaces.

A triangulation from an arc presentation

Fix an arc presentation of a link L with arc index n.

Then there is a triangulation of S^3 with n^2 tetrahedra in which L is simplicial.



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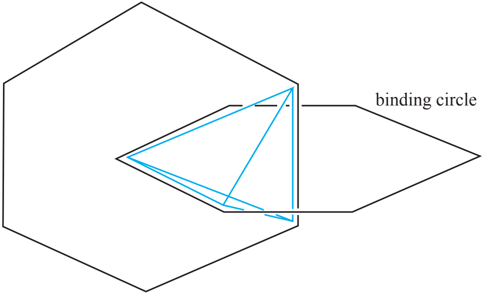

The binding circle is also simplicial.

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- Find a normal spanning disc with at most (roughly) 2^{343n^2} vertices.

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- We know that there is a 3-valent or 2-valent interior vertex or a 1-valent boundary vertex (we can ensure that it has no poles).
- Find large collection of these which have 'parallel' stars.
- Perform a single exchange move and reduce the number of singularities by a factor of roughly

$$\left(1 - \frac{1}{n^2}\right).$$

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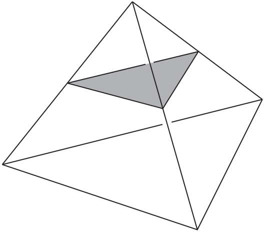
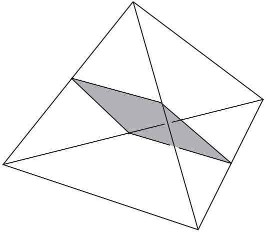


Triangle



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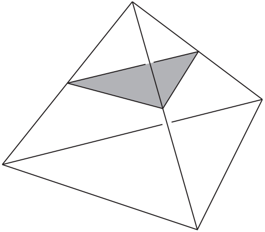

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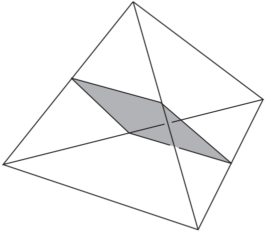

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The same principle applies to the stars of vertices of S.

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So, if there are V vertices of S, we would expect there to be a collection of at least

$$\frac{1}{5(7n)^2} V$$

3-valent/2-valent/1-valent vertices, all of which have parallel stars.

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But how do we prove this??

Exploiting Euler characteristic

The argument implying that there is a 3-valent or 2-valent interior vertex or a 1-valent boundary vertex actually implies that

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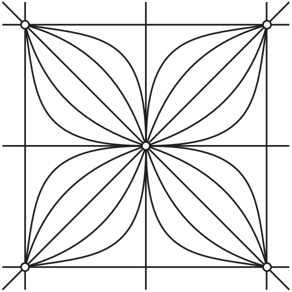
- or there are lots of vertices which 'contribute zero' to the Euler characteristic of S. An example of such a vertex:



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In the latter case, we show that these regions patch together to form a normal torus which forms a summand for the disc, contradicting an assumption that it is fundamental.

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This requires a subtle argument involving branched surfaces.

Split links

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Note that the polynomial is slightly smaller than in the case of the unknot.

This is because the boundary of the spanning disc causes technical difficulties.

Where next?

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- Can one find polynomial upper bounds on Reidemeister moves for other knot types?