HEEGAARD SPLITTINGS, THE VIRTUALLY HAKEN CONJECTURE AND PROPERTY (τ)

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1. Introduction

The behaviour of finite covers of 3-manifolds is a very important, but poorly understood, topic. There are three, increasingly strong, conjectures in the field that have remained open for over twenty years – the virtually Haken conjecture, the positive virtual b_1 conjecture and the virtually fibred conjecture. Any of these would have profound ramifications for 3-manifold theory. In this paper, we explore the interaction of these conjectures with the following seemingly unrelated areas: eigenvalues of the Laplacian, and Heegaard splittings.

We first give a necessary and sufficient condition, in terms of spectral geometry, for a finitely presented group to have a finite index subgroup with infinite abelianisation. This result in geometric group theory may have uses beyond 3-manifold theory. We also show that, for negatively curved 3-manifolds, this is equivalent to a statement about generalised Heegaard splittings. This provides one link between the positive virtual b_1 conjecture, Heegaard splittings and the Laplacian.

In a second direction, we define a new invariant of 3-manifolds: their Heegaard gradient. This measures the growth rate of the Heegaard genus of the manifold's finite covering spaces, as a function of their degree. We formulate a conjecture about Heegaard gradient, and provide some supporting evidence. We show that this, together with a conjecture of Lubotzky and Sarnak about Property (τ), would imply the virtually Haken conjecture for hyperbolic 3-manifolds. Property (τ) is a concept due to Lubotzky and Zimmer [37], which encodes the behaviour of the first eigenvalue of the Laplacian in finite covering spaces.

Along the way, we prove a number of unexpected theorems about 3-manifolds.

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For example, we show that for any closed 3-manifold that fibres over the circle with pseudo-Anosov monodromy, any cyclic cover dual to the fibre of sufficiently large degree has an irreducible, weakly reducible Heegaard splitting. Also, we establish lower and upper bounds on the Heegaard genus of the congruence covers of an arithmetic hyperbolic 3-manifold, which are linear in volume.

The virtually Haken conjecture and related conjectures

The following will motivate much of this paper.

Virtually Haken conjecture. *A compact orientable irreducible 3-manifold with infinite fundamental group is virtually Haken.*

This is important for many reasons. Haken manifolds have played a central rôle in 3-manifold theory for the past 40 years. Although many closed orientable irreducible 3-manifolds with infinite fundamental group are known to be non-Haken, it would be useful and satisfying to know that any such 3-manifold M was finitely covered by a Haken 3-manifold $\tilde{M}$. This would have many implications for M. Firstly, the embedded incompressible surface in $\tilde{M}$ would project to a π_1-injective immersed surface in M. Thus, $\pi_1(M)$ would contain the fundamental group of a closed orientable surface with positive genus. Secondly, many of the important properties that are known to hold for $\tilde{M}$ descend to properties of M. A notable example of this phenomenon is the theorem that virtually Haken 3-manifolds satisfy the geometrisation conjecture. For, Thurston showed that if a 3-manifold is Haken, then it satisfies the geometrisation conjecture [40] and it follows from the equivariant sphere theorem [38], the Seifert fibre space theorem ([7],[18],[54]) and the fact that virtually hyperbolic 3-manifolds are hyperbolic ([19],[20]) that if a compact orientable irreducible 3-manifold with infinite fundamental group is finitely covered by a 3-manifold satisfying the geometrisation conjecture, then it does also. There is the possibility that the geometrisation conjecture has recently been proved by other means, due to the work of Hamilton ([22], [23], [24], [25]) and Perelman ([42], [43], [44]). This is fortuitous, since there seems to be little hope of proving the virtually Haken conjecture without assuming that the manifold is geometric, or making some similar hypothesis. A solution to the virtually Haken conjecture just for hyperbolic 3-manifolds would be a considerable achievement in its own right.

The following implies the virtually Haken conjecture.

Positive virtual b_1 conjecture. *A compact orientable irreducible atoroidal 3-manifold with infinite fundamental group has a finite cover with positive first Betti number.*

A recent analysis by Dunfield and Thurston of the 10986 closed hyperbolic 3-manifolds in the Hodgson-Weeks census showed that they all satisfy this conjecture [13].

The following conjecture, due to Thurston [58], is stronger still.

Virtually fibred conjecture. *A compact orientable irreducible 3-manifold with boundary a (possibly empty) collection of tori, whose fundamental group is infinite and contains no $\mathbb{Z} \times \mathbb{Z}$ subgroup, is finitely covered by a surface bundle over the circle.*

This seems less likely to be true, since there are very few known examples of 3-manifolds that are virtually fibred but not fibred ([48], [33]).

Heegaard splittings

Casson and Gordon [6] introduced the notion of a weakly reducible Heegaard splitting and showed that if a compact orientable 3-manifold admits a splitting that is irreducible but weakly reducible, then it is Haken. This technology was developed by Scharlemann and Thompson [51], who introduced generalised Heegaard splittings, which are a decomposition of the 3-manifold along closed surfaces, together with Heegaard surfaces for the complementary regions. They defined a complexity for such splittings, and termed a generalised Heegaard splitting *thin* when it has minimal complexity. They showed that if an irreducible 3-manifold has a thin generalised Heegaard splitting that is not a Heegaard splitting, then the manifold contains a closed essential surface. In particular, it is Haken.

Our approach in this paper is to study the behaviour of Heegaard genus under finite covers. In fact, it is slightly more convenient to use the Euler characteristic of Heegaard surfaces. We therefore define the *Heegaard Euler characteristic* $\chi_-^h(M)$ of a compact orientable 3-manifold M to be the negative of the maximal Euler characteristic of a Heegaard surface. Of course, since Heegaard surfaces are always connected, $\chi_-^h(M)$ is linearly related to the more familiar Heegaard genus $g(M)$ by the formula $\chi_-^h(M) = 2g(M) - 2$. We also define the *strong Heegaard Euler characteristic* $\chi_-^{sh}(M)$ of M to be the negative of the maximal Euler characteristic of a strongly irreducible Heegaard surface, or infinity if M does not have such a Heegaard surface. One can study many interesting asymptotic quantities associated with these invariants, but we focus on two.

Definition. Let $\{M_i \to M\}$ be a collection of finite covers of a compact orientable 3-manifold M, with degree d_i. The *infimal Heegaard gradient* of $\{M_i \to M\}$ is

$$\inf_i \frac{\chi_-^h(M_i)}{d_i}.$$

The *infimal strong Heegaard gradient* of $\{M_i \to M\}$ is

$$\liminf_i \frac{\chi_-^{sh}(M_i)}{d_i},$$

which may be infinite. When the covers $\{M_i \to M\}$ are not mentioned, they are understood to be all the finite covers of M. For brevity, we sometimes drop the word 'infimal'.

A Heegaard splitting for M lifts to one for M_i, and its Euler characteristic is scaled by d_i. Thus, the Heegaard gradient of M is at most $\chi_-^h(M)$. Heegaard gradient therefore measures the degeneration of Heegaard Euler characteristic in finite sheeted covers. There are variants of the above definitions, where inf is replaced by lim inf, and where only regular covers are considered. We explore these in §3.

There exist hyperbolic 3-manifolds with zero Heegaard gradient. For example, $\chi_-^h(M)$ of a closed orientable 3-manifold M that fibres over the circle with fibre F is at most $2|\chi(F)| + 4$. Thus, by passing to cyclic covers M_i, we can make d_i arbitrarily large, but keep $\chi_-^h(M_i)$ bounded above. So, if a 3-manifold virtually fibres over the circle, then its infimal Heegaard gradient is zero. The following conjecture proposes that this is the only source of this phenomenon for hyperbolic 3-manifolds.

Heegaard gradient conjecture. *A compact orientable hyperbolic 3-manifold has zero Heegaard gradient if and only if it virtually fibres over the circle.*

Along similar lines, we also propose the following.

Strong Heegaard gradient conjecture. *Any closed orientable hyperbolic 3-manifold has positive strong Heegaard gradient.*

As we shall see, establishing either of these conjectures would be a step towards proving the virtually Haken conjecture for hyperbolic 3-manifolds. We will provide some supporting evidence for them in this paper.

Property (τ)

Property (τ) for finitely generated groups was defined by Lubotzky and Zimmer [37]. It has a number of equivalent definitions, relating to graph theory, differential geometry, and group representations, which are recalled in §2. In this paper, we will focus on its interpretation in terms of the smallest eigenvalue of the Laplacian, and the Cheeger constant. The following conjecture is central to this paper.

Conjecture. [Lubotzky-Sarnak] *The fundamental group of a finite volume hyperbolic 3-manifold fails to have Property (τ).*

Evidence for this conjecture is presented in [36]. Its motivation arises from the similarity between Property (τ) and Kazhdan's Property (T) [30], for which the corresponding assertion is true. Many finitely generated groups are known not to have Property (τ). For example, a finitely generated group with a finite index subgroup having infinite abelianisation does not have Property (τ). Hence, the positive virtual b_1 conjecture implies the Lubotzky-Sarnak conjecture. Another class of groups that fail to have Property (τ) are infinite, amenable, residually finite groups.

The main results

In §2, we establish a necessary and sufficient condition for a finitely presented group to have a finite index subgroup with infinite abelianisation, in terms of eigenvalues of the Laplacian, and the geometry of Schreier coset graphs. It is somewhat surprising that such a characterisation should exist.

Theorem 1.1. *Let G be a finitely presented group, and let S be a finite set of generators. Let $\{G_i\}$ be its finite index subgroups, and let $N(G_i)$ be the normaliser of G_i in G. Let X_i be the Schreier coset graph of G/G_i induced by S. Then the following are equivalent, and are independent of the choice of S:*

1. *some G_i has infinite abelianisation;*
2. *G_i has infinite abelianisation for infinitely many i;*
3. *$\lambda_1(X_i)[G : N(G_i)]^2[G : G_i]^2$ has a bounded subsequence;*
4. *$\lambda_1(X_i)[G : N(G_i)]^4[N(G_i) : G_i]$ has zero infimum;*
5. *$h(X_i)[G : N(G_i)][G : G_i]$ has a bounded subsequence;*
6. *$h(X_i)[G : N(G_i)]^2[N(G_i) : G_i]^{1/2}$ has zero infimum.*

Furthermore, if G is the fundamental group of some closed orientable Riemannian manifold M, and M_i is the cover of M corresponding to G_i, then the above are also equivalent to each of the following:

7. *$\lambda_1(M_i)[G : N(G_i)]^2[G : G_i]^2$ has a bounded subsequence;*
8. *$\lambda_1(M_i)[G : N(G_i)]^4[N(G_i) : G_i]$ has zero infimum;*
9. *$h(M_i)[G : N(G_i)][G : G_i]$ has a bounded subsequence;*
10. *$h(M_i)[G : N(G_i)]^2[N(G_i) : G_i]^{1/2}$ has zero infimum.*

For a finite graph or Riemannian manifold, λ_1 denotes the smallest non-zero eigenvalue of the Laplacian, and h is its Cheeger constant. The definitions of these terms are recalled in §2.

Note that $[G : G_i]$ and $[N(G_i) : G_i]$ have the following simple topological interpretations: $[G : G_i]$ is the number of vertices of X_i; and $[N(G_i) : G_i]$ is the number of covering translations of X_i. Note also that if G_i is a normal subgroup of G, then $[G : N(G_i)]$ is one.

Theorem 1.1 should be compared with the definition of Property (τ), which is also recalled in §2.

A sample application of the methods behind Theorem 1.1 gives the following result. Recall that a group presentation is *triangular* if each relation has length three. Any finitely presented group has a finite triangular presentation.

Theorem 1.2. *Let X be the Cayley graph of a finite group, arising from a finite triangular group presentation. Then $h(X) \geq \sqrt{2/(3|V(X)|)}$.*

In §3, we study generalised Heegaard splittings. In [51], Scharlemann and Thompson defined a complexity for a compact orientable 3-manifold M, which measures the minimal complexity of a generalised Heegaard splitting. It is a finite multi-set of integers. The largest integer in this multi-set, minus 1, is denoted by $c_+(M)$. When M is closed, irreducible, non-Haken and not S^3, this is equal to $\chi_-^h(M)$ and $\chi_-^{sh}(M)$. In all cases, other than when M is S^3 or a 3-ball, $c_+(M) \leq \chi_-^h(M)$. More details can be found in §3. In §4, we find an upper bound on the Cheeger constant for a negatively curved 3-manifold M in terms of $c_+(M)$. Using Theorem 1.1, this allows us reformulate the positive virtual b_1 conjecture for negatively curved 3-manifolds in terms of generalised Heegaard splittings.

Theorem 1.3. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric. Let $\{M_i \to M\}$ be the finite covers of M. Then the following are equivalent, and are each equivalent to the conditions in Theorem 1.1:*

1. *$b_1(M_i) > 0$ for infinitely many i;*
2. *$c_+(M_i)[\pi_1 M : N(\pi_1 M_i)]$ has a bounded subsequence;*
3. *$c_+(M_i)[\pi_1 M : N(\pi_1 M_i)][N(\pi_1 M_i) : \pi_1 M_i]^{-1/2}$ has zero infimum.*

Using (3) $\Rightarrow$ (1), this has the following immediate corollary.

Corollary 1.4. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric. Suppose that, for some collection of finite regular covers $\{M_i \to M\}$ with degree d_i, $\inf \chi_-^h(M_i)/\sqrt{d_i} = 0$. Then M satisfies the positive virtual b_1 conjecture.*

As another consequence of the relation of the Cheeger constant to Heegaard Euler characteristic, we obtain the following.

Theorem 1.5. *Let M be an orientable 3-manifold that admits a complete, negatively curved, finite volume Riemannian metric. Let $\{M_i \to M\}$ be a collection of finite covers of M. If $\pi_1(M)$ has Property (τ) with respect to $\{\pi_1(M_i)\}$, then the infimal Heegaard gradient of $\{M_i \to M\}$ is non-zero.*

The fundamental group of an arithmetic hyperbolic 3-manifold has Property (τ) with respect to its congruence subgroups [35]. Hence, we have the following immediate corollary.

Corollary 1.6. *Let M an arithmetic hyperbolic 3-manifold. Then there are positive constants c and C, such that for any congruence cover $M_i \to M$,*

$$c \text{ Volume}(M_i) \leq \chi_-^h(M_i) \leq C \text{ Volume}(M_i).$$

This may be viewed as supporting evidence for the strong Heegaard gradient conjecture, since it provides a large collection of finite covers with non-zero Heegaard gradient and hence non-zero strong Heegaard gradient.

In §5, we relate Property (τ), Heegaard splittings and the virtually Haken conjecture, by proving the following theorem. This may represent a viable approach to the virtually Haken conjecture for hyperbolic 3-manifolds.

We term a collection of covers $\{M_i \to M\}$ a *lattice* if, whenever $M_i \to M$ and $M_j \to M$ lies in the collection, so does the cover corresponding to $\pi_1(M_i) \cap \pi_1(M_j)$. For example, the set of all finite regular covers of M forms a lattice.

Theorem 1.7. *Let M be a compact orientable irreducible 3-manifold, with boundary a (possibly empty) collection of tori. Let $\{M_i \to M\}$ be a lattice of finite regular covers of M. Suppose that*

1. *$\pi_1(M)$ fails to have Property (τ) with respect to $\{\pi_1(M_i)\}$, and*
2. *$\{M_i \to M\}$ has non-zero infimal strong Heegaard gradient.*

Then, for infinitely many i, M_i has a thin generalised Heegaard splitting which is not a Heegaard splitting, and so contains a closed essential surface. In particular, M is virtually Haken.

As a consequence, the virtually Haken conjecture for finite volume hyperbolic 3-manifolds would follow from the Lubotzky-Sarnak conjecture, together with either the Heegaard gradient conjecture or the strong Heegaard gradient conjecture.

An immediate consequence of Theorems 1.5 and 1.7 is the following result.

Theorem 1.8. *Let $\{M_i \to M\}$ be a lattice of finite regular non-Haken covers of a closed orientable negatively curved 3-manifold M. Then the following are equivalent:*

1. *$\{M_i \to M\}$ has non-zero Heegaard gradient;*
2. *$\{M_i \to M\}$ has non-zero strong Heegaard gradient;*
3. *$\pi_1(M)$ has Property (τ) with respect to $\{\pi_1(M_i)\}$.*

It is striking that (3) above is a group-theoretic property of $\pi_1(M)$ and $\{\pi_1(M_i)\}$, whereas this is not at all obvious of either (1) or (2).

In §6, we investigate the diameter of minimal surfaces in negatively curved 3-manifolds. An immediate application is the following, which provides some evidence for the strong Heegaard gradient conjecture.

Theorem 1.9. *Let M be a closed orientable hyperbolic 3-manifold, and let $\{M_i \to M\}$ be the cyclic covers dual to some non-trivial element of $H_2(M)$. Then the strong Heegaard gradient of $\{M_i \to M\}$ is non-zero.*

Thus, in the above case, the covers $\{M_i \to M\}$ satisfy each of the requirements of Theorem 1.7. Of course, the final conclusion of Theorem 1.7 also holds trivially, but this shows that the hypotheses of Theorem 1.7 are often satisfied.

Theorem 1.9 has the following interesting implication for Heegaard splittings of hyperbolic manifolds that fibre over the circle. Rubinstein has independently proved a stronger version of this result [50].

Corollary 1.10. *Let M be a closed orientable 3-manifold that fibres over the circle with pseudo-Anosov monodromy. Let $\{M_i \to M\}$ be the cyclic covers dual to the fibre. Then, for all but finitely many i, M_i has an irreducible, weakly reducible, minimal genus Heegaard splitting.*

In §7, we continue with the study of cyclic coverings. The technical result in §6 is used to prove the following theorem, which supports the Heegaard gradient conjecture.

Theorem 1.11. *Let M be a compact orientable finite volume hyperbolic 3-manifold, and let $\{M_i \to M\}$ be the cyclic covers dual to some non-trivial element*

z of $H_2(M, \partial M)$. Then, the infimal Heegaard gradient of $\{M_i \to M\}$ is zero if and only if z is represented by a fibre.

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2. Positive virtual b_1 and spectral geometry

In this section, we prove Theorem 1.1, which gives a number of equivalent geometric characterisations of positive virtual b_1 for a finitely presented group. It is stated in terms of the Cheeger constant and the first eigenvalue of the Laplacian of Cayley graphs and Riemannian manifolds. We now recall the definitions of these terms.

Given a group G with a finite set S of generators, and a subgroup H, the *Schreier coset graph* (or *coset diagram*) of G/H with respect to S is defined as follows. It has a vertex for each right coset Hg, and vertices Hg_1 and Hg_2 are joined by an oriented edge if and only if $g_2 = g_1 s$ for some $s \in S$. When $H = 1$, this gives the *Cayley graph* of G with respect to S. The Cayley graph of the cyclic group of order 6, with respect to the generators $\{1, 2\}$, is shown in Figure 1.

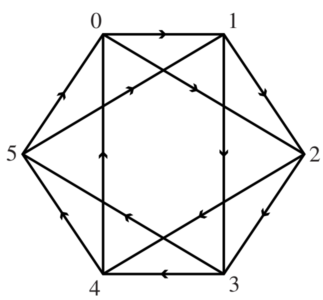

Figure 1.

Note that left multiplication by any element of G induces an automorphism of a Cayley graph for G.

Given a finite graph X, we denote its vertex set by $V(X)$ and its edge set by $E(X)$. The *Cheeger constant* of X, denoted $h(X)$, is defined to be

$$h(X) = \inf_{\substack{A \subset V(X) \\ A \neq \emptyset, V(X)}} \frac{|\partial A|}{\min(|A|, |V(X) - A|)},$$

where, for a subset A of $V(X)$, ∂A is the set of edges connecting a vertex in A to one not in A. (See Figure 2.)

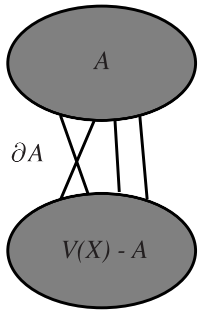

Figure 2.

The Cheeger constant of a Riemannian manifold M is defined similarly. One considers all possible codimension one submanifolds F that separate M into two manifolds M_1 and M_2. Then the *Cheeger constant* $h(M)$ is

$$h(M) = \inf_F \frac{\text{Area}(F)}{\min(\text{Volume}(M_1), \text{Volume}(M_2))}.$$

For a finite graph X, the smallest non-zero eigenvalue of the Laplacian is denoted $\lambda_1(X)$. It is given by the formula

$$\inf \left\{ \frac{||df||^2}{||f||^2} : f \text{ is a real-valued function on } V(X) \text{ such that } \sum_{v \in V(X)} f(v) = 0 \right\},$$

where

$$||f||^2 = \sum_{v \in V(X)} |f(v)|^2$$
$$||df||^2 = \sum_{e \in E(X)} |f(e_+) - f(e_-)|^2.$$

Here, we have picked an arbitrary orientation on each edge e of X, and have denoted its target and source vertex by e_+ and e_-. It is clear that the formula for $\lambda_1(X)$ is independent of this choice of orientation.

Similarly, if M is a closed Riemannian manifold, then $\lambda_1(M)$, the smallest non-zero eigenvalue of the Laplacian, is given by

$$\lambda_1(M) = \inf \left\{ \frac{\int_M |df|^2}{\int_M |f|^2} : f \in C^\infty(M), f \not\equiv 0 \text{ and } \int_M f = 0 \right\}.$$

The above formula for $\lambda_1(X)$ can be interpreted as follows. A subset A of $V(X)$, as in the definition of the Cheeger constant $h(X)$, determines (up to a multiplicative factor) a step function $f: V(X) \rightarrow \mathbb{R}$, with a single positive value on A and a single negative value on $V(X) - A$, such that the sum of $f(v)$ over all vertices v in $V(X)$ is zero. The quantity $||df||^2/||f||^2$, which appears in the definition of $\lambda_1(X)$, is clearly a close relative of $|\partial A| / \min(|A|, |V(X) - A|)$, which appears in the definition of $h(X)$. However, the formula for $\lambda_1(X)$ considers all functions $f: V(X) \rightarrow \mathbb{R}$ that sum to zero, rather than just two-valued step functions. In fact, the Cheeger constant and first eigenvalue of the Laplacian are known to be related by results of Cheeger [8], Buser [5], Brooks [4], Alon [1], Dodziuk [10], Tanner [56], Alon and Milman [2]. A consequence of this relationship is the following result, which can be found in [35]. This gives many equivalent definitions of Property (τ), which was first defined by Lubotzky and Zimmer [37]. This should be compared with Theorem 1.1.

Theorem. [35] *Let G be a finitely generated group, generated by a finite symmetric set of generators S. Let $\{G_i\}$ be a collection of finite index normal subgroups. Then the following conditions are equivalent, are independent of the choice of S, and are known as Property (τ) for G with respect to $\{G_i\}$:*

1. *There exists $\epsilon_1 > 0$ such that if $\rho: G \rightarrow Aut(H)$ is a non-trivial unitary irreducible representation of G whose kernel contains G_i for some i, then for every $v \in H$ with $||v|| = 1$, there exists an $s \in S$ such that $||\rho(s)v - v|| \geq \epsilon_1$.*

2. *There exists $\epsilon_2 > 0$ such that all the Cayley graphs X_i of G/G_i with respect to S are $([G : G_i], |S|, \epsilon_2)$-expanders.*

3. *There exists $\epsilon_3 > 0$ such that $h(X_i) \geq \epsilon_3$.*

4. *There exists* $\epsilon_4 > 0$ *such that* $\lambda_1(X_i) \geq \epsilon_4$.

If, in addition, $G = \pi_1(M)$ *for some compact Riemannian manifold* M*, and* M_i *are the finite sheeted covers corresponding to* G_i*, then the above conditions are equivalent to each of the following:*

5. *There exists* $\epsilon_5 > 0$ *such that* $h(M_i) \geq \epsilon_5$.

6. *There exists* $\epsilon_6 > 0$ *such that* $\lambda_1(M_i) \geq \epsilon_6$.

Property (τ) is usually defined, as above, for finite index *normal* subgroups of a finitely generated group G. But, one can extend it to collections $\{G_i\}$ of finite index subgroups, some of which may not be normal, by taking any of (2) - (6) above as the definition, and where X_i now denotes the Schreier coset diagram of G/G_i with respect to S.

Thus, the failure of Property (τ) asserts that $\lambda_1(X_i)$, $h(X_i)$, $\lambda_1(M_i)$ and $h(M_i)$ all have subsequences that tend to zero. The force of Theorem 1.1 is that if the convergence is fast enough (when measured by $[G : G_i]$ and $[N(G_i) : G_i]$), then we can deduce positive virtual b_1.

We now embark on the proof of Theorem 1.1. Given a finite graph X and two disjoint subsets V_1 and V_2 of $V(X)$, let $e(V_1, V_2)$ denote the number of edges that have an endpoint in V_1 and an endpoint in V_2.

Lemma 2.1. *Let* X *be a Cayley graph of a finite group. Let* A *be a non-empty subset of* $V(X)$ *such that* $|\partial A|/|A| = h(X)$*. Then* $|A| > |V(X)|/4$.

Proof. Consider a smallest non-empty set of vertices A satisfying $|\partial A|/|A| = h(X)$. Suppose that $|A| \leq |V(X)|/4$. Let B be the image of A under the left action of an arbitrary group element. Then

$$\begin{aligned}
&|\partial A| + |\partial B| - |\partial(A \cap B)| \\
&\quad = e(A, A^c) + e(B, B^c) - e(A \cap B, A^c \cup B^c) \\
&\quad = e(A \cap B, A^c \cap B^c) + e(A \cap B, B \backslash A) + e(A \backslash B, A^c \cap B^c) + e(A \backslash B, B \backslash A) \\
&\qquad + e(A \cap B, A^c \cap B^c) + e(A \cap B, A \backslash B) + e(B \backslash A, A^c \cap B^c) + e(B \backslash A, A \backslash B) \\
&\qquad - e(A \cap B, A \backslash B) - e(A \cap B, B \backslash A) - e(A \cap B, A^c \cap B^c)
\end{aligned}$$

$$= e(A \cup B, A^c \cap B^c) + 2e(A\backslash B, B\backslash A)$$
$$= |\partial(A \cup B)| + 2e(A\backslash B, B\backslash A).$$

Now, $|A \cap B| \leq |A|$, and so by the minimality of A,

$$|\partial(A \cap B)| \geq h(X)|A \cap B|,$$

with equality if and only if $A \cap B = \emptyset$ or $A = B$. Therefore, we deduce that

$$|\partial(A \cup B)| = |\partial A| + |\partial B| - |\partial(A \cap B)| - 2e(A\backslash B, B\backslash A)$$
$$\leq h(X)(|A| + |B| - |A \cap B|)$$
$$= h(X)|A \cup B|.$$

This must be an equality, since $A \cup B$ has size at most $|V(X)|/2$. Hence, we deduce that either A equals B or they are disjoint, and in the latter case, there can be no edges joining A to B. Hence, the images of A under the left action of the group form a disjoint union of copies of A with no edges between them. This implies the graph is disconnected, which is impossible. □

Lemma 2.2. *Let X be a Cayley graph of a finite group. Let A be a non-empty subset of $V(X)$ such that $|\partial A|/|A| = h(X)$ and $|A| \leq |V(X)|/2$. Then the subgraphs induced by A and its complement are connected.*

Proof. Suppose that the subgraph induced by A is not connected. Let A_1 be the vertices of one component, and let A_2 be the remaining vertices of A. Then,

$$|\partial A_1| + |\partial A_2| = |\partial A| = h(X)|A| = h(X)|A_1| + h(X)|A_2|.$$

But, $|\partial A_i| \geq h(X)|A_i|$ for each i, and hence, this must be an equality. However some A_i has at most $|V(X)|/4$ vertices, which contradicts Lemma 2.1.

Now suppose that the subgraph induced by the complement of A is not connected. Let B_1 be the vertices of one component, and let B_2 the remaining vertices of the complement. Now, $|B_1|$ and $|B_2|$ must each be less than $|V(X)|/2$. For, if $|B_1| \geq |V(X)|/2$, say, then we may add B_2 to A to obtain a collection of at most $|V(X)|/2$ vertices, having more vertices than $|A|$, but smaller boundary, contradicting the definition of $h(X)$. If we apply the definition of $h(X)$ to B_1 and B_2, we obtain,

$$e(A, B_1) \geq h(X)|B_1|$$
$$e(A, B_2) \geq h(X)|B_2|.$$

Hence,

$$|A| = (e(A, B_1) + e(A, B_2))/h(X) \geq |B_1| + |B_2|.$$

But, $|A| + |B_1| + |B_2| = |V(X)|$, which implies that $|A| \geq |V(X)|/2$. Thus, $|A| = |V(X)|/2$ and each of the above inequalities are equalities. But, some B_i has at most $|V(X)|/4$ vertices, which by Lemma 2.1, implies that $e(A, B_i) > h(X)|B_i|$, a contradiction. □

Lemma 2.3. *Let S and $\overline{S}$ be finite sets of generators for a group G. Let $\{G_i\}$ be a collection of finite index subgroups of G. Let X_i and $\overline{X}_i$ be the Schreier coset graphs of G/G_i with respect to S and $\overline{S}$. Then, there is a constant $C \geq 1$, depending only on S and $\overline{S}$, such that*

$$C^{-1}h(X_i) \leq h(\overline{X}_i) \leq C\ h(X_i).$$

Proof. By first changing S to $S \cup \overline{S}$ and then $S \cup \overline{S}$ to $\overline{S}$, it suffices to consider the case where $S \subset \overline{S}$. Then $h(X_i) \leq h(\overline{X}_i)$, establishing one of the inequalities. Let $S^{(n)}$ be the set of words in $S \cup S^{-1}$ with length at most n. Then $\overline{S} \subset S^{(n)}$ for some n. Since $S^{(n)} \subset (\dots(S^{(2)})^{(2)}\dots)^{(2)}$, it suffices to establish the lemma in the case where $\overline{S} = S^{(2)}$. Let A be a subset of $V(X_i)$ such that $h(X_i) = |\partial A|/|A|$ and $|A| \leq |V(X_i)|/2$. Consider the corresponding subset $\overline{A}$ of $V(\overline{X}_i)$. An edge e of $\partial\overline{A}$ joins a vertex in $\overline{A}$ to one not in $\overline{A}$. These two vertices differ by an element of $\overline{S}$, which is either an element of S, or product of two elements of S. In the first case, we obtain a corresponding edge of ∂A. In the second case, we obtain two edges e_1 and e_2 of X_i, one of which (e_1, say) must be in ∂A. The other, e_2, must be attached to e_1, and so there are at most $4|S| - 2$ choices for e_2, given e_1. Hence, $|\partial\overline{A}| \leq (4|S| - 1)|\partial A|$. So,

$$h(\overline{X}_i) \leq (4|S| - 1)h(X_i),$$

which proves the lemma. □

Recall that any finitely presented group has a finite triangular presentation. This is most easily seen by viewing the group as the fundamental group of a finite 2-complex with a single vertex. One subdivides all faces with more than three edges into triangles, without adding any vertices. One way to deal with faces with two or fewer edges is to add to the complex a 1-cell g and a 2-cell attached along the word g^2g^{-1}. Then we enlarge any face with two or fewer edges to triangles by adding one or more letters g to its boundary. The resulting 2-complex has the same fundamental group and specifies a triangular presentation.

A *meta-cocycle* for a graph is a 1-cochain for which any closed path of length at most three evaluates to zero. If the graph is the 1-skeleton of a 2-complex K where every 2-cell is a triangle, a meta-cocycle determines a genuine cocycle on K.

Lemma 2.4. *Let X be the Cayley graph of a finite group G. Suppose that $h(X) < \sqrt{2/(3|V(X)|)}$. Then X admits a meta-cocycle that is not a co-boundary.*

Proof. Let A be a non-empty subset of $V(X)$ such that $|\partial A|/|A| = h(X)$ and $|A| \leq |V(X)|/2$.

Claim. There are elements $g_1, \ldots, g_4$ of G such that $g_i(\partial A) \cap g_j(\partial A) = \emptyset$ if $i \neq j$.

Since X is a Cayley graph, each edge is oriented. Let C denote initial vertices of ∂A. As $V(X)$ is in one-one correspondence with G, we view C as a set of elements of G. Suppose that $g(\partial A) \cap h(\partial A) \neq \emptyset$ for some g and h in G. Then $gC \cap hC \neq \emptyset$, and so $h^{-1}g \in CC^{-1}$. Suppose that the claim were not true. Then, each 4-tuple $(g_1, \ldots, g_4) \in G^4$ would have $g_iC \cap g_jC \neq \emptyset$ for some $i \neq j$. For $1 \leq i < j \leq 4$, define

$$p_{ij}\colon G^4 \to G$$
$$(g_1, \ldots, g_4) \mapsto g_j^{-1}g_i.$$

Then, the sets $p_{ij}^{-1}(CC^{-1})$ cover G^4. Each set has size $|G|^3|CC^{-1}|$, and so

$$|G|^4 \leq \binom{4}{2} |G|^3|C|^2,$$

which implies that

$$|G| \leq 6|C|^2 \leq 6|\partial A|^2 = 6(h(X)|A|)^2 < 6\left(\sqrt{\frac{2}{3|V(X)|}}\frac{|V(X)|}{2}\right)^2 = |G|,$$

which is a contradiction. This proves the claim.

Since $|A| > |V(X)|/4$, by Lemma 2.1, and i and j range from 1 to 4, we have $g_iA \cap g_jA \neq \emptyset$ for some $i \neq j$. Since $|A| \leq |V(X)|/2$, $g_i(V(X) - A) \cap g_j(V(X) - A) \neq$

$\emptyset$. Let c' be the coboundary of the characteristic function of $g_i A$, considered as a 0-cochain on G. This has support precisely $g_i(\partial A)$. Let c be the cochain that agrees with c' on those edges with both endpoints in $g_j A$, and is zero elsewhere. (See Figure 3.)

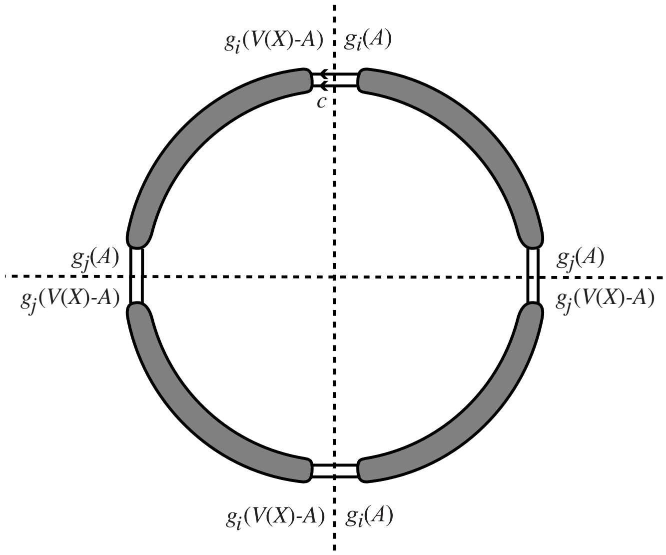

Figure 3.

Claim. c is a meta-cocycle.

Loops of length one or two trivially evaluate to zero. Consider a loop ℓ of length three in X. We must show that it evaluates to zero under c. Since c' is a coboundary, the evaluation of ℓ under c' is zero. As c' only takes values in $\{-1, 0, 1\}$, ℓ must run over exactly two edges in the support of c'. These edges lie in $g_i(\partial A)$. Neither of these edges can lie in $g_j(\partial A)$. Hence, either all or none of the vertices of ℓ lie in $g_j(A)$. In the former case, c agrees with c' on ℓ, and hence evaluates to zero. In the latter case, c is zero on ℓ, by definition.

Claim. There are edges e_1 and e_2 in $g_i(\partial A)$, such that the endpoints of e_1 both lie in $g_j(A)$, whereas neither endpoint of e_2 lies in $g_j(A)$.

There exists a vertex in $g_i(A) \cap g_j(A)$. Now, $g_i(A) \neq g_j(A)$, since $g_i(\partial A) \cap g_j(\partial A) = \emptyset$. So there exists a vertex in $g_j(A) - g_i(A)$. By Lemma 2.2, we may pick a path in the subgraph induced by $g_j(A)$ between these two vertices. This must run across an edge in $g_i(\partial A)$, which we may take to be e_1. To find e_2, apply a similar argument joining vertices in $g_i(V(X) - A) \cap g_j(V(X) - A)$ and

$g_i(A) - g_j(A)$ by a path in $g_j(V(X) - A)$.

Claim. c is not a co-boundary.

It suffices to give a loop in X which has non-zero evaluation. Now, the subgraphs induced by $g_i(A)$ and $g_i(V(X) - A)$ are both connected, by Lemma 2.2, and so there are arcs α_1 and α_2 in these subgraphs, joining the endpoints of e_1 and e_2. The resulting loop $e_1 \cup \alpha_1 \cup e_2 \cup \alpha_2$ has non-zero evaluation under c, since its intersection with the support of c is e_1. □

Proof of Theorem 1.2. Let K be the Cayley 2-complex arising from the finite triangular presentation, and let X be its 1-skeleton. If $h(X) < \sqrt{2/(3|V(X)|)}$, then by Lemma 2.4, X admits a meta-cocycle that is not a co-boundary. Hence, $H^1(K)$ is infinite, but $\pi_1(K)$ is trivial, which is a contradiction. □

At several points in the proof of Theorem 1.1, we will need to consider the following situation. Let G be a finitely generated group, and let $H \triangleleft K \leq G$ be finite index subgroups. Let $X(G/H)$ and $X(G/K)$ be the Schreier coset diagrams for G/H and G/K with respect to some finite generating set S for G. Now the group K/H acts on the right cosets G/H by left multiplication, since H is normal in K, and hence K/H acts freely on $X(G/H)$. The quotient of $X(G/H)$ by this action is $X(G/K)$, and the quotient map $X(G/H) \to X(G/K)$ is a covering map. Pick a maximal tree T in $X(G/K)$. Its inverse image in $X(G/H)$ is a forest F. If we collapse each component of this forest to a single vertex, we obtain a Cayley graph $X(K/H)$ for K/H. (See Figure 4.)

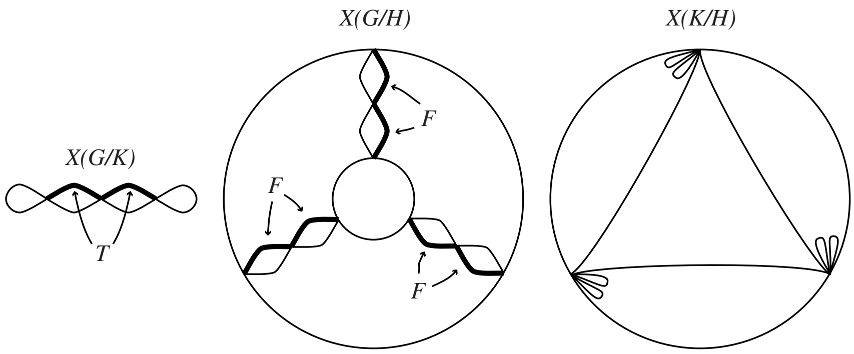

Figure 4.

Lemma 2.5. *With the above notation,*

$$\lambda_1(X(G/H)) \leq \lambda_1(X(K/H))\ [G:K]^{-1}.$$

Provided $h(X(G/H)) \leq [G:K]^{-1}/2$,

$$h(X(K/H)) \leq 4\ h(X(G/H))\ [G:K]^2\ |S|.$$

Proof. Let $p\colon X(G/H) \to X(K/H)$ be the quotient map. Any real-valued function f on $V(X(K/H))$ induces a function $f \circ p = \tilde{f}$ on $V(X(G/H))$. If f sums to zero, so does $\tilde{f}$. This is because every component of F has the same number of vertices, $[G:K]$. Also,

$$||d\tilde{f}||^2 = ||df||^2$$
$$||\tilde{f}||^2 = [G:K]\ ||f||^2.$$

The first inequality of the lemma follows immediately.

We now prove the second inequality of the lemma. Let A be a non-empty subset of $V(X(G/H))$ such that $|A| \leq |V(X(G/H))|/2$ and $h(X(G/H)) = |\partial A|/|A|$. Then $|A| \geq 1/h(X(G/H))$. We use A to construct a subset B of $V(X(K/H))$. If some component of the forest F has all its vertices in A (respectively, all its vertices in A^c), we place the corresponding vertex of $X(K/H)$ in B (respectively, B^c). We place the remaining vertices of $X(K/H)$ in B or B^c, so that $|B|$ is the largest integer less than or equal to $|A|/[G:K]$. So, $|B| \leq |V(X(K/H))|/2$. Also,

$$|B| \geq \frac{|A|}{[G:K]} - 1 \geq \frac{|A|}{2[G:K]}.$$

The second inequality holds, since

$$\frac{|A|}{2[G:K]} \geq \frac{1/h(X(G/H))}{2[G:K]} \geq 1,$$

by assumption.

We now wish to estimate $|\partial B|$. Each edge e' in ∂B comes from a single edge e in $X(G/H)$ with endpoints in distinct components of F. If e lies in ∂A, we have a contribution to ∂A. If e does not lie in ∂A, at least one of the components of F at its endpoints does not lie wholly in A or wholly in A^c. Hence, this component of F contains an edge e'' in ∂A. The number of edges e' that can count towards e'' is at most the number of edges emanating from a component of F, which is less than $2|S|[G:K]$. So, $|\partial B| \leq 2|S|[G:K]|\partial A|$. Hence,

$$h(X(K/H)) \leq \frac{|\partial B|}{|B|} \leq \frac{2|S|[G:K]|\partial A|}{|A|/(2[G:K])} = 4h(X(G/H))\ [G:K]^2\ |S|,$$

which proves the lemma. □

Proof of Theorem 1.1. (1) ⇒ (2): If some G_i has infinite abelianisation, then it admits a non-trivial homomorphism $G_i \to \mathbb{Z}$. Hence, so does any finite index subgroup of G_i. There are infinitely many finite index subgroups, for example arising from the kernel of $G_i \to \mathbb{Z} \xrightarrow{\text{mod } n} \mathbb{Z}_n$.

(2) ⇒ (1): Trivial.

(1) ⇒ (3): Suppose that G_i has infinite abelianisation, and let $\phi\colon G_i \to \mathbb{Z}$ be a surjective homomorphism. Let G_i^n be the kernel of the homomorphism $G_i \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\text{mod } n} \mathbb{Z}_n$. Then $G_i^n \lhd G_i \leq G$ are finite index subgroups. Now, G_i/G_i^n is the cyclic group of order n. Hence, each vertex v of $X(G_i/G_i^n)$ is labelled with an integer $\psi(v)$ mod n. (See Figure 1.)

Let T be the maximal tree in $X(G/G_i)$ defined before Lemma 2.5 (where $K = G_i$). The edges of $X(G/G_i)$ not in T give a set of generators for G_i. Let N be the maximal absolute value of $\phi(g)$, as g varies over these generators. Then, each edge of $X(G_i/G_i^n)$ joins vertices whose labels differ by at most N.

We will now find an upper bound for $\lambda_1(X(G_i/G_i^n))$. Define

$$f\colon V(X(G_i/G_i^n)) \to \mathbb{R}$$
$$v \mapsto \sin(2\pi\psi(v)/n).$$

Note that

$$\sum_{v \in V(X(G_i/G_i^n))} f(v) = 0.$$

Now each edge e of $X(G_i/G_i^n)$, oriented from e_- to e_+, satisfies

$$\begin{aligned}|f(e_+) - f(e_-)| &= |\sin(2\pi\psi(e_+)/n) - \sin(2\pi\psi(e_-)/n)| \\ &= 2|\sin(\pi(\psi(e_+) - \psi(e_-))/n)\cos(\pi(\psi(e_+) + \psi(e_-))/n)| \\ &\leq 2\pi N/n.\end{aligned}$$

Hence,

$$||df||^2 = \sum_{e \in E(X(G_i/G_i^n))} |f(e_+) - f(e_-)|^2 \leq \frac{[G:G_i]}{n}(4\pi^2 N^2 |S|),$$

since $X(G_i/G_i^n)$ has at most $[G:G_i]n|S|$ edges. Also,

$$||f||^2 = \sum_{v \in V(X(G_i/G_i^n))} |f(v)|^2 = \sum_{m=1}^{n} \sin^2(2\pi m/n) \geq n/8,$$

provided $n > 2$. The last inequality holds because at least half the values of $2\pi m/n$ lie in the intervals $[\pi/6, 5\pi/6]$ and $[7\pi/6, 11\pi/6]$. So,

$$\lambda_1(X(G_i/G_i^n)) \leq \frac{||df||^2}{||f||^2} \leq \frac{[G:G_i]}{n^2}(32\pi^2 N^2 |S|).$$

Now apply Lemma 2.5 to deduce that

$$\lambda_1(X(G/G_i^n)) \leq \frac{32\pi^2 N^2 |S|}{n^2}.$$

Since G_i^n is a normal subgroup of G_i, $N(G_i^n)$ contains G_i. So, $[G : N(G_i^n)]$ is bounded independently of n. Hence, $\lambda_1(X(G/G_i^n))[G : N(G_i^n)]^2[G : G_i^n]^2$ is bounded independently of n, as required.

(3) $\Rightarrow$ (4): It is trivial to show that, if X is a finite connected graph, then $\lambda_1(X) \geq 1/|V(X)|^2$. So, consider a subsequence for which $\lambda_1(X_i)[G : N(G_i)]^2[G : G_i]^2$ is bounded above by some constant k. Then,

$$k \geq \lambda_1(X_i)[G : N(G_i)]^2[G : G_i]^2 \geq [G : N(G_i)]^2.$$

So, $[G : N(G_i)]$ is bounded. Since $[G : G_i]$ tends to infinity, so must $[N(G_i) : G_i]$. We deduce that

$$\lambda_1(X_i)[G : N(G_i)]^4[N(G_i) : G_i] \leq k/[N(G_i) : G_i] \to 0.$$

(4) $\Rightarrow$ (6): This follows from the fact ([1],[10]) that if X is a finite graph and each of its vertices has valence at most $2|S|$, then $h(X) \leq \sqrt{4|S|\lambda_1(X)}$.

(1) $\Rightarrow$ (5): Let G_i^n be the subgroups in the proof of (1) $\Rightarrow$ (3). Let B be the vertices of $X(G_i/G_i^n)$ with label between 1 and $n/2$. Let A be their inverse image

in $X(G/G_i^n)$. Then $|A|/|V(X(G/G_i^n))| \to 1/2$ and $|A| \leq |V(X(G/G_i^n))|/2$, but $|\partial A|$ is uniformly bounded. So,

$$h(X(G/G_i^n))\ [G : N(G_i^n)]\ [G : G_i^n] \leq [G : G_i]\ |V(X(G/G_i^n))|\ |\partial A|\ /\ |A|,$$

which is uniformly bounded.

(5) ⇒ (6): This is similar to (3) ⇒ (4). Since $h(X) \geq 2/|V(X)|$ for a finite connected graph X, the hypothesis that $h(X_i)[G : N(G_i)][G : G_i]$ is bounded implies that $[G : N(G_i)]$ is bounded. Hence, $[N(G_i) : G_i]$ tends to infinity, and therefore $h(X_i)[G : N(G_i)]^2[N(G_i) : G_i]^{1/2}$ has zero infimum.

(6) ⇒ (1): This is the heart of Theorem 1.1, where we go from a geometric hypothesis to an algebraic conclusion.

By Lemma 2.3, if (6) holds for some finite presentation of G, it holds for any finite presentation. So, we use a finite triangular presentation.

Since the quantity in (6) has zero infimum, so must $h(X_i)[G : N(G_i)]$. Hence, we may apply Lemma 2.5, with $H = G_i$ and $K = N(G_i)$, to deduce that

$$h(X(N(G_i)/G_i)) \leq 4h(X_i)\ [G : N(G_i)]^2\ |S|.$$

Therefore $h(X(N(G_i)/G_i))[N(G_i) : G_i]^{1/2}$ tends to zero in some subsequence. For sufficiently large i, we may apply Lemma 2.4 to deduce that $X(N(G_i)/G_i)$ has a meta-cocycle that is not a co-boundary. This pulls back to a meta-cocycle for X_i. It also is not a co-boundary. For, there is a closed loop in $X(N(G_i)/G_i)$ that has non-zero evaluation, and we may use this to construct a closed loop in X_i with non-zero evaluation. Since the presentation for G was triangular, this gives a non-trivial cocycle for G_i, which implies that it has infinite abelianisation.

Thus, we have established the equivalence of (1) - (6). In particular, (3) - (6) do not depend on the choice of generators S for G.

Suppose now that G is the fundamental group of some closed orientable Riemannian manifold M, and let M_i be the cover of M corresponding to G_i. In [4], a finite set S of generators for $\pi_1(M)$ are chosen that arise from a fundamental domain. By the argument of Lemma 2 in [4], $h(X_i) \leq c\ h(M_i)$, for a constant c independent of i. Moreover, it is simple to ensure that $h(M_i) \leq c\ h(X_i)$ and that $\lambda_1(M_i) \leq c\lambda_1(X_i)$.

(3) $\Rightarrow$ (7); (4) $\Rightarrow$ (8); (5) $\Leftrightarrow$ (9); (6) $\Leftrightarrow$ (10): These follow immediately.

(7) $\Rightarrow$ (9); (8) $\Rightarrow$ (10): These follow from Cheeger's inequality [8], which states that $\lambda_1(M_i) \geq (h(M_i))^2/4$. □

3. Background on Heegaard splittings

We now recall various definitions relating to Heegaard splittings. We then collate some fairly elementary information about the behaviour of Heegaard splittings when they are lifted to finite covers.

Recall that a *handlebody* is a compact orientable 3-manifold obtained from a 3-ball by attaching a (possibly empty) collection of 1-handles. A *compression body* is a connected compact orientable 3-manifold that either is a handlebody or is obtained from $F \times I$, where F is a possibly disconnected, closed, orientable surface, by attaching 1-handles to $F \times \{1\}$. The *negative boundary* is empty in the case of a handlebody, and $F \times \{0\}$ otherwise. The *positive boundary* is the remaining boundary components. A *meridian disc* in a compression body is a properly embedded disc, whose boundary is an essential curve in the positive boundary.

A *Heegaard splitting* of a compact orientable 3-manifold is a description of the manifold as two compression bodies glued along their positive boundary. The latter forms the *Heegaard surface*. Any compact orientable 3-manifold has a Heegaard splitting, and therefore its Heegaard Euler characteristic, defined in §1, is well-defined. A Heegaard splitting is *reducible* if there are meridian discs in each compression body with equal boundary; otherwise it is *irreducible*. It is *weakly reducible* if there are meridian discs in each compression body with disjoint boundaries; otherwise it is *strongly irreducible*. It is a well known result of Casson and Gordon [6] that if a 3-manifold admits a Heegaard splitting which is irreducible but weakly reducible, it is Haken. Thus, for irreducible non-Haken manifolds M, $\chi_-^h(M)$ and $\chi_-^{sh}(M)$ are equal.

A Heegaard splitting arises from a handle structure on the 3-manifold. For, any closed orientable 3-manifold is obtained from a collection of 0-handles, by attaching 1-handles, then 2-handles, then 3-handles. The 3-manifold obtained after attaching all the 1-handles is a handlebody, as is the closure of its complement. Thus, the boundary of this submanifold is a Heegaard surface.

By considering more general handle structures, Scharlemann and Thompson introduced generalised Heegaard splittings [51]. Here, one views a compact orientable 3-manifold M as built from a collection of 0-handles, and possibly collars on some boundary components of M, by attaching a collection of 1-handles, then a collection of 2-handles, then 1-handles, and so on in an alternating fashion, and ending finally in a collection of 3-handles. If one were to halt this process just after attaching the j^{th} batch of 1-handles or 2-handles, the result would be a 3-manifold embedded in M. Let F_j be the boundary of this 3-manifold, after discarding any 2-sphere components that bound a 0-handle or 3-handle in M. After a small isotopy to make them disjoint, the surfaces F_{2j} (known as *even* surfaces) divide M into 3-manifolds, for which the surfaces F_{2j+1} (known as *odd* surfaces) form Heegaard surfaces. We term the number of such 3-manifolds the *length* of the generalised Heegaard splitting. Thus, a splitting of length one is a genuine Heegaard splitting. A schematic diagram of a generalised Heegaard splitting is shown in Figure 5.

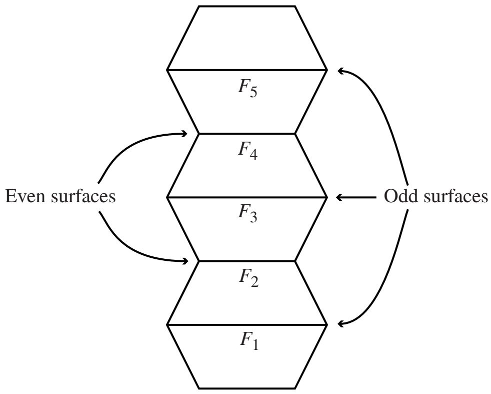

Figure 5.

Define the *complexity* $c(S)$ of a closed orientable connected surface S to be $\max\{0, 1 - \chi(S)\}$. Define the *complexity* $c(S)$ of a closed orientable disconnected surface S to be the sum of the complexities of its components. The *complexity* of a generalised Heegaard splitting $\{F_1, \ldots, F_n\}$ is defined in [51] to be

$$\{c(F_{2j+1}) : 1 \leq 2j + 1 \leq n\},$$

where repetitions are retained. These sets are compared lexicographically, comparing the largest integers first and working down. This is a well-ordering. A decomposition of minimal complexity is known as *thin*. If $\{F_1, \dots, F_n\}$ is a thin decomposition for M, we set $c_+(M)$ to be the maximal value of $c(F_{2j+1}) - 1$, as $2j+1$ ranges from 1 to n.

It is proven in [51], using results of Casson and Gordon [6], that for a thin generalised Heegaard splitting of length more than one, the even surfaces are incompressible. In fact, from any Heegaard surface F for an irreducible 3-manifold M, one can construct a generalised Heegaard splitting $\{F_1, \dots, F_n\}$ with the following properties:

1. each odd surface is strongly irreducible;
2. each even surface is incompressible and has no 2-sphere components;
3. F_{j+1} and F_j are not parallel for any j;
4. $\sum_j (-1)^j \chi(F_j) \leq -\chi(F)$;
5. $\{c(F_{2j+1}) : 1 \leq 2j+1 \leq n\} \leq c(F)$.

This procedure is described in [51]; we summarise it here. At each stage, starting with the given Heegaard surface, a generalised Heegaard splitting is constructed, with lower complexity than the one before, and hence the procedure is guaranteed to terminate. So, let $\{F_1, \dots, F_n\}$ be some generalised Heegaard splitting. If some F_{2j+1} were weakly reducible, then the modification in Rule 3 of [51] would replace F_{2j+1} with three surfaces G_1, G_2 and G_3, where $\chi(G_1) = \chi(G_3) = \chi(F_{2j+1}) + 2$ and $\chi(G_2) = \chi(F_{2j+1}) + 4$. Note that this leaves $\sum_j (-1)^j \chi(F_j)$ unchanged. (See Figure 6.) Thus, by applying this procedure enough times, we end with a generalised Heegaard splitting $\{F_1, \dots, F_n\}$ in which each odd surface is strongly irreducible. Hence, by Rule 5 in [51], each even surface is incompressible. If some even surface has a 2-sphere component, it bounds a 3-ball since M is irreducible. Pass to an innermost such 2-sphere S. A component of an odd surface forms a Heegaard surface for the 3-ball that S bounds. Remove this component and S. This reduces complexity and retains the remaining conditions. If some F_j and F_{j+1} are parallel, they are both discarded. This reduces complexity and preserves the remaining conditions. So, we end with a generalised Heegaard splitting satisfying conditions (1) to (5).

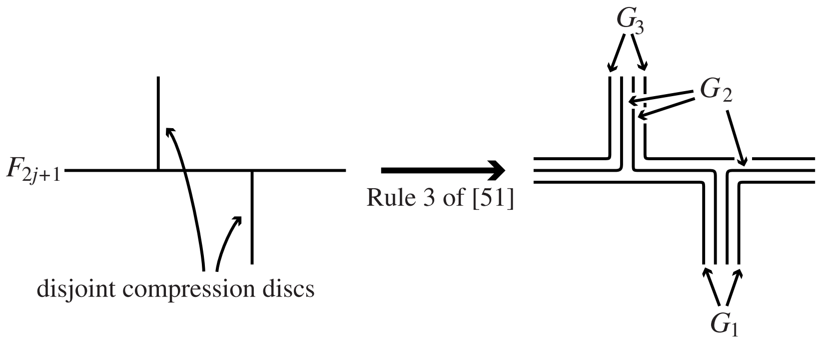

Figure 6.

We now go on to investigate how Heegaard splittings behave under finite covers. None of the results in the remainder of this section will be used later in this paper, but they do form a useful backdrop. The following, fairly well-known, observation was the motivation for much of this paper.

Proposition 3.1. *Let M be a compact orientable 3-manifold. Let F be any Heegaard splitting for M. Then, for all covers of M with sufficiently large degree, F lifts to a weakly reducible Heegaard surface.*

Proof. Pick a maximal collection of non-parallel meridian discs for each side of the Heegaard surface. Suppose that there are n_1 discs on one side, and n_2 on the other. Let k be the total number of intersection points between these discs. Consider a d-fold cover $\tilde{M} \rightarrow M$, and let $\tilde{F}$ be the inverse image of F. Then, the discs lift to dn_1 and dn_2 meridian discs either side of $\tilde{F}$. The total number of intersections between these discs is dk. When $(dn_1)(dn_2) > dk$, at least one pair of discs, one on each side of $\tilde{F}$, are disjoint. Hence, $\tilde{F}$ is weakly reducible. □

However, the above fact is certainly not enough on its own to prove the virtually Haken conjecture, since the lifted Heegaard surface may be reducible. In fact, the following proposition asserts that generically this will be the case.

Proposition 3.2. *Let M be a closed orientable Riemannian 3-manifold, and let F be a Heegaard surface for M. Then, for all finite covers $\tilde{M}$ of sufficiently large injectivity radius, the inverse image $\tilde{F}$ of F is a reducible Heegaard surface.*

Proof. If M is reducible, then so is any cover $\tilde{M}$, and hence so is $\tilde{F}$ [21]. Hence, we may assume that M is irreducible. This implies that $\pi_1(M)$ is not free [27].

We may take F to be the boundary of a small regular neighbourhood of a bouquet of circles W embedded in M. This specifies a set of generators for $\pi_1(M)$. Since $\pi_1(M)$ is not free, there is some non-trivial word in these generators that is trivial in $\pi_1(M)$. Pick a shortest such word. This specifies a loop α in W. Let l be its length in the path metric on W.

Pass to a cover $\tilde{M} \to M$ with injectivity radius more than l. Let the basepoint of M be the basepoint of W, and assign a basepoint for $\tilde{M}$ which lies in the inverse image of the basepoint of M. Consider the ball of radius l in $\tilde{M}$ about this basepoint. This is a 3-ball B. Since α is homotopically trivial in M, it lifts to a loop $\tilde{\alpha}$ based at the basepoint of $\tilde{M}$. As α represents a shortest non-trivial word in the generators that is trivial in $\pi_1(M)$, α is an embedded loop in B. Hence, by a theorem of Frohman [17], $\tilde{F}$ is reducible. □

Corollary 3.3. *The infimal Heegaard gradient of a closed hyperbolic 3-manifold is strictly less than its Heegaard Euler characteristic.*

Proof. Let F be a Heegaard surface for M with $|\chi(F)| = \chi_-^h(M)$. The fundamental group of a hyperbolic 3-manifold is residually finite, and so we may find finite covers with arbitrarily large injectivity radius. By Proposition 3.2, we may find a cover $\tilde{M} \to M$, with degree d, say, such that the inverse image $\tilde{F}$ of F is reducible. However, $\tilde{M}$ is irreducible, and therefore $\tilde{F}$ is not minimal genus. So, $\chi_-^h(\tilde{M}) < d|\chi(F)|$, which proves the corollary. □

The rationale of the Heegaard gradient conjecture is as follows. The above corollary gives that, when passing to a finite cover of a closed hyperbolic 3-manifold, the Heegaard Euler characteristic will in general be scaled by less than the degree of the cover. However, the conjecture proposes that it will not degenerate too much, unless the manifold is virtually fibred.

We now give some elementary information about Heegaard gradient. The first observation to make is that one could very well consider any of the following quantities:

$$c_1 = \inf\{\chi^h_-(M_i)/d_i : M_i \to M \text{ is a cover with degree } d_i\}$$
$$c_2 = \inf\{\chi^h_-(M_i)/d_i : M_i \to M \text{ is a regular cover with degree } d_i\}$$
$$c_3 = \liminf\{\chi^h_-(M_i)/d_i : M_i \to M \text{ is a cover with degree } d_i\}$$
$$c_4 = \liminf\{\chi^h_-(M_i)/d_i : M_i \to M \text{ is a regular cover with degree } d_i\}.$$

It is clear that

$$c_1 = c_2 \leq c_3 = c_4.$$

This is because any finite cover $M_i \to M$ has a further finite cover $M_j \to M_i$ such that $M_j \to M_i \to M$ is regular. Moreover, $\chi^h_-(M_j)/d_j \leq \chi^h_-(M_i)/d_i$, where d_i and d_j are the relevant covering degrees, since any Heegaard splitting for M_i lifts to one for M_j and its Euler characteristic is scaled by d_j/d_i. It is also clear that

$$c_1 = c_2 = c_3 = c_4$$

when M has infinitely many covers. This is because, given any finite cover $M_i \to M$ and an infinite sequence of covers $M_j \to M$, we may consider the finite covers $M_{i,j} \to M$ corresponding to the subgroups $\pi_1(M_i) \cap \pi_1(M_j)$ of $\pi_1(M)$. These have $\chi^h_-(M_{i,j})/d_{i,j} \leq \chi^h_-(M_i)/d_i$, where again $d_{i,j}$ is the relevant covering degree. When M has only finitely many covers, c_3 and c_4 are both infinite, since, by convention, the $\liminf$ of a finite set of numbers is infinite.

One may also consider the quantities defined in the same way as c_1, c_2, c_3 and c_4 but with $\chi^h_-(M_i)$ replaced by $\chi^{sh}_-(M_i)$. In this case, there are again obvious inequalities, but no immediate equalities. This is because a strongly irreducible Heegaard surface may lift to a weakly reducible Heegaard splitting in a finite cover. However, if M_i is irreducible and non-Haken, then $\chi^h_-(M_i) = \chi^{sh}_-(M_i)$. So, for the purposes of proving the virtually Haken conjecture, almost any of these quantities may be used interchangeably.

Note that the Heegaard Euler characteristic of a closed orientable 3-manifold M is always non-negative, unless M is the 3-sphere. Thus, the Heegaard gradient of a closed orientable 3-manifold is negative if and only if its universal cover is S^3. Also, its strong Heegaard gradient is always non-negative.

There do exist manifolds with positive Heegaard gradient, as given by the following proposition.

Proposition 3.4. *Let M be a closed orientable reducible 3-manifold other than $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$. Then the infimal Heegaard gradient of M is at least $\frac{1}{3}$. Moreover, when M has no $\mathbb{R}P^3$ summand, it is at least $\frac{2}{3}$.*

Proof. Pick a maximal collection of disjoint non-parallel essential 2-spheres in M. Consider a degree d cover $\tilde{M} \to M$. The inverse image of the 2-spheres in M is a collection $\mathcal{S}$ of at least d 2-spheres in $\tilde{M}$. If two are parallel, then between them is a copy of $S^2 \times I$. The only orientable manifolds that are covered by $S^2 \times I$ are itself and once-punctured $\mathbb{R}P^3$ [34]. The former is impossible in this case, since the spheres in M were not parallel and M is not $S^2 \times S^1$. We deduce that no three spheres in $\mathcal{S}$ can be parallel, since M is neither $S^2 \times S^1$ nor $\mathbb{R}P^3 \# \mathbb{R}P^3$. Also, when M has no $\mathbb{R}P^3$ summand, no two spheres of $\mathcal{S}$ are parallel. If any sphere in $\mathcal{S}$ is not parallel to any of the others, add in a parallel copy. Let $\mathcal{S}_+$ be the resulting enlargement of $\mathcal{S}$. Let $\{M_i\}$ be the complementary regions of $\mathcal{S}_+$ which are not copies of $S^2 \times I$. Since M is neither $S^2 \times S^1$ nor $\mathbb{R}P^3 \# \mathbb{R}P^3$, each component of $\mathcal{S}_+$ is adjacent to some M_i. Let $\hat{M}_i$ be the closed manifold obtained from M_i by attaching 3-balls to its boundary components. Now, by Haken's theorem [21], a Heegaard surface realizing $\chi_-^h(\tilde{M})$ may be obtained from ones from $\hat{M}_i$ by attaching tubes that intersect each component of $\mathcal{S}_+$ in a single closed curve. Hence,

$$\chi_-^h(\tilde{M}) = \sum_i (\chi_-^h(\hat{M}_i) + |\mathcal{S}_+ \cap M_i|).$$

Now, if M_i has one or two boundary components, the Heegaard Euler characteristic of $\hat{M}_i$ is non-negative. Hence, for each i,

$$\chi_-^h(\hat{M}_i) + 2|\mathcal{S}_+ \cap M_i|/3 \geq 0.$$

Summing over i,

$$\chi_-^h(\tilde{M}) \geq \sum_i |\mathcal{S}_+ \cap M_i|/3 = |\mathcal{S}_+|/3.$$

This is at least $d/3$, and when M has no $\mathbb{R}P^3$ summands, it is at least $2d/3$. □

The above proposition should be compared with the Heegaard gradient conjecture. A closed orientable reducible 3-manifold is virtually fibred if and only if it is $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$. In this case, it has zero Heegaard gradient. But in all other cases, its Heegaard gradient is positive.

We conclude this section with an example, which illustrates that arbitrarily large degeneration of Heegaard Euler characteristic can occur when passing to a finite cover. More precisely, we show that, when $\tilde{M} \to M$ is a degree d cover, the ratio $d\chi_-^h(M)/\chi_-^h(\tilde{M})$ can be arbitrarily large.

Let X be a once-punctured torus bundle with pseudo-Anosov monodromy. Then X admits an obvious genus three Heegaard surface. Since X is not a solid torus, its Heegaard genus is at least two. So its Heegaard Euler characteristic is either two or four. We now construct M by Dehn filling X along a slope s having distance d from the slope of the fibre. If we take d to be large, M will be hyperbolic and hence its Heegaard Euler characteristic will be two or four. Now, the slope s lifts to a simple closed curve in the d-fold cyclic cover $\tilde{X}$ of X. By Dehn filling $\tilde{X}$ along this slope, we obtain a d-fold cover $\tilde{M} \to M$. Since $\tilde{M}$ is obtained by Dehn filling a once-punctured torus bundle, its Heegaard Euler characteristic is at most four. Therefore, the ratio $d\chi_-^h(M)/\chi_-^h(\tilde{M})$ can be made arbitrarily large.

4. The Cheeger constant and generalised Heegaard splittings

In this section, we establish an upper bound on the Cheeger constant of a negatively curved Riemannian 3-manifold in terms of its generalised Heegaard splittings. We then go on to explore some of the consequences of this.

Theorem 4.1. *Let M be a complete, finite volume Riemannian 3-manifold. Let $\kappa < 0$ be its supremal sectional curvature. Then*

$$h(M) \leq \frac{4\pi c_+(M)}{|\kappa|\mathrm{Volume}(M)} \leq \frac{4\pi\chi_-^h(M)}{|\kappa|\mathrm{Volume}(M)}.$$

Proof. Let $\{F_1, \dots, F_n\}$ be a thin generalised Heegaard splitting for M. Since the even surfaces are incompressible and have no 2-sphere components, each component that is not boundary parallel may be isotoped to either a least area minimal surface or the orientable double cover of an embedded non-orientable least area minimal surface ([52], [16], [39]). Furthermore, any two such components are either equal or disjoint after the isotopy. For odd j, let M_j be the manifold between F_{j-1} and F_{j+1} (letting M_1 and M_n be the manifolds separated off by F_2 and F_{n-1} respectively). There is some odd j such that Volume$(M_1 \cup \ldots \cup M_{j-2})$ and

Volume($M_{j+2} \cup \ldots \cup M_n$) are each at most half the volume of M. (When $n = 1$, we take j to be 1). Now, F_j forms a Heegaard surface for M_j, and hence determines a sweepout of M_j. In any such sweepout, there is a surface of maximum area. The infimal possible value for this maximum is known as a *minimax* value. According to the methods of Pitts and Rubinstein [45], there is a minimal surface F whose area is this minimax value, and which is obtained from F_j possibly by removing some boundary parallel components, possibly performing some compressions and then possibly amalgamating parallel components into a single component. Its area, by Gauss-Bonnet, is at most $|\kappa|^{-1} 2\pi |\chi(F_j)| \leq |\kappa|^{-1} 2\pi c_+(M)$. We may therefore find, for any $\epsilon > 0$, a sweepout of M_j whose maximal area is at most $|\kappa|^{-1} 2\pi c_+(M) + \epsilon$. Some surface F_j in this sweepout divides M into two pieces of equal volume. Hence,

$$h(M) \leq \frac{\text{Area}(F_j)}{(1/2)\text{Volume}(M)} \leq \frac{4\pi\, c_+(M)}{|\kappa|\text{Volume}(M)} + \frac{2\epsilon}{\text{Volume}(M)}.$$

Since ϵ was arbitrary, the first inequality of the theorem is established. The second follows from the trivial observation that $c_+(M) \leq \chi_-^h(M)$. □

Theorem 1.3. *Let M be a closed orientable 3-manifold with a negatively curved Riemannian metric. Let $\{M_i \to M\}$ be the finite covers of M. Then the following are equivalent, and are each equivalent to the conditions in Theorem 1.1:*

1. *$b_1(M_i) > 0$ for infinitely many i;*
2. *$c_+(M_i)[\pi_1 M : N(\pi_1 M_i)]$ has a bounded subsequence;*
3. *$c_+(M_i)[\pi_1 M : N(\pi_1 M_i)][N(\pi_1 M_i) : \pi_1 M_i]^{-1/2}$ has zero infimum.*

Proof. Note that (1) is simply a restatement of (2) in Theorem 1.1.

(1) $\Rightarrow$ (2): We pick an M_i with $b_1(M_i) > 0$ and find a closed orientable embedded surface S in M_i that represents a non-trivial element of $H_2(M_i)$. We may assume that S has no 2-sphere components. Let M_i^n be the n-fold cyclic cover dual to $[S]$. We will show that $c_+(M_i^n)[\pi_1 M : N(\pi_1 M_i^n)]$ is bounded independent of n. Since $N(\pi_1 M_i^n)$ contains $\pi_1 M_i$, $[\pi_1 M : N(\pi_1 M_i^n)]$ is bounded above. So, it suffices to show that $c_+(M_i^n)$ is bounded above. Let F be a Heegaard surface for $M_i - \text{int}(\mathcal{N}(S))$ which separates the inward-pointing boundary components from the outward-pointing ones. We view F as a surface in M_i. We will construct a generalised Heegaard splitting for M_i^n from the inverse images of F and S.

Take a copy of S in M_i^n and let $S \times [0,1]$ be its regular neighbourhood. Pick a handle structure on $S \times [0, \frac{1}{2}]$ with no 3-handles. Its 0- and 1-handles form the first handles in the generalised Heegaard splitting on M_i^n, and its 2-handles form the next handles. The remaining copies of F and S form odd and even surfaces, working one way round the cyclic cover. We end by forming a handle structure in $S \times [\frac{1}{2}, 1]$ using 1-, 2- and 3-handles. By picking the handle structures on $S \times [0, \frac{1}{2}]$ and $S \times [\frac{1}{2}, 1]$ appropriately, we can ensure that the complexity of this splitting is less than $\max\{c(F) + c(S) + 1, 2c(S) + |S| + 1\}$, which is independent of n, establishing (2).

(2) $\Rightarrow$ (3): For the subsequence in (2), $[\pi_1 M : \pi_1 M_i]$ tends to infinity, but $[\pi_1 M : N(\pi_1 M_i)]$ is bounded. So, $[N(\pi_1 M_i) : \pi_1 M_i]$ tends to infinity, and therefore $c_+(M_i)[\pi_1 M : N(\pi_1 M_i)][N(\pi_1 M_i) : \pi_1 M_i]^{-1/2}$ tends to zero.

(3) $\Rightarrow$ (1): By Theorem 4.1, (3) implies that

$$\begin{aligned} &h(M_i)[\pi_1 M : N(\pi_1 M_i)]^2[N(\pi_1 M_i) : \pi_1 M_i]^{1/2} \\ &\quad \leq \frac{4\pi c_+(M_i)[\pi_1 M : N(\pi_1 M_i)]^2[N(\pi_1 M_i) : \pi_1 M_i]^{1/2}}{|\kappa| \mathrm{Volume}(M_i)} \\ &\quad = \left(\frac{4\pi}{|\kappa|\mathrm{Volume}(M)}\right) c_+(M_i)[\pi_1 M : N(\pi_1 M_i)][N(\pi_1 M_i) : \pi_1 M_i]^{-1/2}, \end{aligned}$$

which has zero infimum. Now apply (10) $\Rightarrow$ (2) of Theorem 1.1. □

Theorem 1.5. *Let M be an orientable 3-manifold that admits a complete, negatively curved, finite volume Riemannian metric. Let $\{M_i \to M\}$ be a collection of finite covers of M. If $\pi_1(M)$ has Property (τ) with respect to $\{\pi_1(M_i)\}$, then the infimal Heegaard gradient of $\{M_i \to M\}$ is non-zero.*

Proof. Suppose that the Heegaard gradient of $\{M_i \to M\}$ is zero. Then

$$h(M_i) \leq \frac{4\pi c_+(M_i)}{|\kappa|\mathrm{Volume}(M_i)} \leq \frac{4\pi}{|\kappa|\mathrm{Volume}(M)} \frac{\chi_-^h(M_i)}{[\pi_1 M : \pi_1 M_i]},$$

which has zero infimum. So, $\pi_1(M)$ fails to have Property (τ) with respect to $\{\pi_1(M_i)\}$. □

5. Towards the virtually Haken conjecture

In this section, we establish Theorem 1.7. This gives two conditions (which conjecturally always hold for a closed orientable hyperbolic 3-manifold) that imply the existence of a finite Haken cover.

Lemma 5.1. *Let F be a Heegaard surface for a compact orientable 3-manifold M. Suppose that we can find d_1 non-parallel meridian discs on one side of F, and d_2 non-parallel meridian discs on the other side of F, that are all disjoint from each other. Then, either M has a thin generalised Heegaard splitting with length at least two, or $\chi_-^h(M)$ is at most $-\chi(F) - \frac{1}{3}\min\{d_1, d_2\}$.*

Proof. The given Heegaard splitting of M has complexity $\{c(F)\}$. The existence of d_1 and d_2 disjoint discs either side of F allows us to attach d_2 2-handles before adding d_1 1-handles. It is trivially established, by induction on k, that if a closed connected orientable surface F is compressed along a collection of k pairwise non-parallel discs, then the resulting surface F' has $c(F') \leq c(F) - k/3$. Thus, we obtain a generalised Heegaard splitting of complexity at most $\{c(F) - d_1/3, c(F) - d_2/3\}$. This is an upper bound for the complexity of any thin generalised Heegaard splitting. If this has length two or more, the lemma is proved. If not, it is a Heegaard splitting, with Heegaard surface F'', such that

$$\begin{aligned}\chi_-^h(M) = -\chi(F'') \leq c(F'') - 1 &\leq c(F) - 1 - \frac{1}{3}\min\{d_1, d_2\} \\ &= -\chi(F) - \frac{1}{3}\min\{d_1, d_2\}.\end{aligned}$$

□

Theorem 1.7. *Let M be a compact orientable irreducible 3-manifold, with boundary a (possibly empty) collection of tori. Let $\{M_i \to M\}$ be a lattice of finite regular covers of M. Suppose that*

1. *$\pi_1(M)$ fails to have Property (τ) with respect to $\{\pi_1(M_i)\}$, and*
2. *$\{M_i \to M\}$ has non-zero infimal strong Heegaard gradient.*

Then, for infinitely many i, M_i has a thin generalised Heegaard splitting which is not a Heegaard splitting, and so contains a closed essential surface. In particular, M is virtually Haken.

Proof. Note that if $\chi_-^h(M_i) < \chi_-^{sh}(M_i)$, then any thin generalised Heegaard splitting for M_i has length at least two. For if a thin generalised Heegaard splitting has length one, it is a minimal genus Heegaard splitting that is strongly irreducible. Thus, we may assume that, for all but finitely many i, $\chi_-^h(M_i) = \chi_-^{sh}(M_i)$. Hence, the strong Heegaard gradient and the Heegaard gradient of $\{M_i \to M\}$ coincide. We work with the latter.

Suppose that $M' \to M$ is an element of $\{M_i \to M\}$. The set of covers in $\{M_i \to M\}$ that factor through $M' \to M$ forms a sublattice of $\{M_i \to M\}$. It is clear that M' has non-zero Heegaard gradient with respect to this sublattice, and that $\pi_1(M')$ fails to have Property (τ) with respect to $\{\pi_1(M') \cap \pi_1(M_i)\}$. Hence, if we pass to a finite cover M' in $\{M_i\}$, the hypotheses of the theorem still hold. By passing to such a cover if necessary, we may assume that the ratio of $\chi_-^h(M)$ to the infimal Heegaard gradient of $\{M_i \to M\}$ is arbitrarily close to one. In particular, we may ensure that this ratio is at most 25/24.

Fix a Heegaard surface F of M for which $-\chi(F) = \chi_-^h(M)$. This divides M into two compression bodies H^1 and H^2. For $j = 1$ and 2, let $\mathcal{D}^j$ be a collection of disjoint compression discs for F in H^j that cut H^j to a collar on its negative boundary or to a 3-ball. Note that $|\mathcal{D}^j|$ is either $1 - (\chi(F)/2)$ or $-\chi(F)/2$, depending on whether the negative boundary of H^j is empty or not. (Here, we are using the hypothesis that ∂M is a possibly empty collection of tori.) We pick a handle structure on H^1, by first picking a handle structure on this collar or 3-ball, and then enlarging it to a handle structure for H^1 by adding 1-handles dual to $\mathcal{D}^1$. The cores of the 0-handles and 1-handles form a graph Γ embedded in M. We may assume that the boundaries of the discs in $\mathcal{D}^2$ run only over the 0-handles and 1-handles of H^1 and hence have a projection map to Γ. Pick a maximal tree in Γ. If we collapse this tree to a point, a graph $\overline{\Gamma}$ is obtained. The edges of $\overline{\Gamma}$ determine a set S_- of generators for $\pi_1(M)$. Let k be the maximal number of intersection points between a disc in $\mathcal{D}^2$ and the union of the discs in $\mathcal{D}^1$, and let S be the set of words in letters of S_- and their inverses with length at most k.

The fact that $\pi_1(M)$ fails to have Property (τ) with respect to $\{\pi_1(M_i)\}$ implies that the Cayley graphs X_i of $\pi_1(M)/\pi_1(M_i)$ with respect to S have $\liminf_i h(X_i) = 0$. Pass to a subsequence where $h(X_i) \to 0$. Let d_i be the order of the cover M_i. Let F_i, H_i^1, H_i^2, Γ_i, $\overline{\Gamma}_i$, $\mathcal{D}_i^1$ and $\mathcal{D}_i^2$ be the inverse images in M_i of F, H^1, H^2, Γ, $\overline{\Gamma}$, $\mathcal{D}^1$ and $\mathcal{D}^2$, respectively. There is a graph embedding $\overline{\Gamma}_i \to X_i$ which induces a bijection between their vertices. There is also a map $\Gamma_i \to \overline{\Gamma}_i$ that collapses the lifts in Γ_i of the maximal tree.

Pick a non-empty subset A_i of $V(X_i)$ such that $|\partial A_i|/|A_i| = h(X_i)$ and $|A_i| \leq |V(X_i)|/2$. By Lemma 2.1, $|A_i| > |V(X_i)|/4$. Let $\overline{\mathcal{D}}_i^1$ be the set of discs in $\mathcal{D}_i^1$ whose edge in Γ_i is mapped either to an edge in X_i with an endpoint in A_i or to a vertex in A_i. Let $\overline{\mathcal{D}}_i^2$ be those discs in $\mathcal{D}_i^2$ whose boundary maps to a loop in X_i that runs over a vertex not in A_i. (See Figure 7.)

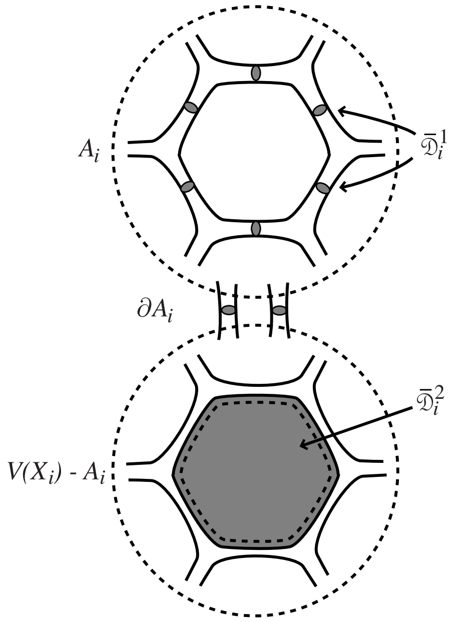

Figure 7.

Claim. $|\overline{\mathcal{D}}_i^1| \geq |\mathcal{D}^1||A_i|$.

Each vertex in A_i corresponds to a vertex in $\overline{\Gamma}_i$. The inverse image of this vertex in Γ_i is a tree. There are $|\mathcal{D}^1|$ edges of Γ_i that either lie in this tree or whose initial vertices lie in the tree. Each corresponds to a disc in $|\overline{\mathcal{D}}_i^1|$.

Claim. $|\overline{\mathcal{D}}_i^2| \geq |\mathcal{D}^2||V(X_i) - A_i|$.

This is similar to the above proof. For each disc in $\mathcal{D}^2$, pick a basepoint in its boundary that maps to the vertex of $\overline{\Gamma}$. Then each disc in $\mathcal{D}_i^2$ inherits a basepoint. Each vertex of $\overline{\Gamma}_i$ is in the image of $|\mathcal{D}^2|$ basepoints. Since $\overline{\mathcal{D}}_i^2$ contains those discs with a basepoint in $V(X_i) - A_i$, the inequality of the claim holds.

Claim. The number of discs in $\overline{\mathcal{D}}_i^1$ having non-empty intersection with some disc in $\overline{\mathcal{D}}_i^2$ is at most $2|\mathcal{D}^1|\,|\partial A_i|$.

Suppose that two discs, D_1 and D_2, in $\overline{\mathcal{D}}_i^1$ and $\overline{\mathcal{D}}_i^2$ intersect. The edge in Γ_i corresponding to D_1 is mapped either to an edge in X_i that is attached to some vertex v in A_i or to a vertex v in A_i. The boundary of D_2 runs over this vertex and at most $k-1$ others, one of which (v', say) is not in A_i. Since S includes all words of length at most k in the generators S_-, there is an edge of X_i joining v to v'. Thus, associated to each such pair of discs D_1 and D_2, there is an edge in ∂A_i. The number of discs D_1 which can correspond to this edge is at most $2|\mathcal{D}^1|$. For the edge uniquely determines v, and there are at most $2|\mathcal{D}^1|$ discs of $\mathcal{D}_i^1$ whose edge in Γ_i maps either to an edge in X_i adjacent to v or to v. This proves the claim.

Hence, if we remove all discs in $\overline{\mathcal{D}}_i^1$ that intersect some disc in $\overline{\mathcal{D}}_i^2$, the result is at least $|\mathcal{D}^1||A_i| - 2|\mathcal{D}^1||\partial A_i|$ discs. Afterwards, none of these discs intersect. So, applying Lemma 5.1, we deduce that either M_i has a thin generalised Heegaard splitting with length at least two, or that

$$\begin{aligned}
\frac{\chi_-^h(M_i)}{d_i} &\leq \frac{-\chi(F_i)}{d_i} - \frac{1}{3}\frac{\min\{|\mathcal{D}^1||A_i| - 2|\mathcal{D}^1||\partial A_i|, |\mathcal{D}^2||V(X_i) - A_i|\}}{d_i} \\
&\leq -\chi(F) + \frac{1}{3}\frac{\chi(F)}{2}\frac{|A_i| - 2|\partial A_i|}{d_i} \\
&= \chi_-^h(M)\left(1 - \frac{1}{6}\frac{|A_i|(1-2h(X_i))}{d_i}\right) \\
&\leq \chi_-^h(M)\left(\frac{23}{24} + \frac{1}{6}h(X_i)\right).
\end{aligned}$$

As $i \to \infty$, the last term tends to zero. Hence, we can find an M_i such that $\chi_-^h(M_i)/d_i < (24/25)\chi_-^h(M)$, which is a contradiction. □

Corollary 5.2. *Let M be a closed orientable irreducible 3-manifold with positive Heegaard gradient. If $\pi_1 M$ is infinite and residually finite, then it is non-amenable and hence has exponential growth.*

Proof. If $\pi_1(M)$ is infinite, amenable and residually finite, then it does not have Property (τ) [35]. Hence, by Theorem 1.7, M is virtually Haken, and therefore satisfies the geometrisation conjecture: it admits a decomposition along a (possibly empty) collection of non-parallel essential tori into Seifert fibred and hyperbolic pieces. None of these pieces can be hyperbolic or Seifert fibred with hyperbolic base orbifold, since $\pi_1(M)$ would then contain a non-abelian free subgroup, contradicting amenability. No base space can be bad or spherical, since M would then be spherical and hence have negative Heegaard gradient. So, the base space of each Seifert fibred piece is Euclidean. If a Seifert fibred piece has more than one boundary component, it must be a copy of $T^2 \times I$. As no tori in the collection are parallel, M must therefore fibre over the circle with fibre a torus. But it then has zero Heegaard gradient, contrary to hypothesis. Thus, we may assume that every Seifert fibred piece has at most one boundary component. Hence, there are either one or two such pieces. If there is only one, M is virtually fibred, and hence has zero Heegaard gradient, which is again a contradiction. Thus, M contains two Seifert fibred pieces M_1 and M_2. If $\pi_1(M_1 \cap M_2)$ has index more than two in either of $\pi_1(M_1)$ or $\pi_1(M_2)$, then $\pi_1(M)$ contains a free non-abelian subgroup, which is a contradiction. Hence, $\pi_1(M_1 \cap M_2)$ has index two in each of $\pi_1(M_1)$ or $\pi_1(M_2)$, and therefore M_1 and M_2 are twisted I-bundles over the Klein bottle. But M is then virtually fibred, which again is a contradiction. □

6. The size of minimal surfaces in negatively curved manifolds

In this section, we will analyse minimal surfaces in closed negatively curved 3-manifolds. We will prove that the diameter of such a surface is bounded by a function of its Euler characteristic, and of the injectivity radius and curvature of the manifold. This will have relevance to Heegaard splittings, since it is a theorem of Pitts and Rubinstein [45] that a strongly irreducible Heegaard surface in a closed orientable Riemannian 3-manifold with a bumpy metric may be isotoped either to a minimal surface or to a double cover of a minimal non-orientable surface with a small tube attached. In the latter case, the tube is vertical in the I-bundle structure on the regular neighbourhood of the surface.

Proposition 6.1. *There is some real-valued function f in two variables, with the following property. Let M be a Riemannian 3-manifold, whose injectivity radius is at least $\epsilon/2 > 0$, and whose sectional curvature is at most $\kappa < 0$. Then the diameter of a connected closed minimal surface F in M is bounded above by $|\chi(F)| f(\kappa, \epsilon)$.*

Proof. We give F its induced Riemannian metric as a submanifold of M. Since F is a minimal surface, its mean curvature is everywhere zero. This is the sum of its principal curvatures. Hence, their product, the extrinsic curvature of F, is non-positive at all points. Now, the intrinsic curvature of F at each point is the sum of its extrinsic curvature and the ambient curvature of M, and is therefore at most $\kappa < 0$.

Let $F_{(0,\epsilon)}$ (respectively, $F_{[\epsilon,\infty)}$) be the set of points in F with injectivity radius less than $\epsilon/2$ (respectively, at least $\epsilon/2$). Here, we are referring to injectivity radius as measured by the Riemannian metric on F, rather than the metric on M. Given $\delta > 0$ and a metric space X, let $C(\delta, X)$ denote the minimal number of δ-balls in X required to cover X. We will show that

$$C(\epsilon, F_{[\epsilon,\infty)}) \leq \frac{1}{\cosh(|\kappa|^{1/2}\epsilon/2) - 1}|\chi(F)|, \tag{1}$$

$$C(\epsilon + \epsilon/(2\pi) + |\kappa|^{-1/2}/2, F_{(0,\epsilon)}) \leq 4|\chi(F)|, \tag{2}$$

which will establish the proposition.

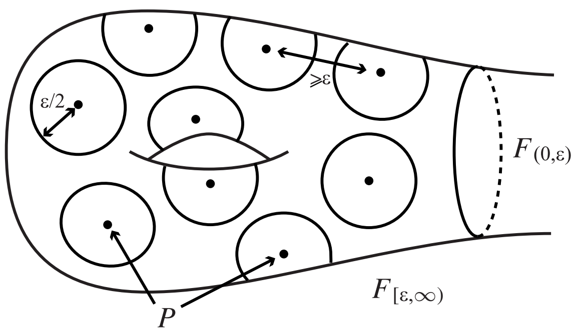

Figure 8.

The first inequality is proved using a well-known argument of Thurston [57]. Pick a maximal collection of points P in $F_{[\epsilon,\infty)}$, no two of which lie less than ϵ apart. (See Figure 8.) Then, the open balls of radius $\epsilon/2$ around these points are disjoint, and each is homeomorphic to a disc, since P lies in $F_{[\epsilon,\infty)}$. By Gunther's comparison theorem, each such disc has area at least $|\kappa|^{-1}2\pi(\cosh(|\kappa|^{1/2}\epsilon/2) - 1)$ since $2\pi(\cosh(\epsilon/2) - 1)$ is the formula for the area of a ball of radius $\epsilon/2$ in the hyperbolic plane. So,

$$\mathrm{Area}(F) \geq \mathrm{Area}(F_{[\epsilon,\infty)}) \geq |P||\kappa|^{-1}2\pi(\cosh(|\kappa|^{1/2}\epsilon/2) - 1).$$

By Gauss-Bonnet,

$$|\kappa|\mathrm{Area}(F) \leq 2\pi|\chi(F)|.$$

By the fact that P is maximal, the balls of radius ϵ about P cover $F_{[\epsilon,\infty)}$, and so

$$C(\epsilon, F_{[\epsilon,\infty)}) \leq |P| \leq \frac{\mathrm{Area}(F)}{|\kappa|^{-1}2\pi(\cosh(|\kappa|^{1/2}\epsilon/2) - 1)} \leq \frac{1}{\cosh(|\kappa|^{1/2}\epsilon/2) - 1}|\chi(F)|,$$

which is the first inequality (1).

We now establish the second inequality (2). Pick a maximal collection of disjoint (not necessarily simple) closed geodesics Γ in F, each with length less than ϵ. Consider a geodesic γ in Γ. Let $\tilde{F}$ be the cover of F, corresponding to the subgroup of $\pi_1(F)$ generated by γ. Then $\tilde{F}$ is homeomorphic to an open annulus, and its core geodesic $\tilde{\gamma}$ is a lift of γ. Let $\tilde{N}$ be the set of points in $\tilde{F}$ at most $\epsilon/(2\pi) + |\kappa|^{-1/2}/2$ from $\tilde{\gamma}$. (See Figure 9.)

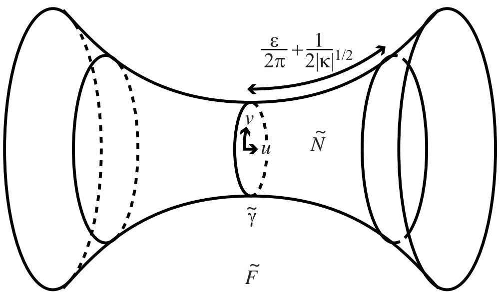

Figure 9.

Claim 1. Any closed curve in $\tilde{F}$ that is disjoint from $\tilde{N}$ and that is freely homotopic to $\tilde{\gamma}$ has length more than ϵ.

We use the terminology of [26]. Choose orthogonal co-ordinates (u, v) on $\tilde{F}$ so that $\tilde{\gamma}$ is the set $\{u = 0\}$ and so that the curves where v is constant are geodesics perpendicular to $\tilde{\gamma}$. So the metric is given by

$$ds^2 = du^2 + J^2(u, v)dv^2,$$

where $J(u, v) > 0$ and $J(0, v) = 1$. The aim is to find a differential inequality satisfied by the lengths of the curves $\{u = \text{constant}\}$. Thereby, we will show that, when $u > \epsilon/(2\pi) + |\kappa|^{-1/2}/2$, each such curve has length more than ϵ.

Let $\pm\lambda$ (where $\lambda \geq 0$) be the principal curvatures of $\tilde{F}$ at a point (u, v). The intrinsic curvature K at this point is the sum of the sectional curvature of M and the product of the principal curvatures, and so

$$K \leq \kappa - \lambda^2. \tag{3}$$

Also, by Formula 10.5.3.3 of [3], the intrinsic curvature is given by

$$K = -\frac{1}{J}\frac{\partial^2 J}{\partial u^2}. \tag{4}$$

Let $\tilde{\alpha}$ be the curve $\{u = c\}$, where u is a constant $c \geq 0$, and let α be its image in F. The geodesic curvature k_g^F in F of α is given by

$$k_g^F = \frac{1}{J}\frac{\partial J}{\partial u}. \tag{5}$$

Now, $\tilde{\alpha}$ is freely homotopic in $\tilde{F}$ to $\tilde{\gamma}$ which has length less than ϵ and hence maps to a homotopically trivial curve in M. It therefore bounds a mapped-in disc D. We may realize this disc as a minimal surface ([11],[12],[46],[47]). The intrinsic curvature K_D of D is then at most $\kappa < 0$. So, applying Gauss-Bonnet to D, we obtain

$$\int_D K_D \; dA + \int_\alpha k_g^D \; ds = 2\pi,$$

where k_g^D is the geodesic curvature of ∂D in D. Here, positive k_g^D at a point in ∂D means that D is locally convex near that point. So,

$$\int_\alpha k_g^D J \; dv = \int_\alpha k_g^D \; ds \geq 2\pi.$$

The geodesic curvature k_g^M of α in M (which, by definition, is non-negative) is at least k_g^D and so

$$\int_\alpha k_g^M J \; dv \geq \int_\alpha k_g^D J \; dv \geq 2\pi. \tag{6}$$

Now, if II is the second fundamental form of F, then

$$\tilde{D}_{\alpha'}\alpha' = II(\alpha', \alpha') + D_{\alpha'}\alpha',$$

where $\tilde{D}$ and D are the covariant derivatives in M and F, respectively. So, applying the triangle inequality,

$$|II(\alpha', \alpha')| \geq |\tilde{D}_{\alpha'}\alpha'| - |D_{\alpha'}\alpha'| = k_g^M J^2 - k_g^F J^2.$$

However,

$$|II(\alpha', \alpha')| \leq \lambda |\alpha'|^2 = \lambda J^2.$$

So,

$$\lambda \geq k_g^M - k_g^F. \tag{7}$$

So, combining (3), (4), (7) and (5), and using the universal inequality $|\kappa| + \lambda^2 \geq 2|\kappa|^{1/2}\lambda$:

$$\frac{1}{J}\frac{\partial^2 J}{\partial u^2} \geq |\kappa| + \lambda^2 \geq 2|\kappa|^{1/2}\lambda \geq 2|\kappa|^{1/2}k_g^M - 2|\kappa|^{1/2}k_g^F = 2|\kappa|^{1/2}k_g^M - |\kappa|^{1/2}\frac{2}{J}\frac{\partial J}{\partial u}.$$

Hence,

$$\frac{\partial^2 J}{\partial u^2} \geq 2|\kappa|^{1/2}k_g^M J - 2|\kappa|^{1/2}\frac{\partial J}{\partial u}. \tag{8}$$

Let $L(u)$ be the length of the curve α. Then

$$L(u) = \int_\alpha J\, dv.$$

So, integrating (8) with respect to v, and applying (6):

$$\frac{d^2 L}{du^2} \geq 4|\kappa|^{1/2}\pi - 2|\kappa|^{1/2}\frac{dL}{du}.$$

So, multiplying by $e^{2|\kappa|^{1/2}u}$ and writing L' for dL/du:

$$\frac{d(L')}{du}e^{2|\kappa|^{1/2}u} + 2|\kappa|^{1/2}L' e^{2|\kappa|^{1/2}u} \geq 4\pi|\kappa|^{1/2}e^{2|\kappa|^{1/2}u}.$$

Integrating with respect to u, we obtain

$$L' e^{2|\kappa|^{1/2}u} \geq 2\pi e^{2|\kappa|^{1/2}u} - 2\pi,$$

and hence

$$L' \geq 2\pi(1 - e^{-2|\kappa|^{1/2}u}).$$

Therefore,

$$L(u) \geq 2\pi u + \frac{\pi}{|\kappa|^{1/2}}e^{-2|\kappa|^{1/2}u} - \frac{\pi}{|\kappa|^{1/2}}.$$

Now, the length of any closed curve in $\tilde{F}$ that is disjoint from $\tilde{N}$ and that is freely homotopic to $\tilde{\gamma}$ has length at least $L(\epsilon/(2\pi) + |\kappa|^{-1/2}/2) \geq \epsilon$, which proves the claim.

Let N_+ be the set of points in F with distance at most $\epsilon/2 + \epsilon/(2\pi) + |\kappa|^{-1/2}/2$ from Γ.

Claim 2. N_+ contains $F_{(0,\epsilon)}$.

Suppose that there is a point x in $F_{(0,\epsilon)}$ not in N_+. There is a closed curve β based at x with length less than ϵ. Now, β must be disjoint from Γ, since Γ is more that $\epsilon/2$ from x. Hence, β can be freely homotoped to a closed (unbased) geodesic $\overline{\beta}$ with length less than ϵ in the surface $\overline{F}$ obtained by cutting F along Γ. Either $\overline{\beta}$ is disjoint from Γ, or it is a component γ of Γ (possibly winding several times round this component). The former case is impossible, since Γ is maximal. In the latter case, β lifts to a closed curve $\tilde{\beta}$ in $\tilde{F}$, the cover corresponding to γ. The basepoint of $\tilde{\beta}$ is at least $\epsilon/2 + \epsilon/(2\pi) + |\kappa|^{-1/2}/2$ from $\tilde{\gamma}$, and hence $\tilde{\beta}$ misses $\tilde{N}$. By Claim 1, $\tilde{\beta}$, and hence β, therefore has length at least ϵ, which is a contradiction, that proves the claim.

Claim 3. The maximum number of disjoint closed geodesics in F is at most $4|\chi(F)|$.

Let Γ^+ be a maximal collection of disjoint closed geodesics. Let Γ^+_s be the simple geodesics in Γ^+ and let Γ^+_{ns} be the non-simple ones. Each complementary region of Γ^+ is an annulus, disc or Möbius band, since any other subsurface of F admits a closed geodesic that is not homotopic to a multiple of a boundary component. Attach the discs to a regular neighbourhood of Γ^+_{ns}, creating a subsurface X of F. Each component of X has negative Euler characteristic, since otherwise the geodesic it contains would be simple. Hence $|\chi(X)| \geq |X| = |\Gamma^+_{ns}|$. Also, $|\partial X| \leq 3|\chi(X)|$. Each complementary region of X is an annulus or Möbius band, that contains a single simple closed geodesic in Γ^+_s. So, $|\Gamma^+| = |\Gamma^+_s| + |\Gamma^+_{ns}| \leq |\partial X| + |X| \leq 4|\chi(X)| = 4|\chi(F)|$, which establishes the claim.

Hence, the number of components of Γ is at most $4|\chi(F)|$. Pick one point on each component of Γ. Each point of N_+, and hence $F_{(0,\epsilon)}$, lies within $\epsilon + \epsilon/(2\pi) + |\kappa|^{-1/2}/2$ of one of these points. This proves inequality (2). □

The following is an immediate application of Proposition 6.1. It provides some evidence for the strong Heegaard gradient conjecture.

Theorem 1.9. *Let M be a closed orientable hyperbolic 3-manifold, and let $\{M_i \to M\}$ be the cyclic covers dual to some non-trivial element of $H_2(M)$. Then the strong Heegaard gradient of $\{M_i \to M\}$ is non-zero.*

Proof. We may perturb the hyperbolic metrics on the covering spaces M_i to bumpy metrics in the sense of [59]. Let $\kappa < 0$ be the supremum of their sectional curvatures. Let S be an incompressible surface embedded in M, representing the non-trivial element of $H_2(M)$. Let M_S be the manifold obtained by cutting M along S. Let δ be the minimal distance between the boundary components of M_S. Let F_i be a strongly irreducible Heegaard surface for M_i (if one exists) with $-\chi(F_i) = \chi_-^{sh}(M_i)$. This may isotoped to a minimal surface or to the boundary of a regular neighbourhood of a non-orientable embedded minimal surface, with a small tube attached [45]. The injectivity radius of M_i is at least that of M, which we denote by $\epsilon/2$. By Proposition 6.1, the diameter of F_i is at most $\chi_-^{sh}(M_i) f(\kappa, \epsilon)$. However, there are lifts of S which are at least $\lfloor i/2 \rfloor \delta$ apart. None of these can be disjoint from F_i, since the complement of F_i is two handlebodies. So,

$$\chi_-^{sh}(M_i) f(\kappa, \epsilon) \geq \lfloor i/2 \rfloor \delta \geq i\delta/3,$$

when $i \geq 2$. Rearranging this gives that the strong Heegaard gradient is bounded below by $\delta/(3f(\kappa, \epsilon))$. □

This theorem does not generalise to cusped hyperbolic 3-manifolds. For example, once-punctured torus bundles have a Heegaard splitting with genus at most three. A minimal genus splitting cannot be weakly reducible. For this would imply that the manifold contained a closed essential surface [41], which does not occur [15]. Hence, if $\{M_i \to M\}$ are the cyclic covers dual to the fibre, then $\chi_-^{sh}(M_i)$ is at most four. So, the strong Heegaard gradient of $\{M_i \to M\}$ is zero.

Corollary 1.10. *Let M be a closed orientable 3-manifold that fibres over the circle with pseudo-Anosov monodromy. Let $\{M_i \to M\}$ be the cyclic covers dual to the fibre. Then, for all but finitely many i, M_i has an irreducible, weakly reducible, minimal genus Heegaard splitting.*

Proof. By Thurston's geometrisation theorem [40], M has a hyperbolic metric. Let S be the fibre. Then $\chi_-^{h}(M_i) \leq 2|\chi(S)| + 4$. However, $\chi_-^{sh}(M_i)/i$ is bounded away from zero by Theorem 1.9. So, for sufficiently large i, $\chi_-^{sh}(M_i) > \chi_-^{h}(M_i)$. Hence, any minimal genus Heegaard splitting for M_i is weakly reducible, but necessarily irreducible. □

7. The Heegaard gradient of cyclic covers

The following result provides some evidence for the Heegaard gradient conjecture.

Theorem 1.11. *Let M be a compact orientable finite volume hyperbolic 3-manifold, and let $\{M_i \to M\}$ be the cyclic covers dual to some non-trivial element z of $H_2(M, \partial M)$. Then, the infimal Heegaard gradient of $\{M_i \to M\}$ is zero if and only if z is represented by a fibre.*

Proof. In one direction, this is trivial. So, suppose that z is not represented by a fibre. Let S be an incompressible representative for z that intersects each component of ∂M in a coherently oriented (possibly empty) collection of curves. Let S_i be its inverse image in M_i.

We first show that $\{\chi_-^h(M_i)\}$ has no bounded subsequence. Suppose it does, and pass to this subsequence. Let F_i be a Heegaard surface realizing $\chi_-^h(M_i)$, which must be irreducible. We construct a generalised Heegaard splitting $F_1^i, \dots, F_n^i$ from F_i satisfying the five conditions in §3. Let F^i be $F_1^i \cup \dots \cup F_n^i$. Because the difference in Euler characteristic between successive odd and even surfaces is at least two, the length of the decomposition is at most $\frac{1}{2}\sum_j (-1)^j \chi(F_j^i)$, which is at most $\frac{1}{2}|\chi(F_i)|$, by Condition 4 in §3. By assumption, $|\chi(F_i)|$ is bounded.

The argument now divides, according to whether M is closed or bounded. We start with the closed case. Since the even surfaces F_{2j}^i are incompressible and contain no 2-sphere components, each component may be isotoped to either a least area minimal surface or the orientable double cover of an embedded non-orientable least area minimal surface ([52], [16], [39]). Furthermore, after this isotopy, any two components of the even surfaces are either equal or disjoint. Each component of the odd surfaces may be also be isotoped to a minimal surface, or to the boundary of a regular neighbourhood of an embedded minimal surface, with a small tube attached. This follows from the argument of Pitts and Rubinstein [45], since the least area minimal surfaces form a barrier for the sweep-outs associated with the odd surfaces. We now follow the argument of Theorem 1.9. The number of components of F^i is bounded, and the Euler characteristic of each component is bounded. Hence, by Proposition 6.1, the diameter of each component is bounded. Therefore, the number of components of S_i that F^i can intersect is bounded.

However, the total number of components of S_i is unbounded. Therefore, if i is sufficiently large, we may find an arbitrarily large number of copies of S who components lie in the same complementary regions of F^i. However, each such region is a compression body, and so these incompressible surfaces must be parallel to components of the even surfaces. We deduce that two copies of S are parallel in M_i. Hence, S is a fibre, contrary to assumption.

We now consider the case where M has non-empty boundary. The argument in the closed case does not work, since the injectivity radius of M is zero, and so Proposition 6.1 does not apply. Instead, we use the theory of normal and almost normal surfaces. Since M is cusped, it has a canonical polyhedral decomposition P [14], which is an angled polyhedral decomposition in the sense of [31]. This lifts to an angled polyhedral decomposition P_i of each M_i. The interior angles of P_i are those of P, and hence they have a uniform upper bound, which is less than π, and a uniform lower bound which is greater than zero.

We may isotope each even surface into normal form in P_i. If P_i were a triangulation (which it is not), then a theorem of Rubinstein [49] and Stocking [55] states that the odd surfaces could be placed in almost normal form with respect to P_i. This was generalised in [32] to ideal polyhedral decompositions, with a suitable generalisation of the notion of almost normal. Two conditions on the ideal polyhedral decomposition are required: that there is no 2-sphere that is normal to one side, and that each face is a triangle or bigon. The former holds, since the ideal polyhedral decomposition is angled, and the latter holds after the faces have been suitably subdivided. Once each F^i_j is in normal or almost normal form, it then inherits a combinatorial area, as in [31]. This is defined additively over each component of intersection between F^i and the polyhedra. The combinatorial area of each such component D is the sum of its exterior angles, minus 2π times its Euler characteristic. The area of D is greater than or equal to zero, with equality if and only if D is the link of an ideal vertex. If the combinatorial area of D is positive, it has a positive lower bound, b say, which applies to each of the M_i. By the argument of Proposition 4.3 in [31], the combinatorial area of a normal or almost normal surface F is $-2\pi\chi(F)$. So the number of normal or almost normal pieces of F^i with positive area is at most $2\pi|\chi(F^i)|/b$, which is bounded.

Now, S_i contains i copies of S, which we denote by $S^i_1, \ldots S^i_i$. These are labelled so that S^i_j is adjacent to S^i_{j-1} and S^i_{j+1}, where the indexing is mod i. We need to consider a closed surface, and so suppose that S has non-empty boundary. Let k be $-\frac{3}{2}\chi(S) + |\partial S| + 1$. Let A^i_j be the annuli in ∂M_i between ∂S^i_j and ∂S^i_{j+k} that intersect $\partial S^i_{j+1}, \ldots, \partial S^i_{j+k-1}$. Then $S^i_j \cup A^i_j \cup S^i_{j+k}$ forms a closed surface $\hat{S}^i_j$ which we may push a little into the interior of M. It is a theorem of Cooper and Long [9] that $\hat{S}^i_j$ is essential, since S is not a fibre. Also, $\hat{S}^i_j$ and $\hat{S}^i_{j'}$ are disjoint providing j and j' differ by more than k mod i. When these surfaces are disjoint, it is clear that they cannot be parallel. When S_i is closed, let $\hat{S}^i_j$ be S^i_j.

Place S in normal form in P. Then S_i is in normal form in P_i. Suppose that each polyhedron of P_i intersects $\hat{S}^i_j$ in at most c components. Then c may be taken to be independent of i and j. We may find $\lfloor i/(k+1) \rfloor$ disjoint $\hat{S}^i_j$ in M_i. Now, the number of polyhedra in P_i containing positive area pieces of F^i is bounded. Hence, if i is sufficiently large, we may find an arbitrarily large number of disjoint $\hat{S}^i_j$ with the property that the polyhedra they lie in contain no positive area pieces of F^i. Since the $\hat{S}^i_j$ are closed, they may be isotoped off F^i. They then lie in the compression bodies of the generalised Heegaard splittings, and hence are parallel to the even surfaces. Since the total number of components of the even surfaces is uniformly bounded, we deduce that two disjoint $\hat{S}^i_j$ are parallel, which is a contradiction.

Thus, in both the case where M is closed and the case where M has boundary, $\chi^h_-(M_i)$ cannot have a bounded subsequence. Hence there is a positive integer n, so that when $i \geq n$, $\chi^h_-(M_i) \geq 36|\chi(S)| + 4|S|$. This implies that the manifold $M(i)$ obtained by cutting M_i along a copy of S has $\chi^h_-(M(i)) \geq 35|\chi(S)|$. For it is not hard construct a Heegaard surface for M_i from one for $M(i)$, changing the Euler characteristic by at most $|\chi(S)| + 4|S|$.

We wish to apply a result of Schultens (Theorem 4.4 of [53]) that gives a lower bound on the Heegaard genus of a Haken manifold in terms of the genus of a properly embedded incompressible surface and the Heegaard genus of its complementary pieces. (See also Proposition 23.40 of [29] for a similar result.) The maximal number of properly embedded disjoint non-parallel essential annuli in the complementary pieces also appears in the formula. We now investigate this.

We claim that in a compact orientable atoroidal irreducible 3-manifold N, there can be no more that $3|\chi(\partial N)|$ properly embedded disjoint non-parallel essential annuli. Pick a maximal such collection A. Suppose that two components of ∂A are parallel in ∂N. We may find two that have no component of ∂A in the annulus A' between them. If these lie in distinct components of A, say A_1 and A_2, then $A_1 \cup A' \cup A_2$ forms an annulus which must be parallel to a component A_3 of A. Then $A_1 \cup A_2 \cup A_3$ separates off a solid torus in N. This has a natural Seifert fibred structure in which A_1, A_2 and A_3 are a union of fibres. Similarly, if the components of $\partial A'$ lie in a single component A_1 of A, then A_1 separates off a Seifert fibred solid torus in N with one exceptional fibre, since N is irreducible and atoroidal. The union of all such Seifert fibred solid tori is a Seifert fibre space V. Each component must have base space that is a disc, and have at most one exceptional fibre, since N is atoroidal. A simple counting argument then gives that at least half the components of $A \cap V$ lie in ∂V. Let A_- denote these annuli. Then ∂A_- forms a collection of curves in ∂N with the property that no three curves are parallel. There can be at most $3|\chi(\partial N)|$ such curves. So there can be at most $\frac{3}{2}|\chi(\partial N)|$ components of A_- and so at most $3|\chi(\partial N)|$ components of A, which proves the claim.

Theorem 4.4 of [53] gives a lower bound on $g(M_i, \partial M_i)$, which is the minimal genus of a Heegaard surface for M_i, subject to the condition that all boundary components for M_i lie on one side of this surface. We wish to relate this to $\chi_-^h(M_i)$. Now, we may assume that S intersects each component of ∂M, for otherwise, M_i contains at least i boundary components, and so $\chi_-^h(M_i) \geq i - 2$, which implies that $\{M_i \rightarrow M\}$ has positive Heegaard gradient. But when S intersects each component of ∂M, there is a uniform upper bound (t, say) on the number of components of ∂M_i. From any Heegaard surface for M_i, we may therefore create one that has all components of ∂M_i on one side, increasing the genus of the surface by at most t. So, denoting the Heegaard genus of M_i by $g(M_i)$, we deduce

$$\chi_-^h(M_i) = 2g(M_i) - 2 \geq 2g(M_i, \partial M_i) - 2 - 2t.$$

Now, for any $i \geq n$, consider the manifold obtained from M_i by cutting along $S_n \cup S_{2n} \cup \ldots \cup S_{n\lfloor \frac{i}{n} \rfloor}$. We obtain $(\lfloor \frac{i}{n} \rfloor - 1)$ copies of $M(n)$ and one copy of

$M(i - \lfloor \frac{i}{n} \rfloor n + n)$. Applying Theorem 4.4 of [53], we obtain

$$\begin{aligned}
\chi^h_-(M_i) &\geq 2g(M_i, \partial M_i) - 2 - 2t \\
&\geq \frac{1}{7}\chi^h_-(M_i - (S_n \cup S_{2n} \cup \dots S_{n\lfloor \frac{i}{n} \rfloor})) + 2 \\
&\quad - 4|\chi(S_n \cup S_{2n} \cup \dots \cup S_{n\lfloor \frac{i}{n} \rfloor})| - 2 - 2t \\
&\geq \left\lfloor \frac{i}{n} \right\rfloor |\chi(S)| - 2t \\
&\geq i\frac{|\chi(S)|}{2n},
\end{aligned}$$

when i is sufficiently large. So the Heegaard gradient of $\{M_i \to M\}$ is at least $|\chi(S)|/2n$, which is non-zero. □

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