FIRST ORDER p-ADIC DEFORMATIONS OF WEIGHT ONE NEWFORMS

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ABSTRACT. This article studies the first-order p-adic deformations of classical weight one newforms, relating their fourier coefficients to the p-adic logarithms of algebraic numbers in the field cut out by the associated projective Galois representation.

CONTENTS

INTRODUCTION

Let g be a classical cuspidal newform of weight one, level N and nebentypus character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Q}_p^\times$, with fourier expansion

$$g(q) = \sum_{n=1}^{\infty} a_n q^n.$$

The *p-stabilisations* of g attached to a rational prime $p \nmid N$ are the eigenforms of level Np defined by

$$(1) \qquad g_\alpha(q) := g(q) - \beta \cdot g(q^p), \qquad g_\beta(q) := g(q) - \alpha \cdot g(q^p),$$

where α and β are the (not necessarily distinct) roots of the Hecke polynomial

$$x^2 - a_p x + \chi(p) =: (x-\alpha)(x-\beta).$$

The forms g_α and g_β are eigenvectors for the Atkin U_p operator, with eigenvalues α and β respectively. Since α and β are roots of unity, these eigenforms are both *ordinary* at p.

An important feature of classical weight one forms is that they are associated to odd, irreducible, two-dimensional Artin representations, via a construction of Deligne-Serre. Let $\varrho_g : G_\mathbb{Q} \longrightarrow \mathbf{GL}_2(\mathbb{C})$ denote this Galois representation, and write V_g for the underlying representation space.

A fundamental result of Hida asserts the existence of a *p-adic family* of ordinary eigenforms specialising to g_α (or to g_β) in weight one. Bellaiche and Dimitrov [BD] later established the uniqueness of this Hida family, under the hypothesis that g is *regular at* p , i.e., that $\alpha \neq \beta$, or equivalently, that the frobenius element at p acts on V_g with distinct eigenvalues. In the intriguing special case where g is the theta series of a character of a real quadratic field F in which the prime p is split, the result of Bellaiche-Dimitrov further asserts that the unique ordinary first-order infinitesimal p-adic deformation of g is an overconvergent (but not classical) modular form of weight one. In [DLR2], the Fourier coefficients of this non-classical form were expressed as p-adic logarithms of algebraic numbers in a ring class field of F, suggesting that a closer examination of such deformations could have some relevance to explicit class field theory for real quadratic fields.

The primary purpose of this note is extend the results of [DLR2] to general weight one eigenforms.

Part A considers the regular setting where $\alpha \neq \beta$, in which the results exhibit a close analogy to those of [DLR2].

Part B takes up the case where g is irregular at p. Here the results are more fragmentary and less definitive. Let $S_1^{(p)}(N,\chi)$ denote the space of p-adic overconvergent modular forms of weight 1, level N, and character χ, and let $S_1^{(p)}(N,\chi)[[g]]$ denote the generalised eigenspace attached to the system of Hecke eigenvalues of an irregular weight one form $g \in S_1(N,\chi)$. The main conjecture of the second part asserts that $S_1^{(p)}(N,\chi)[[g]]$ is always four dimensional, with a two-dimensional subspace consisting of classical forms. Under this conjecture, an explicit description of the elements of the generalised eigenspace in terms of their q-expansions is provided. The resulting concrete description of the generalised eigenspace that emerges from Part B is an indispensible ingredient in the extension of the "elliptic Stark conjectures" of [DLR1] to the irregular setting that will be presented in [DLR3].

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Part A. The regular setting

Let $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ denote the Iwasawa algebra, and let

$$\mathcal{W} := \mathrm{Hom}_{\mathrm{cts}}(1+p\mathbb{Z}_p, \mathbb{C}_p^\times) = \mathrm{Hom}_{\mathrm{alg}}(\Lambda, \mathbb{C}_p)$$

denote the associated *weight space*. For each $k \in \mathbb{Z}_p$, write $\nu_k \in \mathcal{W}$ for the "weight k" homomorphism sending the group-like element $a \in 1+p\mathbb{Z}_p$ to a^{k-1}. The rule $\lambda(k) := \nu_k(\lambda)$ realises elements of Λ as analytic functions on $\mathbb{Z}_p$. The spectrum $\tilde{\mathcal{W}} := \mathrm{Hom}_{\mathrm{alg}}(\tilde\Lambda, \mathbb{C}_p)$ of a finite flat extension $\tilde\Lambda$ of Λ is equipped with a "weight map"

$$\mathrm{w} : \tilde{\mathcal{W}} \longrightarrow \mathcal{W}$$

of finite degree. A $\mathbb{Q}_p$-valued point $x \in \tilde{\mathcal{W}}$ is said to be *of weight* k if $\mathrm{w}(x) = \nu_k$, and is said to be étale over $\mathcal{W}$ if the inclusion $\Lambda \subset \tilde\Lambda$ induces an isomorphism between Λ and the completion of $\tilde\Lambda$ at the kernel of x, denoted $\tilde\Lambda_x$. An element of this completion thus gives rise to an analytic function of $k \in \mathbb{Z}_p$ in a natural way.

A *Hida family* is a formal q-series

$$\mathbf{g} := \sum a_n q^n \in \tilde\Lambda[[q]]$$

with coefficients in a finite flat extension $\tilde{\Lambda}$ of Λ, specialising to a classical ordinary eigenform of weight k at almost all points x of $\tilde{\mathcal{W}}$ of weight $k \in \mathbb{Z}^{\geq 2}$. Two Hida families $\mathbf{g}_1 \in \tilde{\Lambda}_1[[q]]$ and $\mathbf{g}_2 \in \tilde{\Lambda}_2[[q]]$ are regarded as equal if the Λ-algebras $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ can be embedded in a common extension $\tilde{\Lambda}$ in such a way that $\mathbf{g}_1$ and $\mathbf{g}_2$ are identified. A well known theorem of Hida and Wiles asserts the existence of a Hida family specialising to g_α in weight one. The following uniqueness result for this Hida family plays an important role in our study.

Theorem (Bellaiche, Dimitrov)**.** *Assume that the weight one form g is regular at p, and let x_α and x_β denote the distinct points on $\tilde{\mathcal{W}}$ attached to g_α and g_β respectively. Then*

(a) *The curve $\tilde{\mathcal{W}}$ is smooth at x_α and x_β, and in particular there are unique Hida families $\mathbf{g}_\alpha, \mathbf{g}_\beta \in \tilde{\Lambda}[[q]]$ specialising to g_α and g_β at x_α and x_β respectively.*

(b) *The weight map $\mathrm{w} : \tilde{\mathcal{W}} \longrightarrow \mathcal{W}$ is furthermore étale at x_α if any only if*

(†) *g is not the theta series of a character of a real quadratic K in which p splits.*

The setting where g is regular at p but w is not étale at x_α has been treated in [DLR2], and the remainder of Part A will therefore focus on the scenarios where (†) is satisfied. In that case, after viewing elements of the completion $\tilde{\Lambda}_{x_\alpha}$ of $\tilde{\Lambda}$ at x_α as analytic functions of the "weight variable" k, one may consider the *canonical* q-series

$$g'_\alpha := \left(\frac{d}{dk}\mathbf{g}_\alpha\right)_{k=1}$$

representing the first-order infinitesimal ordinary deformation of $\mathbf{g}$ at the weight one point x_α, along this canonical "weight direction". The q-series g'_α is analogous to the overconvergent generalised eigenform considered in [DLR2], with the following differences:

(a) While the overconvergent generalised eigenform of [DLR2] is a (non-classical, but overconvergent) modular form of weight one, such an interpretation is not available for the q-series g'_α, which should rather be viewed as the first order term of a "modular form of weight $1 + \varepsilon$".

(b) In the non-étale setting of [DLR2], the absence of a natural local coordinate with respect to which the derivative would be computed meant that the overconvergent generalised eigenform of loc.cit. could only be meaningfully defined up to scaling by a non-zero multiplicative factor. This ambiguity is not present in the definition of g'_α, whose fourier coefficients are therefore entirely well-defined.

The main results of Part A give explicit formulae for these fourier coefficients: they are stated in Theorems 1.10, 2.1, 2.3, and 3.1 below.

1. The general case

The goal of this section is to describe a general formula for the fourier coefficients of g'_α.

The Artin representation V_g can be realised as a two-dimensional L-vector space, where L is a finite extension of $\mathbb{Q}$, contained in a cyclotomic field. Let $W_g = \hom(V_g, V_g)$ denote the adjoint equipped with its usual conjugation action of $G_\mathbb{Q}$, denoted

$$\sigma \cdot w := \varrho_g(\sigma) w \varrho_g(\sigma)^{-1}, \qquad \sigma \in G_\mathbb{Q}, \quad w \in W_g.$$

Let $H \subset H_g$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by the representations W_g and V_g respectively, and write $G := \mathrm{Gal}\,(H/\mathbb{Q})$.

For notational simplicity, the following assumption is made in the rest of this paper:

Assumption 1.1. *The prime p splits completely in the field L of coefficients of the Artin representation V_g.*

This assumption amounts to a simple congruence condition on p. The choice of an embedding of L into $\mathbb{Q}_p$, which is fixed henceforth, will allow us, when it is convenient, to view V_g and W_g as representations of $G_{\mathbb{Q}}$ with coefficients in $\mathbb{Q}_p$, and the weight one form g as a modular form with fourier coefficients in $\mathbb{Q}_p$ rather than in L. The $\mathbb{Q}_p$-vector spaces V_g and W_g are thus equipped with natural $G_{\mathbb{Q}}$-stable L-rational structures, denoted V_g^L and W_g^L respectively.

An embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ is fixed once and for all, determining a prime $\wp$ of H and of H_g above p, and an associated frobenius element $\tau_\wp$ in $\mathrm{Gal}\,(H_g/\mathbb{Q})$ and in G. Let $G_\wp \subset G$ be the decomposition subgroup generated by $\tau_\wp$.

The representations V_g and W_g admit the following decompositions as $\tau_\wp$-modules:

$$V_g = V_g^\alpha \oplus V_g^\beta, \qquad W_g = W_g^{\alpha\alpha} \oplus W_g^{\alpha\beta} \oplus W_g^{\beta\alpha} \oplus W_g^{\beta\beta},$$

where V_g^α and $\mathcal{V}_g^\beta$ denote the α and β-eigenspaces for the action of $\tau_\wp$ on V_g, and

$$W_g^{\xi\eta} := \hom(V_g^\xi, V_g^\eta), \qquad \text{for } \xi,\eta \in \{\alpha,\beta\}$$

is a $G_\wp$-stable line, on which $\tau_\wp$ acts with eigenvalue η/ξ. Let

$$W_g^{\mathrm{ord}} := \hom(V_g/V_g^\alpha, V_g) = W_g^{\beta\alpha} \oplus W_g^{\beta\beta}.$$

Of course, W_g^{ord} is stable under the action of $G_\wp$ but not under the action of G.

We propose to give a general formula for the ℓ-th fourier coefficient of g'_α as the trace of a certain explicit endomorphism of V_g, which is constructed via a series of lemmas. In the lemma below, we let G act on $\mathcal{O}_H^\times \otimes W_g$ diagonally on both factors in the tensor product.

Lemma 1.2. *The $\mathbb{Q}_p$-vector space $(\mathcal{O}_H^\times \otimes W_g)^G$ of G-invariant vectors is one-dimensional.*

Proof. Let G_∞ be the subgroup of G generated by a complex conjugation c, which has order two, since V_g is odd. By Dirichlet's unit theorem, the global unit group $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$ is isomorphic to $\mathrm{Ind}_{G_\infty}^G(\mathbb{Q}_p) - \mathbb{Q}_p$ as a $\mathbb{Q}_p[G]$-module. Let W_g^0 denote the three-dimensional representation of G consisting of trace zero endomorphisms of V_g. As a representation of G, we have $W_g = W_g^0 \oplus \mathbb{Q}_p$, and W_g^0 does not contain the trivial representation as a constituent. By Frobenius reciprocity,

$$\dim_{\mathbb{Q}_p}((\mathcal{O}_H^\times \otimes W_g)^G) = \dim_{\mathbb{Q}_p}((\mathcal{O}_H^\times \otimes W_g^0)^G) = \dim_{\mathbb{Q}_p}((W_g^0)^{c=1}) = 1.$$

The result follows. □

Assume that the field L of coefficients is large enough so that the semisimple ring $L[G]$ becomes isomorphic to a direct sum of matrix algebras over L. The $L[G]$-module $\mathcal{O}_H^\times \otimes L$ decomposes as a direct sum of V-isotypic components,

$$\mathcal{O}_H^\times \otimes L = \oplus_V \mathcal{O}_H^\times[V],$$

where V runs over the irreducible representations of G, and $\mathcal{O}_H^\times[V]$ denotes the largest subrepresentation of $\mathcal{O}_H^\times \otimes L$ which is isomorphic to a direct sum of copies of V as an $L[G]$-module. For a general, not necessarily irreducible, representation W, the module $\mathcal{O}_H^\times[W]$ is defined as the direct sum of the $\mathcal{O}_H^\times[V]$ as V ranges over the irreducible constituents of W. Because W_g (viewed, for now, as a representation with coefficients in L) is self-dual, Lemma 1.2 can be recast as the assertion that $\mathcal{O}_H^\times[W_g]$ is isomorphic to a single irreducible subrepresentation of W_g. More precisely:

• In the case of "exotic weight one forms" where ϱ_g has non-dihedral projective image (isomorphic to A_4, S_4 or A_5), then

$$\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[W_g^0] \simeq W_g^0, \tag{2}$$

and hence is three-dimensional.

• If ϱ_g is induced from a character ψ_g of an imaginary quadratic field K, then

$$W_g = L \oplus L(\chi_K) \oplus V_\psi,$$

where χ_K is the odd quadratic Dirichlet character associated to K and V_ψ is the two-dimensional representation of G induced from the ring class character $\psi = \psi_g/\psi_g'$ which cuts out the abelian extension H of K. The representation V_ψ is irreducible if and only if ψ is non-quadratic, and in that case,

$$\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[V_\psi] \simeq V_\psi. \tag{3}$$

In the special case where ψ is quadratic, the representation V_ψ further decomposes as the direct sum of one-dimensional representations attached to an even and an odd quadratic Dirichlet character, denoted χ_1 and χ_2 respectively. That special case, in which V_g is also induced from a character of the real quadratic field cut out by χ_1, is thus subsumed under (4) below.

• If ϱ_g is induced from a character ψ_g of a real quadratic field F, then

$$W_g = L \oplus L(\chi_F) \oplus V_\psi, \qquad V_\psi := \mathrm{Ind}_F^{\mathbb{Q}}(\psi), \quad \psi := \psi_g/\psi_g',$$

and one always has

$$\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[\chi_F] \simeq L(\chi_F), \tag{4}$$

i.e., $\mathcal{O}_H^\times[W_g]$ is generated by a fundamental unit of F.

Let $U_g^\times$ be any generator of the one-dimensional $\mathbb{Q}_p$-vector space $(\mathcal{O}_H^\times \otimes W_g)^G$ and let

$$U_g := (\log_\wp \otimes \mathrm{id})(U_g^\times) \in H_\wp \otimes W_g \tag{5}$$

be the image of this vector under the linear map

$$\log_\wp \otimes \mathrm{id} : \mathcal{O}_H^\times \otimes W_g \longrightarrow H_\wp \otimes W_g,$$

where $\log_\wp$ is the p-adic logarithm on the $\wp$-adic completion $H_\wp$ of H at $\wp$.

Lemma 1.3. *There exists a non-zero endomorphism $A \in H_\wp \otimes W_g$ satisfying the following conditions:*

(a) $\mathrm{Trace}(AU_g) = 0$.
(b) *A belongs to $H_\wp \otimes W_g^{\mathrm{ord}}$, i.e., $A(V_g^\alpha) = 0$.*

This endomorphism is unique up to scaling.

Proof. The space $H_\wp \otimes W_g$ is four-dimensional over $H_\wp$ and the conditions in Lemma 1.3 amount to three linear conditions on A. More precisely, choose a $\tau_\wp$-eigenbasis (v_α, v_β) for V_g for which

$$\tau_\wp v_\alpha = \alpha v_\alpha, \qquad \tau_\wp v_\beta = \beta v_\beta.$$

Relative to this basis, the endomorphism U_g is represented by a matrix of the form

$$U_g : \begin{pmatrix} \log_\wp(u_1) & \log_\wp(u_{\beta/\alpha}) \\ \log_\wp(u_{\alpha/\beta}) & -\log_\wp(u_1) \end{pmatrix},$$

where $u_1, u_{\alpha/\beta}$, and $u_{\beta/\alpha}$ are generators of $\mathcal{O}_H^\times[W_g]$ which (when non-zero) are eigenvectors for $\tau_\wp$, satisfying

$$\tau_\wp(u_1) = u_1, \qquad \tau_\wp(u_{\beta/\alpha}) = (\beta/\alpha)u_{\beta/\alpha}, \qquad \tau_\wp(u_{\alpha/\beta}) = (\alpha/\beta)u_{\alpha/\beta}.$$

The endomorphism A satisfies condition (b) above if and only if the matrix representing it in the basis (v_α, v_β) is of the form

$$A : \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}, \qquad x, y \in H_\wp,$$

and condition (a) implies the further linear relation

$$\log_\wp(u_{\alpha/\beta}) \cdot x - \log_\wp(u_1) \cdot y = 0. \tag{6}$$

The injectivity of $\log_\wp : \mathcal{O}_H^\times \otimes L \longrightarrow H_\wp$, which follows from the linear independence over $\bar{\mathbb{Q}}$ of logarithms of algebraic numbers, implies that the coefficients $\log_\wp(u_{\alpha/\beta})$ and $\log_\wp(u_1)$ in (6) vanish simultaneously if and only if

$$u_{\alpha/\beta} = u_1 = 0$$

in $\mathcal{O}_H^\times \otimes L$, i.e., if and only if $\mathcal{O}_H^\times[W_g]$ is one-dimensional over L and generated by $u_{\beta/\alpha}$. This immediately rules out (2) and (3) as scenarios for the structure of $\mathcal{O}_H^\times[W_g]$, leaving only (4). Hence, V_g is induced from a character of a real quadratic field F. In that case, the lines spanned by $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ are interchanged under the action of any reflection in G, and hence the condition $u_{\alpha/\beta} = 0$ implies that $u_{\beta/\alpha} = 0$ as well, thus forcing the vanishing of the full $\mathcal{O}_H^\times[W_g]$. This contradiction to Lemma 1.2 shows that (6) imposes a non-trivial linear condition on x and y, and therefore that A is unique up to scaling. □

Lemma 1.4. *Let A be any element of $H_\wp \otimes W_g$ satisfying the conditions in Lemma 1.3. Then the following are equivalent:*

(a) $\mathrm{Trace}(A) \neq 0$*;*
(b) *The representation ϱ_g is not induced from a character of a real quadratic field in which the prime p splits.*

Proof. The vanishing of $\mathrm{Trace}(A)$ is equivalent to the vanishing of the entry y in (6), and hence to the vanishing of $\log_\wp(u_{\alpha/\beta})$, and therefore of $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ as well. This implies that $\mathcal{O}_H^\times[W_g]$ is one-dimensional and generated by u_1. As in the proof of Lemma 1.3, this rules out (2) and (3), leaving only (4) as a possibility, i.e., V_g is necessarily induced from a character of a real quadratic field F. Furthermore, $\tau_\wp$ fixes the group $\mathcal{O}_H^\times[W_g]$ generated by the fundamental unit of F, which occurs precisely when p splits in F. The lemma follows. □

Assume from now on that the equivalent conditions of Lemma 1.4 hold. One can then define $A_g \in H_\wp \otimes W_g$ to be the unique $H_\wp^\times$- multiple of A satisfying

$$\mathrm{Trace}(A_g) = 1.$$

As in lemma 1.2, $H_\wp \otimes W_g$ is endowed with the diagonal action of $G_\wp$ which acts on both $H_\wp$ and on W_g in a natural way. Given $A \in H_\wp \otimes W_g$ and $\sigma \in G_\wp$, let us write ${}^\sigma A$ for the image of A by the action of σ on the first factor $H_\wp$, and $\sigma \cdot A_g$ for the image of A by the action of σ by conjugation on the second factor W_g.

Lemma 1.5. *The endomorphism A_g belongs to the space $(H_\wp \otimes W_g)^{G_\wp}$ of $G_\wp$-invariants for the diagonal action of $G_\wp$ on $H_\wp \otimes W_g$, i.e.,*

$${}^{\tau_\wp}A_g = \tau_\wp^{-1} \cdot A_g.$$

Proof. Relative to the $\mathbb{Q}_p$-basis for V_g used in the proof of Lemma 1.3, the endomorphism A_g is represented by a matrix of the form

$$\begin{pmatrix} 0 & \frac{\log_\wp(u_1)}{\log_\wp(u_{\alpha/\beta})} \\ 0 & 1 \end{pmatrix}.$$

The lemma follows immediately from this in light of the fact that conjugation by $\varrho_g(\tau_\wp)$ preserves the diagonal entries in such a matrix representation while multiplying its upper right hand entry by α/β, whereas $\tau_\wp$ acts on the upper right-hand entry of the above matrix as multiplication by β/α. □

The matrix A_g gives rise to a G-equivariant homomorphism $\Phi_g : H^\times \longrightarrow H_\wp \otimes W_g$ by setting

$$\Phi_g(x) = \sum_{\sigma\in G} \log_\wp({}^\sigma x)\cdot(\sigma^{-1}\cdot A_g), \tag{7}$$

where, just as above, the group G acts on $H_\wp \otimes W_g$ trivially on the first factor and through the usual conjugation action induced by ρ_g on the second factor.

Lemma 1.6. *The homomorphism Φ_g takes values in W_g.*

Proof. For any $x \in (H\otimes\mathbb{Q}_p)^\times$ we have

$$\begin{aligned} {}^{\tau_\wp}\Phi_g(x) &= \sum_{\sigma\in G} \log_\wp({}^{\tau_\wp\sigma}x)\cdot(\sigma^{-1}\cdot{}^{\tau_\wp}A_g) \\ &= \sum_{\sigma\in G} \log_\wp({}^{\tau_\wp\sigma}x)\cdot(\sigma^{-1}\cdot{\tau_\wp}^{-1}\cdot A_g) \\ &= \sum_{\sigma\in G} \log_\wp({}^{\tau_\wp\sigma}x)\cdot((\tau_\wp\sigma)^{-1}\cdot A_g) \\ &= \Phi_g(x), \end{aligned}$$

where Lemma 1.5 has been used to derive the second equation. ☐

By a slight abuse of notation, we shall continue to denote with the same symbol the homomorphism

$$\Phi_g : (H\otimes\mathbb{Q}_p)^\times \longrightarrow H_\wp \otimes W_g$$

obtained from (7) by extending scalars. Note that $H_\wp^\times$ embeds naturally in $(H\otimes\mathbb{Q}_p)^\times$.

Lemma 1.7. *The homomorphism Φ_g vanishes on $\mathcal{O}_H^\times\otimes\mathbb{Q}_p$ and $\Phi_g(H_\wp^\times)\subseteq H_\wp\otimes W_g^{\mathrm{ord}}$.*

Proof. Picking $u\in\mathcal{O}_H^\times$ and an arbitrary $B\in W_g$, set

$$U_g^\times := \sum_{\sigma\in G} {}^\sigma u\otimes(\sigma\cdot B) \in (\mathcal{O}_H^\times\otimes W_g)^G, \qquad U_g := (\log_\wp\otimes\mathrm{id})(U_g)$$

as in the statement of Lemma 1.3. Note that $U_g^\times$ is either trivial or a generator of the one-dimensional space $(\mathcal{O}_H^\times\otimes W_g)^G$. We have

$$\begin{aligned} \mathrm{Trace}(\Phi_g(u)\cdot B) &= \mathrm{Trace}\left(\left(\sum_{\sigma\in G}\log_\wp({}^\sigma u)\cdot(\sigma^{-1}\cdot A_g)\right)\cdot B\right) \\ &= \mathrm{Trace}\left(A_g\cdot\left(\sum_{\sigma\in G}\log_\wp({}^\sigma u)\cdot(\sigma\cdot B)\right)\right) \\ &= \mathrm{Trace}\left(A_g\cdot(\log_\wp\otimes\mathrm{Id})(U_g^\times)\right) = \mathrm{Trace}\left(A_g\cdot U_g\right). \end{aligned}$$

It follows from Property (a) satisfied by A (and hence A_g in particular) in Lemma 1.3 that

$$\mathrm{Trace}\left(\Phi_g(u)\cdot B\right) = 0, \qquad \text{for all } B\in H_\wp\otimes W_g.$$

The first assertion in the lemma follows from the non-degeneracy of the $H_\wp$-valued trace pairing on $H_\wp\otimes W_g$. The second assertion follows from Property (b) satisfied by A and by A_g in Lemma 1.3.

☐

Let now $\ell \nmid Np$ be a rational prime, and let λ be a prime of H above ℓ. Let $u(\lambda)\in\mathcal{O}_H[1/\lambda]^\times\otimes\mathbb{Q}$ be a λ-unit of H satisfying

$$\mathrm{Norm}_{\mathbb{Q}}^H(u(\lambda)) = \ell. \tag{8}$$

This condition makes $u(\lambda)$ well-defined up to the addition of elements in $\mathcal{O}_H^\times \otimes \mathbb{Q}$, and hence the element

$$A_g(\lambda) := \Phi_g(u(\lambda)) = \sum_{\sigma \in G} \log_\wp({}^\sigma u(\lambda)) \cdot (\sigma^{-1} \cdot A_g)$$

is well-defined, by Lemma 1.7. Although $A_g(\lambda)$ only belongs to $H_\wp \otimes W_g$ a priori, we have:

Lemma 1.8. *The trace of the endomorphism $A_g(\lambda)$ is equal to $\log_p(\ell)$.*

Proof. Since the trace of A_g and its conjugates are all equal to 1, we have

$$\begin{aligned} \mathrm{Trace}(A_g(\lambda)) &= \sum_{\sigma \in G} \log_\wp({}^\sigma u(\lambda)) \cdot \mathrm{Trace}(\sigma^{-1} \cdot A_g) \\ &= \sum_{\sigma \in G} \log_\wp({}^\sigma u(\lambda)) \\ &= \log_\wp \left(\mathrm{Norm}_{\mathbb{Q}}^H (u(\lambda))\right). \end{aligned}$$

The latter expression is equal to $\log_p(\ell)$, by (8). □

Remark 1.9. Although $A_g(\lambda)$ belongs to W_g by Lemma 1.6, the entries of the matrix representing $A_g(\lambda)$ relative to an L-basis for V_g^L are L-linear combinations of products of $\wp$-adic logarithms of units and ℓ-units in H, and in particular $A_g(\lambda)$ need not lie in W_g^L. (In fact, it never does, since its trace is not algebraic.)

In addition to the invariant $A_g(\lambda)$, the choice of the prime λ of H above ℓ also determines a well-defined Frobenius element τ_λ in $G = \mathrm{Gal}\,(H/\mathbb{Q})$, and even in $\mathrm{Gal}\,(H_g/\mathbb{Q})$, since $\mathrm{Gal}\,(H_g/H)$ lies in the center of this group.

We are now ready to state the main theorem of this section:

Theorem 1.10. *For all rational primes $\ell \nmid Np$,*

$$a_\ell(g'_\alpha) = \mathrm{Trace}(\varrho_g(\tau_\lambda) A_g(\lambda)).$$

Remark 1.11. This invariant does not depend on the choice of a prime λ of H above ℓ, since replacing λ by another such prime has the effect of conjugating the endomorphisms $\varrho_g(\tau_\lambda)$ and $A_g(\lambda)$ by the same element of $\mathrm{Aut}(V_g)$.

Proof of Theorem 1.10. Let $\mathbb{Q}[\varepsilon]$ denote the ring of dual numbers over $\mathbb{Q}_p$, with $\varepsilon^2 = 0$, and let

$$\tilde{\varrho}_g : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(\mathbb{Q}_p[\varepsilon])$$

be the unique first order α-ordinary deformation of ϱ_g satisfying

$$\det \tilde{\varrho}_g = \chi_g(1 + \log_p \chi_{\mathrm{cyc}} \cdot \varepsilon).$$

This representation may be written as

$$\tilde{\varrho}_g = (1 + \varepsilon \cdot \kappa_g) \cdot \varrho_g \quad \text{for some} \quad \kappa_g : G_{\mathbb{Q}} \longrightarrow W_g. \tag{9}$$

The multiplicativity of $\tilde{\varrho}_g$ implies that the function κ_g is a 1-cocycle on $G_{\mathbb{Q}}$ with values in W_g, whose class in $H^1(\mathbb{Q}, W_g)$ (which shall be denoted with the same symbol, by a slight abuse of notation) depends only on the isomorphism class of $\tilde{\varrho}_g$. Furthermore,

$$a_\ell(g_\alpha) + \varepsilon \cdot a_\ell(g'_\alpha) = \mathrm{Trace}(\tilde{\varrho}_g(\tau_\lambda)) = a_\ell(g) + \varepsilon \cdot \mathrm{Trace}(\kappa_g(\tau_\lambda)\varrho_g(\tau_\lambda)),$$

and hence

$$a_\ell(g'_\alpha) = \mathrm{Trace}(\varrho_g(\tau_\lambda)\kappa_g(\tau_\lambda)). \tag{10}$$

To make $\kappa_g(\tau_\lambda)$ explicit, observe that the inflation-restriction sequence combined with global class field theory for H gives rise to a series of identifications

$$\begin{aligned} H^1(\mathbb{Q}, W_g) &\stackrel{\mathrm{res}_H}{\longrightarrow} \hom(G_H, W_g)^G \\ &= \hom_G\left(\frac{(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times}{\mathcal{O}_H^\times \otimes \mathbb{Q}_p}, W_g\right). \end{aligned}$$

Under this identification, the class κ_g can be viewed as an element of the space

$$H^1_{\mathrm{ord}}(\mathbb{Q}, W_g) = \left\{\Phi \in \hom_G\left(\frac{(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times}{\mathcal{O}_H^\times \otimes \mathbb{Q}_p}, W_g\right) \text{ such that } \Phi(H_\wp^\times) \subset W_g^{\mathrm{ord}}\right\}.$$

But the homomorphism Φ_g of (7) belongs to the same one-dimensional space, by Lemma 1.6 and 1.7. By global class field theory, the endomorphism $\kappa_g(\tau_\lambda)$ is therefore a $\mathbb{Q}_p^\times$-multiple of $\Phi_g(u_g(\lambda)) = A_g(\lambda)$. The fact that these endomorphisms are actually equal now follows by comparing their traces and noting that

$$\mathrm{Trace}(\kappa_g(\tau_\lambda)) = \log_p \chi_{\mathrm{cyc}}(\ell) = \log_p(\ell),$$

while

$$\mathrm{Trace}(A_g(\lambda)) = \log_p(\ell),$$

by Lemma 1.8. Theorem 1.10 follows. □

Corollary 1.12. *If the rational prime $\ell \nmid Np$ splits completely in $H/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = (1/2) \cdot a_\ell(g) \cdot \log_p(\ell).$$

Proof. The hypothesis implies that $\varrho_g(\tau_\lambda)$ is a scalar, and hence that $\varrho_g(\tau_\lambda) = \frac{1}{2}a_\ell(g)$. It follows that

$$\mathrm{Trace}(\varrho_g(\tau_\lambda)A_g(\lambda)) = (1/2) \cdot a_\ell(g) \cdot \mathrm{Trace}(A_g(\lambda)) = (1/2) \cdot a_\ell(g) \cdot \log_p(\ell).$$

The corollary now follows from Theorem 1.10. □

Example 1.13. Let χ be a Dirichlet character of conductor 171 with order 3 at 9 and 2 at 19. Then $S_1(171, \chi)$ is a $\mathbb{Q}(\chi)$-vector space of dimension 2. It is spanned by an eigenform

$$g = q + \zeta q^2 + \zeta^3 q^3 - \zeta^2 q^5 + (\zeta^2 - 1)q^6 + \cdots$$

defined over $L := \mathbb{Q}(\zeta)$, with ζ a primitive 12th root of unity, and its Galois conjugate. (See [BL] for all weight one eigenforms of level at most 1500.) The associated projective representation ϱ_g has A_4-image and factors through the field

$$H = \mathbb{Q}(a),\ a^4 + 10a^3 + 45a^2 + 81a + 81 = 0.$$

Let $p = 13$, which splits completely in L. The representation ϱ_g is regular at 13, with eigenvalues $\alpha = \zeta$ and $\beta = -\zeta^3$. We computed the first order deformations through each of g_α and g_β to precision 13^{10}, and q-adic precision $q^{37{,}000}$, using methods based upon the algorithms in [La].

The predictions made from Theorem 1.10 for $a_\ell(g'_\alpha)$ depend upon the conjugacy class of the Frobenius at ℓ in $\mathrm{Gal}\,(H/\mathbb{Q})$. For all primes $\ell < 37{,}000$ which split completely in H, such as $\ell = 109, 179, 449, 467, 521, \ldots$, we verified that

$$a_\ell(g'_\alpha) = (1/2) \cdot a_\ell(g) \cdot \log_{13}(\ell) \pmod{13^{10}},$$

as asserted by Corollary 1.12.

2. CM forms

This section focuses on the case where $g = \theta_{\psi_g}$ is the CM theta series attached to a character

$$\psi_g : G_K \longrightarrow L^\times$$

of a quadratic imaginary field K. The main theorems are Theorems 2.1 and 2.3 below, which will be derived in two independent ways, both "from first principles" and by specialising Theorem 1.10.

As in the previous section, the choice of an embedding of L into $\mathbb{Q}_p$ allows us to view ψ_g as a $\mathbb{Q}_p^\times$-valued character, and the weight one form g as a modular form with coefficients in $\mathbb{Q}_p$.

For a character $\psi : G_K \longrightarrow L^\times$, the notation ψ' will be used to designate the composition of ψ with conjugation by the non-trivial element in $\mathrm{Gal}\,(K/\mathbb{Q})$:

$$\psi'(\sigma) = \psi(\tau\sigma\tau^{-1}),$$

where τ is any element of $G_\mathbb{Q}$ which acts non-trivially on K.

The Artin representation ϱ_g is induced from ψ_g and its restriction to G_K is the direct sum $\psi_g \oplus \psi_g'$ of two characters of K, which are *distinct* by the irreducibility of ϱ_g resulting from the fact that g is a cusp form. In this case, the field H is the ring class field of K which is cut out by the non-trivial ring class character $\psi := \psi_g/\psi_g{}'$. The Galois group $G := \mathrm{Gal}\,(H/\mathbb{Q})$ is a generalised dihedral group containing $Z := \mathrm{Gal}\,(H/K)$ as its abelian normal subgroup of index two.

The case of CM forms can be further subdivided into two sub-cases, depending on whether p is split or inert in K.

2.1. **The case where p splits in K.** Write $p\mathcal{O}_K = \mathfrak{p}\,\mathfrak{p}'$, and fix a prime $\wp$ of $\bar{\mathbb{Q}}$ above $\mathfrak{p}$. The roots of the p-th Hecke polynomial of g are

$$\alpha = \psi_g(\mathfrak{p}), \qquad \beta = \psi_g(\mathfrak{p}').$$

This case is notable in that the Hida family $\mathbf{g}$ passing through g_α can be written down explicitly as a family of theta series. Its weight k specialisation $\mathbf{g}_k$ is the theta-series attached to the character $\psi_g\Psi^{k-1}$, where Ψ is a CM Hecke character of weight $(1,0)$ which is unramified at $\mathfrak{p}$. For all rational primes $\ell \nmid Np$, the ℓ-th fourier coefficient of $\mathbf{g}_k$ is given by

$$a_\ell(\mathbf{g}_k) = \begin{cases} \psi_g(\lambda)\Psi^{k-1}(\lambda') + \psi_g(\lambda')\Psi^{k-1}(\lambda) & \text{if } \ell = \lambda\lambda' \text{ splits in } K; \\ 0 & \text{if } \ell \text{ is inert in } K. \end{cases} \tag{11}$$

Letting h be the class number of K and t the cardinality of the unit group $\mathcal{O}_K^\times$, the character Ψ satisfies

$$\Psi(\lambda)^{ht} = u_\lambda^t, \qquad \text{where } (u_\lambda) := \lambda^h,$$

for any prime ideal λ of $\mathcal{O}_K$ whose norm is the rational prime $\ell = \lambda\lambda'$. Let u_λ' denote the conjugate of u_λ in $K/\mathbb{Q}$. It follows that

$$\frac{d}{dk}\Psi^{k-1}(\lambda)_{k=1} \;=\; \log_{\mathfrak{p}}(u(\lambda)), \qquad \text{where } u(\lambda) := u_\lambda \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q},$$

and likewise that

$$\frac{d}{dk}\Psi^{k-1}(\lambda')_{k=1} \;=\; \log_{\mathfrak{p}}(u(\lambda)'), \qquad \text{where } u(\lambda)' := u_\lambda' \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q}.$$

In light of (11), we have obtained:

Theorem 2.1. *For all rational primes ℓ that do not divide Np,*

$$a_\ell(g_\alpha') = \begin{cases} \big(\psi_g(\lambda)\log_{\mathfrak{p}}(u(\lambda')) + \psi_g(\lambda')\log_{\mathfrak{p}}(u(\lambda))\big) & \textit{if } \ell = \lambda\lambda' \textit{ splits in } K; \\ 0 & \textit{if } \ell \textit{ is inert in } K. \end{cases} \tag{12}$$

Thus, the prime fourier coefficients of g'_α are supported at the primes ℓ which are split in K, where they are (algebraic multiples of) the $\mathfrak{p}$-adic logarithms of ℓ-units in this quadratic field. This general pattern will persist in the other settings to be described below, with the notable feature that the fourier coefficients of g'_α will be more complicated expressions involving, in general, the p-adic logarithms of units and ℓ-units in the full ring class field H.

The reader will note Theorem 2.1 is consistent with Theorem 1.10, and could also have been deduced from it. More precisely, choose a basis of V_g consisting of eigenvectors for the action of G_K (and hence also, of $\tau_\wp$) which are interchanged by some element $\tau \in G_{\mathbb{Q}} - G_K$. Relative to such a basis, the endomorphisms U_g and A_g are represented by the following matrices, in which u_ψ and τu_ψ are generators of the spaces of ψ and ψ^{-1}-isotypic vectors in the group of elliptic units in $\mathcal{O}_H^\times \otimes L$:

$$U_g : \begin{pmatrix} 0 & \log_\wp(u_\psi) \\ \log_\wp(u'_\psi) & 0 \end{pmatrix}, \qquad A_g : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that, if $\ell = \lambda\lambda'$ is split in K, then $A_g(\lambda)$ is represented by the matrix

$$A_g(\lambda) : \begin{pmatrix} \log_\wp(u(\lambda')) & 0 \\ 0 & \log_\wp(u(\lambda)) \end{pmatrix},$$

while $A_g(\lambda) = \frac{1}{2}\log_p(\ell)$ is the scalar matrix with trace equal to $\log_p(\ell)$ if ℓ is inert in K.

2.2. **The case where p is inert in K.** We now turn to the more interesting case where p is inert in K. Let $\sigma_\wp := \tau_\wp^2$ denote the frobenius element in G_K attached to the prime $\wp$ of H (which is well-defined modulo the inertia subgroup at $\wp$). Note that the prime p splits completely in H/K, since the image of $\tau_\wp$ in G is a reflection in this generalised dihedral group. The image of $\sigma_\wp$ in $\mathrm{Gal}\,(H_g/K)$ therefore belongs to the subgroup $\mathrm{Gal}\,(H_g/H)$ whose image under ϱ_g consists of scalar matrices. Similar notations and remarks apply to any rational prime ℓ which is inert in $K/\mathbb{Q}$.

Relative to an eigenbasis (e_1, e_2) for the action of G_K on V_g, the Galois representation ϱ_g takes the form

$$\varrho_g(\sigma) = \begin{pmatrix} \psi_g(\sigma) & 0 \\ 0 & \psi'_g(\sigma) \end{pmatrix} \quad \text{for } \sigma \in G_K. \tag{13}$$

The homomorphisms $\psi_g, \psi'_g : G_K \longrightarrow \mathbb{Q}_p^\times$ factor through $\mathrm{Gal}\,(H_g/K)$ and satisfy

$$\psi_g(\tau\sigma\tau^{-1}) = \psi'_g(\sigma), \qquad \text{for all } \tau \in G_{\mathbb{Q}} - G_K, \quad \sigma \in G_K.$$

It follows that $\varrho_g(\tau)$ interchanges the lines spanned by e_1 and e_2, for any element $\tau \in G_{\mathbb{Q}} - G_K$. The restriction of ϱ_g to $G_{\mathbb{Q}} - G_K$ can therefore be described in matrix form by

$$\varrho_g(\tau) = \begin{pmatrix} 0 & \eta_g(\tau) \\ \eta'_g(\tau) & 0 \end{pmatrix} \quad \text{for } \tau \in G_{\mathbb{Q}} - G_K, \tag{14}$$

where η_g and η'_g are L-valued functions on $G_{\mathbb{Q}} - G_K$ that satisfy

$$\eta_g(\tau_1)\eta'_g(\tau_2) = \psi_g(\tau_1\tau_2) = \psi'_g(\tau_2\tau_1), \qquad \text{for all } \tau_1, \tau_2 \in G_{\mathbb{Q}} - G_K, \tag{15}$$

as well as the relations

$$\begin{matrix} \eta_g(\sigma\tau) = \psi_g(\sigma)\eta_g(\tau), & \eta_g(\tau\sigma) = \psi'_g(\sigma)\eta_g(\tau), \\ \eta'_g(\sigma\tau) = \psi'_g(\sigma)\eta'_g(\tau), & \eta'_g(\tau\sigma) = \psi_g(\sigma)\eta'_g(\tau), \end{matrix} \qquad \text{for all } \sigma \in G_K, \quad \tau \in G_{\mathbb{Q}} - G_K. \tag{16}$$

After re-scaling e_1 and e_2 if necessary, we may assume that $\tau_\wp \in G_{\mathbb{Q}} - G_K$ is sent to the matrix

$$\varrho_g(\tau_\wp) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}, \qquad \text{with } \zeta^2 = -\chi_g(p). \tag{17}$$

The eigenvalues of $\varrho_g(\tau_\wp)$ are equal to $\alpha := \zeta$ and $\beta := -\zeta$, and hence g is *always regular* in this setting.

Let

$$\tilde{\varrho}_g : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}(\tilde{V}_g)$$

denote the first-order infinitesimal deformation of ϱ_g attached to the Hida family $\mathbf{g}$ passing through a choice of p-stabilization g_α of g, where $\alpha \in \{\zeta, -\zeta\}$. The module $\tilde{V}_g$ is free of rank two over the ring $\mathbb{Q}_p[\varepsilon]/(\varepsilon^2) = \mathbb{Q}_p[[T]]/(T^2)$ arising from the mod T^2 reduction of the representation $\varrho_{\mathbf{g}}$ attached to $\mathbf{g}$. Choose any $\mathbb{Q}_p[\varepsilon]$-basis $(\tilde{e}_1, \tilde{e}_2)$ of $\tilde{V}_g$ lifting (e_1, e_2), and note that the restriction of $\tilde{\varrho}_g$ to G_K is given by:

$$\tilde{\varrho}_g(\sigma) = \begin{pmatrix} \psi_g(\sigma)\cdot(1+\kappa(\sigma)\cdot\varepsilon) & \psi'_g(\sigma)\cdot\kappa_\psi(\sigma)\cdot\varepsilon \\ \psi_g(\sigma)\cdot\kappa'_\psi(\sigma)\cdot\varepsilon & \psi'_g(\sigma)\cdot(1+\kappa'(\sigma)\cdot\varepsilon) \end{pmatrix}, \qquad \text{for all } \sigma \in G_K. \tag{18}$$

In this expression,

(a) The functions κ and κ' are continuous homomorphisms from G_K to $\mathbb{Q}_p$, i.e., elements of $H^1(K, \mathbb{Q}_p)$, which are interchanged by conjugation by the involution in $\mathrm{Gal}\,(K/\mathbb{Q})$:
$$\kappa(\tau\sigma\tau^{-1}) = \kappa'(\sigma), \qquad \tau \in G_{\mathbb{Q}} - G_K, \quad \sigma \in G_K.$$

(b) The functions $\kappa_\psi, \kappa'_\psi : G_K \longrightarrow \mathbb{Q}_p$ are one-cocycles with values in $\mathbb{Q}_p(\psi)$, and give rise to well defined classes
$$\kappa_\psi \in H^1(K, \mathbb{Q}_p(\psi)), \qquad \kappa'_\psi \in H^1(K, \mathbb{Q}_p(\psi^{-1})),$$
which also satisfy
$$\kappa_\psi(\tau\sigma\tau^{-1}) = \kappa'_\psi(\sigma), \qquad \tau \in G_{\mathbb{Q}} - G_K, \quad \sigma \in G_K.$$

For each rational prime $\ell \nmid Np$, the ℓ-th fourier coefficient $a_\ell(g'_\alpha)$ is given by

$$a_\ell(g'_\alpha) = \frac{d}{dk}\mathrm{Trace}(\varrho_{\mathbf{g}}(\tau_\lambda))_{k=1} \tag{19}$$

Observe that the spaces $H^1(K, \mathbb{Q}_p)$ and $H^1(K, \mathbb{Q}_p(\psi))$ are of dimensions two and one respectively over $\mathbb{Q}_p$, since $\psi \neq 1$. More precisely, restriction to the inertia group at p combined with local class field theory induces an isomorphism

$$H^1(K, \mathbb{Q}_p) = \hom(\mathcal{O}_{K_p}^\times, \mathbb{Q}_p) = \mathbb{Q}_p \log_p(z) \oplus \mathbb{Q}_p \log_p(z'). \tag{20}$$

Let $\mathcal{O}_H^{\times,\psi}$ denote the (one-dimensional) ψ-isotypic component of $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$ on which $\mathrm{Gal}\,(H/K)$ acts through the character ψ, and denote by $\wp$ the prime of H above p arising from our chosen embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. Restriction to the inertia group at $\wp$ in G_H likewise gives rise to an identification

$$H^1(K, \mathbb{Q}_p(\psi)) = \hom(\mathcal{O}_{H_\wp}^\times/\mathcal{O}_H^{\times,\psi}, \mathbb{Q}_p) = \mathbb{Q}_p \cdot (\log_\wp(u'_\psi)\log_\wp(z) - \log_\wp(u_\psi)\log_\wp(z')). \tag{21}$$

In the above equation, u_ψ is to be understood as the natural image in $\mathcal{O}_{H_\wp}^\times = \mathcal{O}_{K_p}^\times$ of an element of the form

$$\sum_{\sigma \in Z} \psi^{-1}(\sigma) u^\sigma \in (\mathcal{O}_H^\times \otimes L)^\psi,$$

where u is an $L[G]$-module generator of $\mathcal{O}_H^\times \otimes L$, and u'_ψ is the image of u_ψ under the conjugation action $K_p \longrightarrow K_p$. Note that replacing u by λu for some $\lambda \in L[G]$ has the effect of multiplying both u_ψ and u'_ψ by $\psi(\lambda) \in \mathbb{Q}_p$, so that the $\mathbb{Q}_p$-line spanned by the right-hand side of (21) is independent of the choice of $u \in \mathcal{O}_H^\times$.

It follows from (20) and (21) that the total deformation space of ϱ_g (before imposing any ordinarity hypotheses, or restrictions on the determinant) is three dimensional.

Let $v_g^+ := e_1 + e_2$ and $v_g^- := e_1 - e_2$ be the eigenvectors for $\tau_\wp$ acting on V_g, with eigenvalues ζ and $-\zeta$ respectively. Let κ_p and $\kappa_{\psi,\wp}$ denote the restrictions κ and κ_ψ to the inertia groups at p and $\wp$ in G_H and $G_K$respectively. Both can be viewed as characters of $K_p^\times = H_\wp^\times$ after identifiying the abelianisations of G_{K_p} and $G_{H_\wp}$ with a quotient of $K_p^\times$ via local class field theory.

Lemma 2.2. *The following are equivalent:*

(a) *The inertia group at $\wp$ acts as the identity on some lift $\tilde{v}_g^+$ of v_g^+ to $\tilde{V}_g$;*
(b) *The inertia group at $\wp$ acts as the identity on all lifts $\tilde{v}_g^+$ of v_g^+ to $\tilde{V}_g$;*
(c) *The restrictions κ_p and $\kappa_{\psi,\wp}$ satisfy*

$$\kappa_p(x) = -\kappa_{\psi,\wp}(x), \qquad \text{for all } x \in \mathcal{O}_{K_p}^\times.$$

Similar statements hold when v_g^+ is replaced by v_g^-, where the conclusion is that $\kappa_p = \kappa_{\psi,\wp}$.

Proof. The equivalence of the first two conditions follows from the fact that $\varepsilon\tilde{V}_g \simeq V_g$ is unramified at p and hence that inertia acts as the identity on the kernel of the natural map $\tilde{V}_g \longrightarrow V_g$. To check the third, note that the inertia group I_p at p is contained in G_K, since K is unramified at p, and observe that any $\sigma \in I_p$ sends $\tilde{e}_1 + \tilde{e}_2$ to

$$\begin{aligned} \tilde{\varrho}_g(\sigma)(\tilde{e}_1+\tilde{e}_2) &= \tilde{e}_1+\tilde{e}_2+\varepsilon\cdot(\kappa(\sigma)\tilde{e}_1+\kappa'_\psi(\sigma)\tilde{e}_2+\kappa_\psi(\sigma)\tilde{e}_1+\kappa'(\sigma)\tilde{e}_2) \\ &= \tilde{e}_1+\tilde{e}_2+\varepsilon\cdot((\kappa(\sigma)+\kappa_\psi(\sigma))\tilde{e}_1+(\kappa'(\sigma)+\kappa'_\psi(\sigma))\tilde{e}_2). \end{aligned}$$

The lemma follows. □

A lift $\tilde{\varrho}_g$ of ϱ_g is ordinary relative to the space spanned by v_g^+ if and only if it satisfies the equivalent conditions of Lemma 2.2. This lemma merely spells out the proof of the Bellaiche-Dimitrov theorem on the one-dimensionality of the tangent space of the eigencurve at the point associated to g_α. More precisely, the general ordinary first-order deformation of ϱ_g is completely determined by the pair $(\kappa_p, \kappa_{\psi,\wp})$, which depends on a single linear parameter $\mu \in \bar{\mathbb{Q}}_p$ and is given by the rule

$$\kappa_p(z) = \mu(\log_\wp(u'_\psi)\cdot\log_\wp(z) - \log_\wp u_\psi \cdot \log_\wp(z')), \tag{22}$$

$$\kappa_{\psi,p}(z) = \pm\mu(\log_\wp(u'_\psi)\cdot\log_\wp(z) - \log_\wp u_\psi \cdot \log_\wp(z')), \tag{23}$$

where the sign in the second formula depends on whether one is working with the ordinary deformation of g_α or g_β.

Let us now make use of the fact that

$$\det(\tilde{\varrho}_g) = 1 + \varepsilon\log_p\chi_{\rm cyc} = 1+\varepsilon\log_p(zz').$$

Since $\det(\tilde{\varrho}_g) = 1+\varepsilon(\kappa+\kappa')$, this condition implies that

$$\mu = \frac{1}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)},$$

and hence that κ_p and $\kappa_{\psi,\wp}$ are given by

$$\kappa_p(z) = \frac{\log_\wp(u'_\psi)\cdot\log_\wp(z) - \log_\wp(u_\psi)\cdot\log_\wp(z')}{\log_\wp(u'_\psi)-\log_\wp(u_\psi)}, \tag{24}$$

$$\kappa_{\psi,\wp}(z) = \pm\frac{\log_\wp(u'_\psi)\cdot\log_\wp(z) - \log_\wp(u_\psi)\cdot\log_\wp(z')}{\log_\wp(u'_\psi)-\log_\wp(u_\psi)}. \tag{25}$$

Equations (24) and (25) give a completely explicit description of the first order deformation $\tilde{\varrho}_{g_\alpha}$ and $\tilde{\varrho}_{g_\beta}$, from which the fourier coefficients of g'_α and g'_β shall be readily calculated.

The formula for the ℓ-th fourier coefficient of g'_α involves the unit u_ψ above as well as certain ℓ-units in $\mathcal{O}_H[1/\ell]^\times \otimes L$ whose definition depends on whether or not the prime ℓ is split or inert in $K/\mathbb{Q}$.

If $\ell = \lambda\lambda'$ splits in $K/\mathbb{Q}$, let $u(\lambda)$ and $u(\lambda')$ denote, as before, the ℓ-units in $\mathcal{O}_K[1/\ell]^\times \otimes \mathbb{Q}$ of norm ℓ with prime factorisation λ and λ' respectively. Set

$$u_g(\lambda) := u(\lambda) \otimes \psi_g(\lambda) + u(\lambda') \otimes \psi_g(\lambda'), \qquad u_g(\lambda') := u(\lambda') \otimes \psi_g(\lambda) + u(\lambda) \otimes \psi_g(\lambda').$$

In other words, $u_g(\lambda)$ is the unique element of $\mathcal{O}_K[1/\ell]^\times \otimes L$ whose prime factorisation is equal to $\psi_g(\lambda) \cdot \lambda + \psi_g(\lambda') \cdot \lambda'$. Note that, if ℓ splits completely in $H/\mathbb{Q}$, i.e., if $\varrho_g(\tau_\lambda)$ is equal to a scalar ζ, then $u_g(\lambda) = u_g(\lambda') = \ell \otimes \zeta$, but that otherwise $u_g(\lambda)$ and ℓ generate the L-vector space $\mathcal{O}_K[1/\ell]^\times \otimes L$ of ℓ-units of K (tensored with L).

If ℓ is inert in $K/\mathbb{Q}$, choose a prime λ of H lying above ℓ, and let $u(\lambda) \in \mathcal{O}_H[1/\lambda]^\times \otimes \mathbb{Q}$ be any λ-unit of H satisfying $\mathrm{ord}_\lambda(u(\lambda)) = 1$, which is well defined up to units in $\mathcal{O}_H^\times$. Define the elements

$$\begin{aligned} u_\psi(\lambda) &= \sum_{\sigma \in Z} \psi^{-1}(\sigma) \otimes {}^\sigma u(\lambda) \quad \in \quad L \otimes \mathcal{O}_H[1/\ell]^\times, \\ u'_\psi(\lambda) &= \tau_\wp u_\psi(\lambda) \quad \in \quad L \otimes \mathcal{O}_H[1/\ell]^\times. \end{aligned}$$

Thus $u_\psi(\lambda)$ lies in the ψ-component $\mathcal{O}_H[1/\ell]^\times[\psi]$ and is well-defined up to the addition of multiples of u_ψ, where

$$u_\psi := \sum_{\sigma \in Z} \psi^{-1}(\sigma) \otimes {}^\sigma u \quad \in \quad L \otimes \mathcal{O}_H[1/\ell]^\times,$$

for any unit $u \in \mathcal{O}_H^\times$, while $u'_\psi(\lambda)$ lies in the ψ^{-1} component and is well-defined up to the addition of multiples of u'_ψ, where

$$u'_\psi = \tau_\wp u_\psi.$$

Recall the function $\eta'_g : G_\mathbb{Q} \setminus G_K$ introduced in (14), with values in the roots of unity of $L^\times$. The main result of this section is:

Theorem 2.3. *Let $\ell \nmid Np$ be a rational prime.*

(a) *If $\ell = \lambda\lambda'$ splits in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = a_\ell(g'_\beta) = \frac{\log_\wp(u'_\psi) \cdot \log_\wp(u_g(\lambda)) - \log_\wp(u_\psi) \cdot \log_\wp(u_g(\lambda'))}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)}. \tag{26}$$

(b) *If ℓ remains inert in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = \eta'_g(\tau_\lambda) \frac{\log_\wp(u'_\psi) \log_\wp(u_\psi(\lambda)) - \log_\wp(u_\psi) \log_\wp(u'_\psi(\lambda))}{\log_\wp(u'_\psi) - \log_\wp(u_\psi))}.$$

Proof. Let us first compute first the fourier coefficients at primes $\ell \nmid Np$ that split as $\ell = \lambda\lambda'$ in K. Let σ_λ and $\sigma_{\lambda'}$ be the frobenius elements associated to λ and λ' respectively. They are well-defined elements in the Galois group of any abelian extension of K in which ℓ is unramified.

It follows from (19) and the matrix expression for $\tilde{\varrho}_{g|G_K}$ given in (18) that

$$\begin{aligned} a_\ell(g'_\alpha) &= \psi_g(\lambda)\kappa(\lambda) + \psi_g(\lambda')\kappa(\lambda') \\ &= \psi_g(\lambda)\kappa_p(u(\lambda)) + \psi_g(\lambda')\kappa_p(u'(\lambda)). \end{aligned}$$

Equation (26) then follows from the formula for $\kappa_p(z)$ given in (24).

We now turn now to the computation of the fourier coefficients of g'_α at primes $\ell \nmid Np$ that remain inert in K. Let τ_λ denote the Frobenius element in $\mathrm{Gal}\,(M/\mathbb{Q})$ associated to the choice of a prime ideal λ above ℓ in $\bar{\mathbb{Q}}$, and let $\sigma_\lambda := \tau_\lambda^2$ denote the associated frobenius element in $\mathrm{Gal}\,(M/K)$.

Since τ_λ belongs to $G_{\mathbb{Q}} - G_K$, it follows from (14) that the matrix $\tilde{\varrho}_g(\tau_\lambda)$ is of the form

$$\tilde{\varrho}_g(\tau_\lambda) = \begin{pmatrix} r_\ell \cdot \varepsilon & \eta_g(\tau_\lambda)(1 + s_\ell \cdot \varepsilon) \\ \eta'_g(\tau_\lambda)(1 + t_\ell \cdot \varepsilon) & u_\ell \cdot \varepsilon \end{pmatrix},$$

for suitable scalars r_ℓ, s_ℓ, t_ℓ, and $u_\ell \in \bar{\mathbb{Q}}_p$. Since

$$a_\ell(g_\alpha) + a_\ell(g'_\alpha)\varepsilon = \mathrm{Trace}(\tilde{\varrho}_g(\tau_\lambda)) = (r_\ell + u_\ell) \cdot \varepsilon,$$

it follows that

$$a_\ell(g'_\alpha) = r_\ell + u_\ell. \tag{27}$$

In order to compute this trace, we observe that it arises in the upper right-hand and lower left-hand entries of the matrix

$$\tilde{\varrho}_g(\sigma_\lambda) = \tilde{\varrho}_g(\tau_\lambda)^2 = \begin{pmatrix} \psi_g(\sigma_\lambda)(1 + (s_\ell + t_\ell) \cdot \varepsilon) & \eta_g(\tau_\lambda)(r_\ell + u_\ell) \cdot \varepsilon \\ \eta'_g(\tau_\lambda)(r_\ell + u_\ell) \cdot \varepsilon & \psi_g(\sigma_\lambda)(1 + (s_\ell + t_\ell) \cdot \varepsilon) \end{pmatrix}. \tag{28}$$

On the other hand, since σ_λ belongs to G_K it follows from (18) that

$$\tilde{\varrho}_g(\sigma_\lambda) = \begin{pmatrix} \psi_g(\sigma_\lambda) \cdot (1 + \kappa(\sigma_\lambda) \cdot \varepsilon) & \psi'_g(\sigma_\lambda) \cdot \kappa_\psi(\sigma_\lambda) \cdot \varepsilon \\ \psi_g(\sigma_\lambda) \cdot \kappa'_\psi(\sigma_\lambda) \cdot \varepsilon & \psi'_g(\sigma_\lambda) \cdot (1 + \kappa'(\sigma_\lambda) \cdot \varepsilon). \end{pmatrix} \tag{29}$$

By comparing upper-right entries in the matrices in (28) and (29) and invoking (27) together with the relation $\psi'_g(\sigma_\lambda)\eta_g(\tau_\lambda)^{-1} = \eta'_g(\tau_\lambda)$ arising from (15), we deduce that

$$a_\ell(g'_\alpha) = \eta'_g(\tau_\lambda)\kappa_\psi(\sigma_\lambda).$$

It is worth noting that each of the expressions $\eta'_g(\tau_\lambda)$ and $\kappa_\psi(\sigma_\lambda)$ depend on the choice of a prime λ of H above ℓ that was made to define τ_λ and σ_λ, since changing this prime replaces τ_λ and σ_λ by their conjugates $\sigma\tau_\lambda\sigma^{-1}$ and $\sigma\sigma_\lambda\sigma^{-1}$ by some $\sigma \in G_K$. More precisely, by (16) and the cocycle property of κ_ψ,

$$\eta'_g(\sigma\tau_\lambda\sigma^{-1}) = \psi^{-1}(\sigma)\eta'_g(\tau_\lambda), \qquad \kappa_\psi(\sigma\sigma_\lambda\sigma^{-1}) = \psi(\sigma)\kappa_\psi(\sigma_\lambda).$$

In particular, the product $\eta'_g(\tau_\lambda)\kappa_\psi(\sigma_\lambda)$ is independent of the choice of a prime above ℓ, as it should be. Note that $\eta'_g(\tau_\lambda)$ is a simple root of unity belonging to the image of ψ_g, while $\kappa_\psi(\sigma_\lambda)$ represents the interesting "transcendental" contribution to the fourier coefficient $a_\ell(g'_\alpha)$.

By the description of $\kappa_\psi(\sigma_\lambda)$ arising from local and global class field theory, we conclude from (25) that

$$a_\ell(g'_\alpha) = \eta'_g(\tau_\lambda)\frac{\log_\wp(u'_\psi)\log_\wp(u_\psi(\lambda)) - \log_\wp(u_\psi)\log_\wp(u'_\psi(\lambda))}{\log_\wp(u'_\psi) - \log_\wp(u_\psi)}, \tag{30}$$

as was to be shown. □

A more efficient (but somewhat less transparent) route to the proof of Theorem 2.3 is to specialise Theorem 1.10 to this setting. Relative to a basis of the form $(v, \tau_\wp v)$ for V_g, where v spans a G_K-stable subspace of V_g on which G_K acts via ψ_g, the matrix for U_g is proportional to one of the form

$$U_g : \begin{pmatrix} 0 & \log_\wp(u_\psi) \\ \log_\wp(\tau_\wp u_\psi) & 0 \end{pmatrix},$$

and the ordinarity condition implies that the matrix representing A_g is proportional to a matrix of the form

$$A : \begin{pmatrix} x & -x \\ y & -y \end{pmatrix}.$$

The relations $\mathrm{Trace}(A_g U_g) = 0$ and $\mathrm{Trace}(A_g) = 1$ show that A_g is represented by the matrix

$$A_g : \frac{1}{\log_\wp(u_\psi) - \log_\wp(\tau_\wp u_\psi)} \cdot \begin{pmatrix} \log_\wp(u_\psi) & -\log_\wp(u_\psi) \\ \log_\wp(\tau_\wp u_\psi) & -\log_\wp(\tau_\wp u_\psi) \end{pmatrix},$$

and Theorem 2.3 is readily deduced from the general formula for the fourier coefficients of g'_α given in Theorem 1.10. The details are left to the reader.

2.3. **Numerical examples.** We begin with an illustration of Theorem 2.3 in which the image of ϱ_g is isomorphic to the symmetric group S_3.

Example 2.4. Let χ be the quadratic character of conductor 23 and

$$g = q - q^2 - q^3 + q^6 + q^8 + \cdots \in S_1(23, \chi)$$

be the theta series attached to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-23})$. The Hilbert class field H of K is

$$H = \mathbb{Q}(\alpha) \qquad \text{where} \qquad \alpha^6 - 6\alpha^4 + 9\alpha^2 + 23 = 0.$$

Write $\mathrm{Gal}\,(H/K) = \langle\sigma\rangle$. The smallest prime which is inert in K is $p = 5$. The deformations g'_1 and g'_{-1} were computed to a 5-adic precision of 5^{40} (and q-adic precision q^{600}).

Consider the inert prime $\ell = 7$ in K. Let $u(7) = (2\alpha^4 - 7\alpha^2 + 5)/9$, a root of $x^3 - x^2 + 2x - 7 = 0$. Taking ω a primitive cube root of unity we have

$$\begin{aligned} \log_5(u_\psi(7)) &= \log_5(u(7)) + \omega \log_5(u(7)^\sigma) + \omega^2 \log_5(u(7)^{\sigma^2}) \\ \log_5(u'_\psi(7)) &= \log_5(u(7)) + \omega^2 \log_5(u(7)^\sigma) + \omega \log_5(u(7)^{\sigma^2}). \end{aligned}$$

Let $u = (\alpha^2 - 1)/3$ be the elliptic unit in H, a root of $x^3 - x^2 + 1 = 0$. Then likewise we have

$$\begin{aligned} \log_5(u_\psi) &= \log_5(u) + \omega \log_5(u^\sigma) + \omega^2 \log_5(u^{\sigma^2}) \\ \log_5(u'_\psi) &= \log_5(u) + \omega^2 \log_5(u^\sigma) + \omega \log_5(u^{\sigma^2}). \end{aligned}$$

Now

$$a_7(g'_1) = -a_7(g'_{-1}) = 4083079847610157092272537548 \cdot 5 \bmod 5^{40}$$

and one checks to 40 digits of 5-adic precision that

$$a_7(g'_1) = \frac{\log_5(u_\psi(7)) \log_5(u'_\psi) - \log_5(u'_\psi(7)) \log_5(u_\psi)}{\log_5(u_\psi) - \log_5(u'_\psi)}$$

as predicted by part (b) of Theorem 2.3.

Consider next the prime $\ell = 13$, which splits in K and factors as $(l) = \lambda\lambda'$, where $\lambda^3 = (u_\lambda)$ is a principal idead generated by $u_\lambda = -6\alpha^3 + 18\alpha - 37$, a root of $x^2 + 74x + 2197$. After setting $u(\lambda) = u_\lambda \otimes \frac{1}{3}$, we let

$$\begin{aligned} \log_5(u_g(\lambda)) &= \left(\omega \log_5(u(\lambda)) + \omega^2 \log_5(u'(\lambda))\right) \\ \log_5(u'_g(\lambda)) &= \tfrac{1}{3}\left(\omega^2 \log_5(u(\lambda)) + \omega \log_5(u'(\lambda))\right). \end{aligned}$$

We have

$$a_{13}(g'_1) = a_{13}(g'_{-1}) = -6388941316808301988520085292 \cdot 5 \bmod 5^{40}$$

and one sees that

$$a_{13}(g'_{\pm 1}) = \frac{\log_5(u'(\psi)) \log_5(u_g(\lambda)) - \log_5(u(\psi)) \log_5(u'_g(\lambda))}{\log_5(u'_\psi) - \log_5(u_\psi)}$$

to 40 digits of 5-adic precision, confirming Part (a) of Theorem 2.3.

The experiment below focuses on the case where ψ_g is a quartic ring class character, so that ϱ_g has image isomorphic to the dihedral group of order 8. The associated ring class character $\psi = \psi_g/\psi'_g = {\psi_g}^2$ of K is quadratic, i.e., a genus character which cuts out a biquadratic exension H of $\mathbb{Q}$ containing K. Let F denote the unique real quadratic subfield of H, and let K' the unique imaginary quadratic subfield of H which is distinct from K. The unit u_ψ is a power of the fundamental unit of F. Observe that the prime p is necessarily inert in $K'/\mathbb{Q}$, since otherwise ϱ_g would be induced from a character of the real quadratic field F in which p splits. It follows that $u'_\psi = u_\psi^{-1}$, so that by Theorem 2.3,

$$a_\ell(g'_\alpha) = \frac{-\log_p u_\psi \cdot (\log_p(u_g(\ell)) + \log_p(u'_g(\ell)))}{-2\log_p(u_\psi)} = \frac{1}{2}\cdot(\log_p(u_g(\ell)) + \log_p(u'_g(\ell))).$$

It follows from the definition of $u_g(\ell)$ that

$$a_\ell(g'_\alpha) = \mathrm{trace}(\varrho_g(\lambda)) \cdot \log_p(\ell).$$

In particular, we obtain

$$a_\ell(g'_\alpha) = \begin{cases} \log_p(\ell) & \text{if } \psi_g(\lambda) = 1, \\ 0 & \text{if } \psi_g(\lambda) = \pm i, \\ -\log_p(\ell) & \text{if } \psi_g(\lambda) = -1, \end{cases} \tag{31}$$

in perfect agreement with the experiments below.

Example 2.5. Let $\chi = \chi_3\chi_{13}$ where χ_3 and χ_{13} are the quadratic characters of conductors 3 and 13, respectively. The space $S_1(39,\chi)$ is one dimensional and spanned by the form $g = q - q^3 - q^4 + q^9 + \cdots$. The representation ρ_g has projective image D_4 and is induced from characters of two imaginary quadratic fields and one real quadratic field. In particular, it is induced from the quadratic character ψ_g of the Hilbert class field

$$H = \mathbb{Q}(\sqrt{-39}, a),\ a^4 + 4a^2 - 48 = 0$$

of $\mathbb{Q}(\sqrt{-39})$ (and also ramified characters of ray class fields of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{13})$). Let $p = 7$, which is inert in $\mathbb{Q}(\sqrt{-39})$. We computed $g'_{\pm 1}$ to 20-digits of 7-adic precision (and to q-adic precision q^{900}).

First consider the case of $\ell = \lambda\lambda'$ split in $\mathbb{Q}(\sqrt{-39})$. Then one observes to 20-digits of 7-adic precision and all such $\ell < 900$ that both $a_\ell(g'_1)$ and $a_\ell(g'_{-1})$ satisfy (31). Next we consider the case that ℓ is inert in $\mathbb{Q}(\sqrt{-39})$. Here one observes numerically that the Fourier coefficients are zero when ℓ is inert in $\mathbb{Q}(\sqrt{-3})$. When ℓ is split in $\mathbb{Q}(\sqrt{-3})$ the Fourier coefficients of the two stabilisations are opposite in sign and both equal to the p-adic logarithm of a fundamental ℓ unit of norm 1 in $\mathbb{Q}(\sqrt{-3})$. (Observe that p is split in $\mathbb{Q}(\sqrt{-3})$ and our numerical observations are consistent in this example with both our theorems for p split and p inert in the CM case.)

3. RM forms

Consider now the case where g is the theta series attached to a character

$$\psi_g : G_K \longrightarrow L^\times$$

of mixed signature of a real quadratic field K. As before, assume for simplicity that the field L may be embedded into $\mathbb{Q}_p$ and fix one such embedding. We also continue to denote H_g the abelian extensions of K which is cut out by ϱ_g, and let H be the ring class field of K cut out by the non-trivial ring class character $\psi := \psi_g/\psi_g{}'$. Since ψ_g has mixed signature, it follows that ψ is totally odd and thus H is totally imaginary. As before, write $G := \mathrm{Gal}\,(H/\mathbb{Q})$ and $Z := \mathrm{Gal}\,(H/K)$.

As explained in the introduction, the case where p splits in K was already dealt with in [DLR2], so in this section we only consider the case where p is inert in $K/\mathbb{Q}$. The prime p then splits completely in H/K and we fix a prime $\wp$ of H above p. This choice determines an embedding $H \longrightarrow H_\wp = K_p$ and we write $z \mapsto z'$ for the conjugation action of $\mathrm{Gal}\,(K_p/\mathbb{Q}_p)$. Let $u_K \in \mathcal{O}_K^\times$ denote the fundamental unit of $\mathcal{O}_K$ of norm 1, which we regard as an element of $K_p^\times = H_\wp^\times$ through the above embedding, and let $u'_K = u_K^{-1}$ denote its algebraic conjugate.

Let (v_1, v_2) be a basis for V_g consisting of eigenvectors for the action of G_K, and which are interchanged by the frobenius element $\tau_\wp$. Just as in the previous section, relative to this basis the Galois representation ϱ_g takes the form

(32)
$$\varrho_g(\sigma) = \begin{pmatrix} \psi_g(\sigma) & 0 \\ 0 & \psi'_g(\sigma) \end{pmatrix} \text{ for } \sigma \in G_K, \qquad \varrho_g(\tau) = \begin{pmatrix} 0 & \eta_g(\tau) \\ \eta'_g(\tau) & 0 \end{pmatrix} \text{ for } \tau \in G_{\mathbb{Q}} - G_K,$$

where η_g and η'_g are functions taking values in the group of roots of unity in $L^\times$. The element $U_g \in (H_\wp \otimes W_g)$ of (5) is thus represented the matrix

$$U_g : \begin{pmatrix} \log_\wp(u_K) & 0 \\ 0 & \log_\wp(u'_K) \end{pmatrix},$$

and hence the endomorphism A_g of Lemma 1.5 is represented by the particularly simple matrix

$$A_g : \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It follows that, if $\ell = \lambda\lambda'$ is split in $K/\mathbb{Q}$, we have

$$A_g(\lambda) : \frac{1}{2} \begin{pmatrix} \log_p(\ell) & 0 \\ 0 & \log_p(\ell) \end{pmatrix}, \tag{33}$$

while if ℓ is inert in $K/\mathbb{Q}$ and λ is a prime of H lying above ℓ,

$$A_g(\lambda) : \frac{1}{2} \begin{pmatrix} \log_p(\ell) & -\log_\wp(u_\psi(\lambda)) \\ -\log_\wp(u'_\psi(\lambda)) & \log_p(\ell) \end{pmatrix}. \tag{34}$$

Theorem 3.1. *For all rational primes $\ell \nmid Np$,*

(a) *If ℓ is split in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = \frac{1}{2} a_\ell(g) \cdot \log_p(\ell).$$

(b) *If ℓ is inert in $K/\mathbb{Q}$, then*

$$a_\ell(g'_\alpha) = -\left(\eta_g(\lambda) \log_\wp(u'_\psi(\lambda)) + \eta_g(\lambda') \log_\wp(u_\psi(\lambda))\right).$$

Proof. This follows directly from Theorem 1.10 in light of (33) and (34). □

Remark 3.2. As already remarked in [DLR2], Theorem 3.1 above (and also Theorem 7.1 of Part B) display a striking analogy with Theorem 1.1. of [DLi] concerning the fourier expansions of mock modular forms whose shadows are weight one theta series attached to characters of imaginary quadratic fields. The underlying philosophy is that the p-adic deformations considered in this paper behave somewhat like mock modular forms of weight one, "with ∞ replaced by p". This explains why the analogy remains compelling when the quadratic imaginary fields of [DLi] are replaced by real quadratic fields in which p is inert (these fields being "imaginary" from a p-adic perspective).

We illustrate Theorem 3.1 on the form of smallest level whose associated Artin representation is induced from a character of a real quadratic field, but of no imaginary quadratic field. The projective image in this example is the dihedral group D_8 of order 8:

Example 3.3. Let $\chi = \chi_5\chi_{29}$ where χ_5 and χ_{29} are quadratic and quartic characters of conductor 5 and 29, respectively. Then $S_1(145, \chi)$ is one dimensional and spanned by the eigenform

$$g = q + iq^4 - iq^5 + iq^9 + (-i-1)q^{11} - q^{16} + (-i-1)q^{19} + \cdots.$$

The form g is induced from a quartic character of a ray class group of $K = \mathbb{Q}(\sqrt{5})$ (see [DLR1, Example 4.1] for a further discussion on this form). The relevant ring class field H is

$$H = \mathbb{Q}(\alpha) \text{ where } \alpha^8 - 2\alpha^7 + 4\alpha^6 - 26\alpha^5 + 94\alpha^4 - 212\alpha^3 + 761\alpha^2 - 700\alpha + 980 = 0.$$

Write $\mathrm{Gal}(H/K) = \langle\psi\rangle$. Take $p = 13$, and note that $\chi(p) = 1$ and so $\alpha = i$ and $\beta = -i$. We compute $g'_{\pm i}$ to 10 digits of 13-adic precision (and q-adic precision $q^{28,000}$).

Consider first the prime $\ell = 7$ which is inert in K. We take the 7-unit $u(7) \in H$ to satisfy $x^4 + 13x^3 + 38x^2 + 5x + 343 = 0$ and define

$$\log_{13}(u(7, \pm i)) := \log_{13}(u(7)) \mp i \log_{13}(u(7)^{\psi}) \mp \log_{13}(u(7)^{\psi^2}) \pm i \log_{13}(u(7)^{\psi^3})$$

and so $\log_{13}(u(7,i)) + \log_{13}(u(7,-i)) = \log_{13}(v)$ where $v = u(7)/u(7)^{\psi^2} \in H$. Then one checks that to 10 digits of 13-adic precision

$$a_7(g'_i) = -\frac{1}{6} \cdot \log_{13}(v)$$

which is in line with Theorem 3.1. Next we take the prime $\ell = 11$ which is split in K. Then to 10-digits of 13-adic precision

$$a_{11}(g'_i) = -\frac{i+1}{2} \cdot \log_{13}(11)$$

exactly as predicted by Theorem 3.1.

Part B. **The irregular setting**

Denote by $S_k(Np, \chi)$ (resp. by $S_k^{(p)}(N, \chi)$) the space of classical (resp. p-adic overconvergent) modular forms of weight k, level Np (resp. tame level N) and character χ, with coefficients in $\mathbb{Q}_p$. The Hecke algebra $\mathbb{T}$ of level Np generated over $\mathbb{Q}$ by the operators T_ℓ with $\ell \nmid Np$ and U_ℓ with $\ell \mid Np$ acts naturally on the spaces $S_k(Np, \chi)$ and $S_k^{(p)}(N, \chi)$.

As in the introduction, let $g \in S_1(N, \chi)$ be a newform and let $g_\alpha \in S_1(Np, \chi)$ be a p-stabilisation of g. The eigenform g_α gives rise to a ring homomorphism $\varphi_{g_\alpha} : \mathbb{T} \longrightarrow L$ to the field L generated by the fourier coefficients of g_α, satisfying

$$\varphi_{g_\alpha}(T_\ell) = a_\ell(g_\alpha) \text{ if } \ell \nmid Np, \qquad \varphi_{g_\alpha}(U_\ell) = \begin{cases} a_\ell(g_\alpha) & \text{if } \ell \mid N; \\ \alpha & \text{if } \ell = p. \end{cases} \tag{35}$$

For any ideal I of a ring R and any R-module M, denote by $M[I]$ the I-torsion in M. Let $I_{g_\alpha} \lhd \mathbb{T}$ be the kernel of φ_{g_α}, and set

$$S_1(Np, \chi)[g_\alpha] := S_1(Np, \chi)[I_{g_\alpha}], \qquad S_1(Np, \chi)[[g_\alpha]] := S_1(Np, \chi)[I_{g_\alpha}^2].$$

Our main object of study is the subspace

$$S_1^{(p)}(N, \chi)[[g_\alpha]] := S_1^{(p)}(N, \chi)[I_{g_\alpha}^2].$$

of the space of overconvergent p-adic modular forms of weight one, which is contained in the generalised eigenspace attached to I_{g_α}. An element of $S_1^{(p)}(N, \chi)[[g_\alpha]]$ is called an *overconvergent generalised eigenform* attached to g_α, and it is said to be *classical* if it belongs to $S_1(Np, \chi)[[g_\alpha]]$. The theorem of Bellaiche and Dimitrov stated in the opening paragraphs of Part A implies that the natural inclusion

$$S_1(Np, \chi)[[g_\alpha]] \hookrightarrow S_1^{(p)}(N, \chi)[[g_\alpha]]$$

is an isomorphism, i.e., every overconvergent generalised eigenform is classical, except possibly in the following cases:

(a) g is the theta series attached to a finite order character of a real quadratic field in which the prime p splits, or
(b) g is *irregular* at p, i.e., $\alpha = \beta$.

The study of $S_1^{(p)}(N,\chi)[[g_\alpha]]$ in scenario (a) was carried out in [DLR2] when $\alpha \neq \beta$. The main result of loc.cit. is the description of a basis $(g_\alpha, g_\alpha^\flat)$ for $S_1^{(p)}(N,\chi)[[g_\alpha]]$ which is *canonical up to scaling*, and an expression for the fourier coefficients of the non-classical $g_\alpha^\flat$ (or rather, of their ratios) in terms of p-adic logarithms of certain algebraic numbers.

Assume henceforth that g is not regular at p, i.e., that $\alpha = \beta$. In that case, the form g admits a unique p-stabilisation $g_\alpha = g_\beta$. The Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid N$ act semisimply (i.e., as scalars) on the two-dimensional vector space

$$S_1(Np,\chi)[[g_\alpha]] = \mathbb{Q}_p g_\alpha \oplus \mathbb{Q}_p g', \qquad g'(q) := g(q^p),$$

but the Hecke operator U_p acts non-semisimply via the formulae

$$U_p g_\alpha = \alpha g_\alpha, \qquad U_p g' = g_\alpha + \alpha g'.$$

Because

$$(a_1(g_\alpha), a_p(g_\alpha)) = (1,\alpha), \qquad (a_1(g'), a_p(g')) = (0,1),$$

the classical subspace $S_1(Np,\chi)[[g_\alpha]]$ has a natural linear complement in $S_1^{(p)}(N,\chi)[[g_\alpha]]$, consisting of the generalised eigenforms $\tilde{g}$ whose q-expansions satisfy

$$a_1(\tilde{g}) = a_p(\tilde{g}) = 0. \tag{36}$$

A modular form satisfying (36) is said to be *normalised*, and the space of normalised generalised eigenforms is denoted $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$. The main goal of Part B is to study this space and give an explicit description of its elements in terms of their fourier expansions. The idoneous fourier coefficients will be expressed as determinants of 2×2 matrices whose entries are p-adic logarithms of algebraic numbers the number field H cut out by the projective Galois representation attached to g (cf. Theorems 5.3, 6.1 and 7.1).

4. Generalised eigenspaces

We begin by recalling some of the notations that were already introduced in Part A. Let

$$\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V_g) \simeq \mathbf{GL}_2(\mathbb{Q}_p)$$

be the odd, two-dimensional Artin representation associated to g by Deligne and Serre (but viewed as having p-adic rather than complex coefficients; as in Part A, we assume for simplicity that the image of ϱ_g can be embedded in $\mathbf{GL}_2(\mathbb{Q}_p)$ and not just in $\mathbf{GL}_2(\bar{\mathbb{Q}}_p)$).

The four-dimensional $\mathbb{Q}_p$-vector space $W_g := \mathrm{Ad}(V_g) := \mathrm{End}(V_g)$ of endomorphisms of V_g is endowed with the conjugation action of $G_{\mathbb{Q}}$,

$$\sigma\cdot M := \varrho_g(\sigma)\circ M \circ \varrho_g(\sigma)^{-1}, \qquad \text{for any } \sigma \in G_{\mathbb{Q}}, \quad M \in W_g.$$

Let H be the field cut out by this Artin representation. The action of $G_{\mathbb{Q}}$ on W_g factors through a faithful action of the finite quotient $G := \mathrm{Gal}\,(H/\mathbb{Q})$. Let $W_g^\circ := \mathrm{Ad}^0(V_g)$ denote the three-dimensional $G_{\mathbb{Q}}$-submodule of W_g consisting of trace zero endomorphisms. The exact sequence

$$0 \longrightarrow W_g^\circ \longrightarrow W_g \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

of G-modules admits a canonical G-equivariant splitting

$$p : W_g \longrightarrow W_g^\circ, \qquad p(A) := A - 1/2\cdot \mathrm{Tr}(A).$$

Because the action of $G_{\mathbb{Q}}$ on V_g also factors through a finite quotient, the field $L \subset \mathbb{Q}_p$ generated by the traces of ϱ_g is a finite extension of $\mathbb{Q}$, and ϱ_g maps the semisimple algebra $L[G_{\mathbb{Q}}]$ to a central simple algebra of rank 4 over L. By eventually enlarging L, it can be assumed that $\varrho_g(L[G_{\mathbb{Q}}]) \simeq M_2(L)$, and therefore that ϱ_g is realised on a two-dimensional L-vector space V_g^L equipped with an identification $\iota : V_g^L \otimes_L \mathbb{Q}_p \longrightarrow V_g$. The spaces

$$W_g^L := \mathrm{Ad}(V_g^L), \qquad W_g^{\circ L} := \mathrm{Ad}^0(V_g^L)$$

likewise correspond to G-stable L-rational structures on W_g and W_g° respectively, equipped with identifications

$$\iota : W_g^L \otimes_L \mathbb{Q}_p \longrightarrow W_g, \qquad \iota : W_g^{\circ L} \otimes_L \mathbb{Q}_p \longrightarrow W_g^\circ.$$

The spaces W_g and W_g° (as well as W_g^L and $W_g^{\circ L}$) are equipped with the Lie bracket $[\ ,\ ]$ and with a symetric non-degenerate pairing $\langle\ ,\ \rangle$ defined by the usual rules

$$[A,B] := AB - BA, \qquad \langle A,B\rangle := \mathrm{Tr}(AB),$$

which are compatible with the G-action in the sense that

$$[\sigma\cdot A, \sigma\cdot B] = \sigma\cdot[A,B], \qquad \langle \sigma\cdot A,\sigma\cdot B\rangle = \langle A,B\rangle, \qquad \text{for all } \sigma\in G.$$

These operations can be combined to define a G-invariant determinant function—i.e., a non-zero, alternating trilinear form—on W_g° and on $W_g^{\circ L}$ by setting

$$\det(A,B,C) := \langle [A,B], C\rangle.$$

The rule described in (35) gives rise to natural identifications

$$S_1(Np,\chi)[g_\alpha] \simeq \mathrm{Hom}(\mathbb{T}/I_{g_\alpha}, \mathbb{Q}_p), \quad S_1^{(p)}(N,\chi)[[g_\alpha]] \simeq \mathrm{Hom}(\mathbb{T}/I_{g_\alpha}^2, \mathbb{Q}_p),$$

and hence the dual of the short exact sequence

$$0 \to I_{g_\alpha}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha} \to 0$$

can be identified with

$$0 \longrightarrow S_1(Np,\chi)[g_\alpha] \longrightarrow S_1^{(p)}(N,\chi)[[g_\alpha]] \longrightarrow S_1^{(p)}(N,\chi)[[g_\alpha]]_0 \longrightarrow 0.$$

In particular, one has the isomorphism

$$S_1^{(p)}(N,\chi)[[g_\alpha]]_0 \simeq \mathrm{Hom}(I_{g_\alpha}/I_{g_\alpha}^2, \mathbb{Q}_p). \tag{37}$$

Let $\mathbb{Q}_p[\varepsilon] = \mathbb{Q}_p[x]/(x^2)$ denote the ring of dual numbers. Given $g^\flat \in S_1^{(p)}(N,\chi)[[g_\alpha]]_0$, the modular form $\tilde g := g_\alpha + \varepsilon\cdot g^\flat$ is an eigenform for $\mathbb{T}$ with coefficients in $\mathbb{Q}_p[\varepsilon]$. Its associated Galois representation

$$\varrho_{\tilde g} : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(\mathbb{Q}_p[\varepsilon])$$

satisfies

(i) $\varrho_{\tilde g} = \varrho_g \pmod{\varepsilon}$ and $\det(\varrho_{\tilde g}) = \chi$,

(ii) for every prime number $\ell \nmid Np$, the trace of an arithmetic Frobenius τ_ℓ at ℓ is

$$\mathrm{Tr}(\varrho_{\tilde g}(\tau_\ell)) = a_\ell(g_\alpha) + \varepsilon\cdot a_\ell(g^\flat). \tag{38}$$

Conjecture 4.1. *Assume that g is irregular at p. Then the assignment $g^\flat \mapsto \varrho_{\tilde g}$ gives rise to a canonical isomorphism between $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ and the space $\mathrm{Def}^0(\varrho_g)$ of isomorphism classes of deformations of ϱ_g to the ring of dual numbers, with constant determinant.*

We now derive some consequences of this conjecture.

Proposition 4.2. *Assume Conjecture 4.1. If g is irregular at p, then the space $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ is two-dimensional over $\mathbb{Q}_p$.*

Proof. Since any $\tilde\varrho \in \mathrm{Def}^0(\varrho_g)$ has constant determinant, it may be written as

$$\tilde\varrho = (1+\varepsilon\cdot c)\cdot\varrho_g \quad \text{for some} \quad c = c(\tilde\varrho) : G_{\mathbb{Q}} \longrightarrow W_g^\circ. \tag{39}$$

The multiplicativity of $\tilde\varrho$ implies that the function c is a 1-cocycle of $G_{\mathbb{Q}}$ with values in W_g°, whose class in $H^1(\mathbb{Q}, W_g^\circ)$ (which shall be denoted with the same symbol, by a slight abuse of notation) depends only on the isomorphism class of $\tilde\varrho$. The assignment $\tilde\varrho \mapsto c(\tilde\varrho)$ realises an isomorphism (cf. for instance [Ma, §1.2])

$$\mathrm{Def}^0(\varrho_g) \longrightarrow H^1(\mathbb{Q}, W_g^\circ).$$

Under Conjecture 4.1, this yields an isomorphism

$$S_1^{(p)}(N,\chi)[[g_\alpha]]_0 \xrightarrow{\sim} H^1(\mathbb{Q}, W_g^\circ), \quad g^\flat \mapsto c_{g^\flat}. \tag{40}$$

The inflation-restriction sequence combined with global class field theory for H now gives rise to a series of identifications

$$\begin{aligned} H^1(\mathbb{Q}, W_g^\circ) &\overset{\mathrm{res}_H}{\longrightarrow} \hom(G_H, W_g^\circ)^G \\ &= \hom_G\left(\frac{(\mathcal{O}_H\otimes\mathbb{Z}_p)^\times}{\mathcal{O}_H^\times\otimes\mathbb{Z}_p}, W_g^\circ\right) \\ &= \hom_G\left(\frac{H_p}{U}, W_g^\circ\right) \\ &= \ker\left(\hom_G(H_p, W_g^\circ) \overset{\mathrm{res}_U}{\longrightarrow} \hom_G(U, W_g^\circ)\right), \end{aligned} \tag{41}$$

where U denotes the natural image of $\mathcal{O}_H^\times\otimes\mathbb{Z}_p$ in $H_p := H\otimes\mathbb{Q}_p$ under the p-adic logarithm map

$$\log_p : H_p^\times \longrightarrow H_p.$$

As representations for G, the space H_p is isomorphic to the regular representation

$$H_p \simeq \mathrm{Ind}_1^G \mathbb{Q}_p,$$

while U, by the Dirichlet unit theorem, is induced from the trivial representation of the subroup $G_\infty \subset G$ generated by a complex conjugation:

$$U \simeq \mathrm{Ind}_{G_\infty}^G \mathbb{Q}_p.$$

Complex conjugation acts on W_g° with eigenvalues 1, -1 and-1, and hence by Frobenius reciprocity,

$$\dim_{\mathbb{Q}_p}\hom_G(H_p, W_g^\circ) = 3, \qquad \dim_{\mathbb{Q}_p}\hom_G(U, W_g^\circ) = 1. \tag{42}$$

It follows from (41) that $H^1(\mathbb{Q}, W_g^\circ)$ is two-dimensional over $\mathbb{Q}_p$. Proposition 4.2 follows. □

For any $\ell \nmid Np$, the ℓ-th fourier coefficient of $g^\flat$ is given in terms of the associated cocycle $c_{g^\flat}$ by the rule

$$a_\ell(g^\flat) = \mathrm{Tr}(c_{g^\flat}(\sigma_\lambda)\varrho_g(\sigma_\lambda)) \tag{43}$$

where $\lambda|\ell$ is any prime above ℓ and σ_λ denotes the arithmetic Frobenius associated to it. Note that the right-hand side of (43) does not depend on the choice of λ.

Our next goal is to parametrise the elements of (41) explicitly, and then to derive concrete formulae for the fourier expansions of the associated modular forms in $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ via (40) and (43). After treating the general case in Section 5, Sections 6 and 7 focus on the special features of the scenarios where W_g° is reducible, i.e.,

(i) the CM case where V_g is induced from a character of an imaginary quadratic field;
(ii) the RM case where V_g is induced from a character of a real quadratic field.

5. The general case

The Galois representation W_g° is irreducible if and only if $G := \mathrm{Gal}(H/\mathbb{Q})$ is isomorphic to A_4, S_4, or A_5. Otherwise, the representation ϱ_g has dihedral projective image and G is isomorphic to a dihedral group.

The irregularity assumption implies that the prime p splits completely in H, and H can therefore be viewed as a subfield of $\mathbb{Q}_p$ after fixing an embedding $H \hookrightarrow \mathbb{Q}_p$ once and for all. This amounts to choosing a prime $\wp$ of H above p. Let $\log_\wp : H_p^\times \longrightarrow \mathbb{Q}_p$ denote the associated $\wp$-adic logarithm map, which factors through $\log_p$.

The Dirichlet unit theorem implies (via the second equation in (42)) that

$$\dim_L(\mathcal{O}_H^\times \otimes W_g^{\circ L})^G = 1.$$

In particular, for all $\mathfrak{u} \in \mathcal{O}_H^\times$ and all $w \in W_g^{\circ L}$, the element

$$\xi(\mathfrak{u}, w) := \frac{1}{\#G} \times \sum_{\sigma\in G} (\sigma\mathfrak{u}) \otimes (\sigma \cdot w) \in (\mathcal{O}_H^\times \otimes W_g^{\circ L})^G \tag{44}$$

only depends on the choices of $\mathfrak{u}$ and w up to scaling by a (possibly zero) factor in L. As $\mathfrak{u}$ varies over $\mathcal{O}_H^\times$ and w over $W_g^{\circ L}$, the elements

$$\xi_\wp(\mathfrak{u}, w) := (\log_\wp \otimes \mathrm{id})\xi(\mathfrak{u}, w) = \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma\mathfrak{u}) \cdot (\sigma \cdot w) \in W_g^\circ \tag{45}$$

therefore lie in a one-dimensional L-vector subspace of W_g°. Choose a generator $w(1)$ for this space. The coordinates of $w(1)$ relative to a basis (e_1, e_2, e_3) for $W_g^{\circ L}$ are $\wp$-adic logarithms of units in $\mathcal{O}_H$, namely, we can write

$$w(1) = \log_\wp(\mathfrak{u}_1)e_1 + \log_\wp(\mathfrak{u}_2)e_2 + \log_\wp(\mathfrak{u}_3)e_3, \tag{46}$$

for appropriate $\mathfrak{u}_i \in (\mathcal{O}_H^\times) \otimes_{\mathbb{Z}} L$.

Let $\ell \nmid Np$ be a rational prime. For any prime λ of H above ℓ, let $\tilde{\mathfrak{u}}_\lambda$ be a generator of the principal ideal λ^h, where h is the class number of H, and set

$$\mathfrak{u}_\lambda := \tilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in (\mathcal{O}_H[1/\ell]^\times) \otimes_{\mathbb{Z}} L.$$

Let

$$\tilde{w}_\lambda := \varrho_g(\sigma_\lambda) \in W_g^L, \qquad w_\lambda := p(\tilde{w}_\lambda) \in W_g^{\circ L} \tag{47}$$

be the endomorphisms of V_g arising from the image of σ_λ under ϱ_g. The element $\mathfrak{u}_\lambda$ is well-defined up to multiplication by elements of $\mathcal{O}_H^\times$, and hence the elements

$$\begin{aligned} \xi(\mathfrak{u}_\lambda, w_\lambda) &:= \frac{1}{\#G} \times \sum_{\sigma\in G} (\sigma\mathfrak{u}_\lambda) \otimes (\sigma \cdot w_\lambda) \in (\mathcal{O}_H[1/\ell]^\times \otimes W_g^{\circ L})^G, \\ w(\ell) = \xi_\wp(\mathfrak{u}_\lambda, w_\lambda) &:= \frac{1}{\#G} \times \sum_{\sigma\in G} \log_\wp(\sigma\mathfrak{u}_\lambda) \cdot (\sigma \cdot w_\lambda) \in W_g^\circ \end{aligned} \tag{48}$$

are defined up to translation by elements of the one-dimensional L-vector spaces $(\mathcal{O}_H^\times \otimes W_g^{\circ L})^G$ and $L \cdot w(1)$ respectively. Furthermore, the image of $w(\ell)$ in the quotient $W_g^\circ/(L \cdot w(1))$ does not depend on the choice of the prime λ of H above ℓ that was made to define it. The Lie bracket

$$\mathfrak{W}(\ell) := [w(1), w(\ell)] \in W_g^\circ$$

is thus independent of the choices that were made in defining $w(\ell)$.

Remark 5.1. The coordinates of $w(\ell)$ relative to a basis (e_1, e_2, e_3) for $W_g^{\circ L}$ are $\wp$-adic logarithms of ℓ-units in H, i.e., one can write

$$w(\ell) = \log_\wp(\mathfrak{v}_1)e_1 + \log_\wp(\mathfrak{v}_2)e_2 + \log_\wp(\mathfrak{v}_3)e_3, \tag{49}$$

with $\mathfrak{v}_i \in (\mathcal{O}_H[1/\ell]^\times)_L$ for $i = 1, 2, 3$. A direct computation shows that

$$\dim_{\mathbb{Q}_p}(W_g^\circ)^{\sigma_\lambda=1} = \begin{cases} 1 & \text{if } g \text{ is regular at } \ell; \\ 3 & \text{if } g \text{ is irregular at } \ell. \end{cases}$$

It follows that for all regular primes ℓ,

$$\dim_L(\mathcal{O}_H[1/\ell]^\times \otimes W_g^{\circ L})^G = 2,$$

and therefore that the element $\xi(\mathfrak{v}, w)$ attached to any pair $(\mathfrak{v}, w) \in \mathcal{O}[1/\ell]^\times \times W_g^{\circ L}$ as in (48) is well-defined up to scaling by L and up to translation by elements of the one-dimensional space $(\mathcal{O}_H^\times \otimes W_g^{\circ L})^G$. In particular, the associated vector $\mathfrak{W}(\ell)$ lies in a canonical one-dimensional subspace of W_g°, namely, the orthogonal complement in W_g° of

$$(\log_\wp \otimes \mathrm{Id})(\mathcal{O}_H[1/\ell]^\times \otimes W_g^\circ)^G \subset W_g^\circ.$$

If the basis $(e_1 e_2, e_3)$ for $W_g^{\circ L}$ in (46) and (49) is taken to be the standard basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$\mathfrak{W}(\ell) \;=\; \det\begin{pmatrix} \mathfrak{u}_2 & \mathfrak{u}_3 \\ \mathfrak{v}_2 & \mathfrak{v}_3 \end{pmatrix} \cdot e_1 \;+\; 2\det\begin{pmatrix} \mathfrak{u}_1 & \mathfrak{u}_2 \\ \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot e_2 \;-\; 2\det\begin{pmatrix} \mathfrak{u}_1 & \mathfrak{u}_3 \\ \mathfrak{v}_1 & \mathfrak{v}_3 \end{pmatrix} \cdot e_3.$$

Remark 5.2. Observe that if the prime ℓ is irregular for g, the vector $\tilde{w}_\lambda$ is a scalar endomorphism in W_g and hence $w_\lambda = w(\ell) = \mathfrak{W}(\ell) = 0$.

Our main result is

Theorem 5.3. *Assume Conjecture 4.1. For all $w \in W_g^\circ$, there exists an overconvergent generalised eigenform $g_w^\flat \in S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ satisfying*

$$a_\ell(g_w^\flat) = \langle w, \mathfrak{W}(\ell)\rangle = \det(w, w(1), w(\ell)),$$

for all primes $\ell \nmid Np$. The assignment $w \mapsto g_w^\flat$ induces an isomorphism between W_g°/U and $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$.

Proof. The semi-local field $H_p = H \otimes_{\mathbb{Q}} \mathbb{Q}_p = \oplus_{\wp|p} \mathbb{Q}_p$ is naturally identified with the set of vectors $h = (h_\wp)_{\wp|p}$ with entries $h_\wp \in \mathbb{Q}_p$, indexed by the primes of H above p. The function which to $w \in W_g^\circ$ associates the linear transformation

$$\tilde{\varphi}_w : H_p \longrightarrow W_g^\circ, \qquad \tilde{\varphi}_w(h) = \frac{1}{\#G} \times \sum_{\sigma \in G} (\sigma^{-1} h)_\wp \cdot (\sigma \cdot w)$$

identifies W_g° with $\hom_G(H_p, W_g^\circ)$. The linear function $\tilde{\varphi}_w$ is trivial on $U := \log_p(\mathcal{O}_H^\times) \subset H_p$ if and only if, for all $\mathfrak{u} \in \mathcal{O}_H^\times$ and all $w' \in W_g^\circ$,

$$\langle \tilde{\varphi}_w(\log_p(\mathfrak{u})), w'\rangle = 0.$$

But

$$\begin{aligned}
\langle \tilde{\varphi}_w(\log_p(\mathfrak{u})), w'\rangle &= \frac{1}{\#G} \times \left\langle \sum_{\sigma \in G} \log_\wp(\sigma^{-1}(\mathfrak{u})) \cdot (\sigma \cdot w),\ w' \right\rangle \\
&= \frac{1}{\#G} \times \sum_{\sigma \in G} \log_\wp(\sigma^{-1}(\mathfrak{u})) \cdot \langle \sigma \cdot w, w'\rangle \\
&= \frac{1}{\#G} \times \sum_{\sigma \in G} \log_\wp(\sigma^{-1}(\mathfrak{u})) \cdot \langle w, \sigma^{-1} \cdot w'\rangle \\
&= \langle w, \xi_\wp(\mathfrak{u}, w')\rangle,
\end{aligned}$$

and hence $\tilde{\varphi}_w$ is trivial on $U = \log_p(\mathcal{O}_H^\times)$ if and only if w is orthogonal in W_g° to the line spanned by $w(1)$. It follows that the G-equivariant linear function

$$\varphi_w := \tilde{\varphi}_{[w, w(1)]} : H_p \longrightarrow W_g^\circ$$

factors through H_p/U. The assignment $w \mapsto \varphi_w$ identifies $W_g^\circ/(L \cdot w(1))$ with $\hom_G(H_p/U, W_g^\circ)$, and gives an explicit description of the latter space.

Let $\tilde{g}_w = g+\varepsilon g_w^\flat$ be the eigenform with coefficients in $\mathbb{Q}_p[\varepsilon]$ which is attached to the cocycle $\varphi_w \in \hom_G(H_p/U, W_g^\circ) = H^1(\mathbb{Q}, W_g^\circ)$. Equation (43) with $g^\flat = g_w^\flat$ (and hence $c_{g^\flat} = \varphi_w$) combined with (47) shows that the ℓ-th the fourier coefficient of $g_w^\flat$ at a prime $\ell \nmid Np$ is equal to

$$a_\ell(g_w^\flat) = \mathrm{Tr}(\varphi_w(\sigma_\lambda)\cdot\varrho_g(\sigma_\lambda)) = \langle\varphi_w(\sigma_\lambda), \tilde{w}_\lambda\rangle = \langle\varphi_w(\sigma_\lambda), w_\lambda\rangle. \tag{50}$$

Class field theory for H implies that

$$\varphi_w(\sigma_\lambda) = \frac{1}{\#G}\times\sum_{\sigma\in G}\log_\wp(\sigma^{-1}\mathfrak{u}_\lambda)\cdot\sigma\cdot[w, w(1)].$$

Hence

$$\begin{aligned} a_\ell(g_w^\flat) &= \frac{1}{\#G}\times\left\langle\sum_{\sigma\in G}\log_\wp(\sigma^{-1}\mathfrak{u}_\lambda)\cdot\sigma\cdot[w,w(1)],\ w_\lambda\right\rangle \\ &= \frac{1}{\#G}\times\sum_{\sigma\in G}\log_\wp(\sigma^{-1}\mathfrak{u}_\lambda)\cdot\langle\sigma\cdot[w,w(1)],\ w_\lambda\rangle \\ &= \frac{1}{\#G}\times\sum_{\sigma\in G}\log_\wp(\sigma^{-1}\mathfrak{u}_\lambda)\cdot\langle[w,w(1)],\ \sigma^{-1}\cdot w_\lambda\rangle \\ &= \langle[w,w(1)], w(\ell)\rangle = \det(w, w(1), w(\ell)) = \langle w, \mathfrak{W}(\ell)\rangle. \end{aligned}$$

The theorem follows. □

If w in a vector in $W_g^{\circ L}$, Theorem 5.3 shows that the associated overconvergent generalised eigenform $g_w^\flat$ has fourier coefficients which are L-rational linear combinations of determinants of 2×2 matrices whose entries are the $\wp$-adic logarithms of algebraic numbers in H. In the CM and RM cases to be discussed below, the representation W_g° is reducible and decomposes further into non-trivial irreducible representations. In that case the choice of an L-basis for $W_g^{\circ L}$ which is compatible with this decomposition leads to canonical elements of $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ which can sometimes be re-scaled so that their fourier expansions admit even simpler expressions, as will be described in the next two sections.

6. CM forms

Assume that g is the theta series attached to a character of a quadratic imaginary field K, i.e., that

$$V_g^L = \mathrm{Ind}_K^{\mathbb{Q}}\,\psi_g,$$

where $\psi_g : \mathrm{Gal}\,(\bar{K}/K)\longrightarrow L^\times$ is a finite order character. Let ψ_g' denote the character deduced from ψ_g by composing it with the involution in $\mathrm{Gal}\,(K/\mathbb{Q})$. The irreducibility assumption on V_g^L implies that the characters ψ_g and ψ_g' are distinct, and therefore the representations V_g^L and V_g decompose canonically as a direct sum of two G_K-stable one-dimensional subspaces

$$V_g^L = \mathcal{L}_{\psi_g}^L \oplus \mathcal{L}_{\psi_g'}^L, \qquad V_g = \mathcal{L}_{\psi_g}\oplus\mathcal{L}_{\psi_g'}$$

on which G_K acts via the characters ψ_g and ψ_g' respectively. The representations W_g^L and W_g also decompose as direct sums of four G_K-stable lines

$$\begin{aligned} W_g^L &= \left(\hom(\mathcal{L}_{\psi_g}^L, \mathcal{L}_{\psi_g}^L)\oplus\hom(\mathcal{L}_{\psi_g'}^L, \mathcal{L}_{\psi_g'}^L)\right) \oplus \left(\hom(\mathcal{L}_{\psi_g'}^L, \mathcal{L}_{\psi_g}^L)\oplus\hom(\mathcal{L}_{\psi_g}^L, \mathcal{L}_{\psi_g'}^L)\right), \\ W_g &= \left(\hom(\mathcal{L}_{\psi_g}, \mathcal{L}_{\psi_g})\oplus\hom(\mathcal{L}_{\psi_g'}, \mathcal{L}_{\psi_g'})\right) \oplus \left(\hom(\mathcal{L}_{\psi_g'}, \mathcal{L}_{\psi_g})\oplus\hom(\mathcal{L}_{\psi_g}, \mathcal{L}_{\psi_g'})\right). \end{aligned}$$

The direct summands in parentheses are also stable under $G_{\mathbb{Q}}$ and are isomorphic to the induced representations $\mathrm{Ind}_K^{\mathbb{Q}} 1$ and $\mathrm{Ind}_K^{\mathbb{Q}} \psi$ respectively, where $\psi := \psi_g/\psi'_g$, is the *ring class character* of K associated to ψ_g. It follows that

$$W_g^{\circ L} = L(\chi_K) \oplus Y_g^L, \qquad W_g^{\circ} = \mathbb{Q}_p(\chi_K) \oplus Y_g, \qquad Y_g^L := \mathrm{Ind}_K^{\mathbb{Q}} \psi, \quad Y_g := Y_g^L \otimes_L \mathbb{Q}_p.$$

It will be convenient to choose a basis $(e_1, e_2) \in \mathcal{L}_{\psi_g}^L \times \mathcal{L}_{\psi'_g}^L$ for V_g^L, and to denote by $e_{11}, e_{12}, e_{21}, e_{22}$ the resulting basis of W_g^L, where e_{ij} is the elementary matrix whose (i', j')-entry is $\delta i = i'\delta_{j=j'}$. Relative to the identification of $W_g^{\circ L}$ with the space of 2×2 matrices of trace zero with entries in L via this basis, the representation $L(\chi_K) = L \cdot (e_{11} - e_{22})$ is identified with the space of diagonal matrices of trace 0, while $Y_g^L = L \cdot e_{12} \oplus L \cdot e_{21}$ is identified with the space of off-diagonal matrices in $M_2(L)$. Fix an element $\tau \in G_{\mathbb{Q}} = G_K$ once and for all. By eventually re-scaling e_1 and e_2, it can (and shall, henceforth) be assumed that $\varrho_g(\tau)$ is represented by the matrix $\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$ in this basis, where $-t^2 := \chi(\tau)$.

Let $Z := \mathrm{Gal}\,(H/K)$ be the maximal abelian normal subgroup of the dihedral group $G = \mathrm{Gal}\,(H/\mathbb{Q})$. Note that every element in $G - Z$ (such as the image of τ in G) is an involution, and that Z operates transitively on $G - Z$ by either left or right multiplication.

The field H through which W_g° factors is the ring class field of K attached to the character ψ. The group $\mathcal{O}_H^{\times} \otimes \mathbb{Q}$ of units of H is isomorphic to the regular representation of Z minus the trivial representation, and a finite index subgroup of $\mathcal{O}_H^{\times}$ can be constructed explicity from the elliptic units arising in the theory of complex multiplication. Let

$$e_{\psi} := \frac{1}{\#Z} \sum_{\sigma \in Z} \psi^{-1}(\sigma)\sigma$$

be the idempotent in the group ring of Z giving rise to the projection onto the ψ-isotypic component for the action of Z. Choose a unit $\mathfrak{u} \in \mathcal{O}_H^{\times}$ and let

$$\mathfrak{u}_{\psi} := e_{\psi}\mathfrak{u}, \qquad \tau\mathfrak{u}_{\psi} = e_{\psi'}(\tau\mathfrak{u})$$

be elements of $\mathcal{O}_H^{\times} \otimes L$ on which Z acts via the characters ψ and $\psi' = \psi^{-1}$ respectively. With these choices, we can let

$$w(1) = \begin{pmatrix} 0 & \log_{\wp}(\mathfrak{u}_{\psi}) \\ \log_{\wp}(\tau\mathfrak{u}_{\psi}) & 0 \end{pmatrix}. \tag{51}$$

The description of the canonical vectors $w(\ell), \mathfrak{W}(\ell) \in W_g^{\circ}$ attached to a rational prime $\ell \nmid Np$ depends in an essential way on whether ℓ is split or inert in $K/\mathbb{Q}$.

If $\ell = \lambda\lambda'$ is split in K and ℓ is regular for g, i.e., $\psi_g(\sigma_{\lambda}) \neq \psi_g(\lambda')$, then the natural map

$$(\mathcal{O}_K[1/\ell]^{\times} \otimes W_g^{\circ L})^G \subset \left(\frac{\mathcal{O}_H[1/\ell]^{\times}}{\mathcal{O}_H^{\times}} \otimes W_g^{\circ L}\right)^G$$

is an isomorphism of L-vector spaces.

Let $\tilde{\mathfrak{u}}_{\lambda}$ be a generator of λ^h where h is the class number of K, and set

$$\mathfrak{u}_{\lambda} := \tilde{\mathfrak{u}}_{\lambda} \otimes h^{-1}.$$

Since

$$\tilde{w}_{\lambda} = \begin{pmatrix} \psi_g(\lambda) & 0 \\ 0 & \psi_g(\lambda') \end{pmatrix}, \qquad w_{\lambda} = \frac{\psi_g(\lambda) - \psi_g(\lambda')}{2} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

a direct calculation shows that

$$w(\ell) = \log_{\wp}(\mathfrak{u}_{\lambda}/\mathfrak{u}'_{\lambda}) \times \frac{(\psi_g(\lambda) - \psi_g(\lambda'))}{2} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows that

$$\mathfrak{W}(\ell) = \log_\wp(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda) \times (\psi_g(\lambda) - \psi_g(\lambda')) \times \begin{pmatrix} 0 & -\log_\wp(\mathfrak{u}_\psi) \\ \log_\wp(\mathfrak{u}'_\psi) & 0 \end{pmatrix}. \tag{52}$$

If ℓ is inert in K then ℓ is always regular for g since $\varrho_g(\tau_\ell)$ has trace 0 and hence has distinct eigenvalues. The prime ℓ splits completely in H/K, and hence the group $(\mathcal{O}_H[1/\ell]^\times) \otimes L$ is isomorphic to two copies of the regular representation of Z minus a trivial representation. The choice of a prime λ of H_g above ℓ determines a matrix (and not just a conjugacy class)

$$\tilde{w}_\lambda = w_\lambda = \varrho(\sigma_\lambda) = \begin{pmatrix} 0 & b_\lambda \\ c_\lambda & 0 \end{pmatrix}$$

with entries in L. Let $\mathfrak{u}_\lambda$ be an element of $(\mathcal{O}_H[1/\ell]^\times/\mathcal{O}_H^\times) \otimes L$ whose prime factorisation is given by

$$(\mathfrak{u}_\lambda) = b_\lambda \lambda + c_\lambda(\tau\lambda). \tag{53}$$

This ℓ-unit is only well defined by (53) up to translation by $\mathcal{O}_H^\times \otimes L$, and the defining equation (53) of course depends crucially on the choice of the prime λ above ℓ. However, the ψ-isotypic projection

$$\mathfrak{u}_\psi(\ell) := e_\psi \mathfrak{u}_\lambda \tag{54}$$

is independent of this choice. A direct calculation shows that

$$w(\ell) = \frac{1}{2} \times \begin{pmatrix} 0 & \log_\wp(\mathfrak{u}_\psi(\ell)) \\ \log_\wp(\tau\mathfrak{u}_\psi(\ell)) & 0 \end{pmatrix}.$$

It follows that

$$\mathfrak{W}(\ell) = \begin{pmatrix} R_\psi(\ell) & 0 \\ 0 & -R_\psi(\ell) \end{pmatrix}, \tag{55}$$

where

$$R_\psi(\ell) = \det \begin{pmatrix} \log_\wp(\mathfrak{u}_\psi) & \log_\wp(\tau\mathfrak{u}_\psi) \\ \log_\wp(\mathfrak{u}_\psi(\ell)) & \log_\wp(\tau\mathfrak{u}_\psi(\ell)) \end{pmatrix}$$

is an ℓ-unit regulator attached to ψ, which is independent of the choice of prime λ of H above ℓ. The function $\ell \mapsto R_\psi(\ell)$ does depend on the choice of the unit $\mathfrak{u}$, but only up to scaling by $L^\times$.

Theorem 6.1. *The space $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ has a canonical basis $(g_1^\flat, g_2^\flat)$ which is characterised by the properties:*

(i) *The fourier coefficients $a_\ell(g_1^\flat)$ are 0 for all primes $\ell \nmid Np$ that are inert in K. If $\ell = \lambda\lambda'$ is split in K, then*

$$a_\ell(g_1^\flat) = (\psi_g(\lambda) - \psi_g(\lambda')) \times \log_\wp(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda)$$

is a simple algebraic multiple of the p-adic logarithm of the fundamental ℓ-unit of norm 1 in K.

(ii) *The fourier coefficients of $g_2^\flat$ are 0 at all the primes $\ell \nmid Np$ that are split in K. If ℓ is inert in K , then*

$$a_\ell(g_s^\flat) = R_\psi(\ell).$$

Proof. This follows directly from the calculation of the matrices $\mathfrak{W}(\ell)$ in (52) and (55) in light of Theorem 5.3. $\square$

Example 6.2. Let χ be the quadratic character of conductor 59. The space $S(59,\chi)$ is one dimensional and spanned by the theta series

$$g = q - q^3 + q^4 - q^5 - q^7 - q^{12} + q^{15} + q^{16} + 2q^{17} - \cdots .$$

Here $K = \mathbb{Q}(\sqrt{-59})$ and the ring class field attached to ψ is

$$H = K(\alpha) \text{ where } \alpha^3 - 3\alpha + 46\sqrt{-59} = 0.$$

The inert primes ℓ in K are $2, 3, 13, 23, \cdots$ and the unit and first few ℓ-units are

$$\begin{aligned} u &= \tfrac{1}{612}\left(13\alpha^2 - 7\sqrt{-59}\alpha - 26\right), & u_2 &= \tfrac{1}{612}\left(-5\alpha^2 - 13\sqrt{-59}\alpha - 194\right) \\ u_{11} &= \tfrac{1}{306}\left(5\alpha^2 + 13\sqrt{-59}\alpha - 112\right), & u_{13} &= \tfrac{1}{612}\left(13\alpha^2 - 7\sqrt{-59}\alpha - 1250\right) \\ u_{23} &= \tfrac{1}{204}\left(-\alpha^2 + 11\sqrt{-59}\alpha + 138\right). \end{aligned}$$

Let $p = 17$, an irregular prime for g. We computed a basis of q-expansions for the generalised eigenspace modulo p^{20} and $q^{30{,}000}$. One observes that it contains the classical space spanned by the forms $g_\alpha(q)$ and $g(q^p)$ and in addition a complementary space of dimension two. This space is canonically spanned by two normalised generalised eigenforms

$$\tilde{g}_1^\flat = q^3 + \cdots + 0 \cdot q^p + \cdots \quad \text{and} \quad \tilde{g}_2^\flat = q^2 + 0 \cdot q^3 + \cdots + \cdots + 0 \cdot q^p + \cdots .$$

Note that the natural scaling of the forms output by our algorithm is with leading Fourier coefficients equal to 1. By Theorem 6.1 one expects that for ℓ inert in K, or ℓ split in K but irregular, we have $a_\ell(\tilde{g}_1^\flat) = 0$; and for ℓ split in K we have that

$$a_\ell(\tilde{g}_1^\flat) = \frac{\log_p(u_\ell)}{\log_p(u_3)}$$

where u_ℓ is a fundamental ℓ-unit in K (the logarithm of this is well-defined up to sign). We checked this to 20-digits of 17-adic precision for primes $\ell < 1000$. Further, one expects that

$$a_\ell(\tilde{g}_2^\flat) = \frac{R_\psi(\ell)}{R_\psi(2)} \text{ for } \ell \text{ inert in } K, \text{ and } a_\ell(\tilde{g}_2^\flat) = 0 \text{ for } \ell \text{ split in } K.$$

We checked this for all split primes $\ell < 30{,}000$ and for the inert primes $\ell = 2, 3, 11$ and 23, constructing $R_\psi(\ell)$ using the unit u and ℓ-unit u_ℓ above.

7. RM forms

We now turn to the RM setting where F is a real quadratic field and

$$V_g = \mathrm{Ind}_F^{\mathbb{Q}} \psi_g,$$

where $\psi_g : \mathrm{Gal}\,(\bar{F}/F) \longrightarrow L^\times$ is a finite order character of mixed signature. Letting ψ_g' denote the character deduced from ψ_g by composing it with the involution in $\mathrm{Gal}\,(F/\mathbb{Q})$, the ratio $\psi := \psi_g/\psi_g'$ is a totally odd L-valued ring class character of F.

As before, let H denote the ring class field of F which is fixed by the kernel of ψ, and set $Z := \mathrm{Gal}\,(H/F)$ and $G := \mathrm{Gal}\,(H/\mathbb{Q})$. Just as in the previous section,

$$W_g^\circ = \chi_K \oplus Y_g, \qquad Y_g := \mathrm{Ind}_K^{\mathbb{Q}} \psi,$$

and we can set

$$w(1) = \begin{pmatrix} \log_\wp(\mathfrak{u}_F) & 0 \\ 0 & -\log_\wp(\mathfrak{u}_F) \end{pmatrix},$$

where $\mathfrak{u}_F$ is a fundamental unit of F.

If ℓ is split in $K/\mathbb{Q}$, it is easy to see that the vector $w(\ell)$ is proportional to $w(1)$, and hence that

$$\mathfrak{W}(\ell) = 0. \tag{56}$$

If ℓ is inert in K, let U_g and $U_g(\ell)$ denote the subspaces $(\mathcal{O}_H^\times \otimes Y_g)^{G_\mathbb{Q}}$ and $(\mathcal{O}_H[1/\ell]^\times \otimes Y_g)^{G_\mathbb{Q}}$. The the dimensions of these spaces are 0 and 1 respectively. Choose a prime λ of H above ℓ, and let $\mathfrak{u}_\lambda$ and $\mathfrak{u}_\psi(\ell)$ be the elements of $\mathcal{O}_H[1/\ell]^\times$ determined by the relations

$$(\mathfrak{u}_\lambda) = b_\lambda\lambda + c_\lambda\tau\lambda, \qquad \mathfrak{u}_\psi(\ell) = e_\psi(\mathfrak{u}_\lambda), \qquad \mathfrak{u}'_\psi(\ell) = \tau\mathfrak{u}_\psi(\ell),$$

where

$$\varrho_g(\sigma_\lambda) = \begin{pmatrix} 0 & b_\lambda \\ c_\lambda & 0 \end{pmatrix}.$$

The $\wp$-adic logarithms

$$\log_\wp(\mathfrak{u}_\psi(\ell)), \qquad \log_\wp(\mathfrak{u}'_\psi(\ell))$$

are well-defined invariants of ℓ and ϱ which do not depend on the choice of a prime λ lying above ℓ, and

$$w(\ell) = \frac{1}{2} \times \begin{pmatrix} 0 & \log_\wp(\mathfrak{u}_\psi(\ell)) \\ \log_\wp(\mathfrak{u}'_\psi(\ell)) & 0 \end{pmatrix}.$$

It follows that

$$\mathfrak{W}(\ell) = \log_\wp(\mathfrak{u}_F) \times \begin{pmatrix} 0 & \log_\wp(\mathfrak{u}_\psi(\ell)) \\ -\log_\wp(\mathfrak{u}'_\psi(\ell)) & 0 \end{pmatrix}. \tag{57}$$

Theorem 7.1. *The space $S_1^{(p)}(N,\chi)[[g_\alpha]]_0$ has a canonical basis $(g_1^\flat, g_2^\flat)$ which is characterised by the properties:*

(i) *The fourier coefficients of $g_1^\flat$ and $g_2^\flat$ are 0 at all primes $\ell \nmid Np$ that are split in F.*
(ii) *If ℓ is inert in F, then*

$$a_\ell(g_1^\flat) = \log_\wp(\mathfrak{u}_\psi(\ell)), \qquad a_\ell(g_2^\flat) = \log_\wp(\mathfrak{u}'_\psi(\ell)).$$

Proof. This follows directly from Theorem 5.3 in light of equations (56) and (57). □

Example 7.2. Let χ_8 and χ_7 denote the quadratic characters of conductors 8 and 7, respectively, and define $\chi := \chi_8\chi_7$. Then $S_1(56,\chi)$ is one-dimensional and spanned by the form

$$g = q - q^2 + q^4 - q^7 - q^8 - q^9 + q^{14} + q^{16} + q^{18} + 2q^{23} - \cdots .$$

We take $p = 23$, an irregular prime for g, and compute a basis for the generalised eigenspace modulo (p^{15}, q^{3000}). The two dimensional space complementary to the classical space has a natural basis

$$\tilde{g}_1^\flat = q^3 + \cdots + 0 \cdot q^p + \cdots \quad \text{and} \quad \tilde{g}_2^\flat = q^2 + 0 \cdot q^3 + \cdots + \cdots + 0 \cdot q^p + \cdots .$$

Take

$$g_1^\flat := \frac{1}{2} \cdot \log_p(u_2) \cdot \tilde{g}_1^\flat \quad \text{and} \quad g_2^\flat := \log_p(u_3) \cdot \tilde{g}_2^\flat.$$

Here u_ℓ, $\ell = 2$ and 3, denotes a fundamental ℓ-unit of norm 1 in $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-56})$, respectively. One finds that the coefficients at primes ℓ which are split in $\mathbb{Q}(\sqrt{8})$ of both forms $g_1^\flat$ and $g_2^\flat$ are zero. At inert primes the coefficients of $g_1^\flat$ are the logarithms of fundamental ℓ-units of norm 1 in $\mathbb{Q}(\sqrt{-7})$, and those of $g_2^\flat$ are the logarithms of fundamental ℓ-units of norm 1 in $\mathbb{Q}(\sqrt{-56})$ (such logarithms are well-defined up to sign; interestingly, the forms $g_j^\flat$ single out a consistent choice of signs).

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