ELLIPTIC STARK CONJECTURES AND EXCEPTIONAL WEIGHT ONE FORMS

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To Joël Bellaïche, with affection and admiration

Abstract. A classical point of the Coleman-Mazur eigencurve is said to be *exceptional* if the map to weight space is non-étale at that point. The goal of this paper is to revisit the p-adic elliptic Stark conjecture of [DLR1] concerning a triple (f, g, h) of classical modular forms of weights $(2, 1, 1)$, and extend it to the setting where the p-stabilised eigenform g corresponds to such an exceptional point.

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Introduction

A classical point of the Coleman-Mazur eigencurve is said to be *exceptional* if the map to weight space is non-étale at that point, and a p-stabilised eigenform is called exceptional if it corresponds to such a point. By a theorem of Hida, a classical ordinary eigenform that is exceptional is necessarily of weight one. A result of Bellaïche and Dimitrov [BDi1] characterises the ordinary exceptional eigenforms in terms of the odd two-dimensional Artin representation attached to them by the construction of Deligne-Serre. More precisely, g is exceptional if and only if its Artin representation ϱ_g satisfies one of the following mutually exclusive conditions:

(i) ϱ_g is induced from a finite order mixed signature character of a real quadratic field in which the prime p splits, and maps the frobenius element at p to a linear transformation with distinct eigenvalues. In this case, Cho and Vatsal showed that the Coleman-Mazur eigencurve is smooth but *not étale* over weight space [CV] at the two p-stabilisations of g.

(ii) ϱ_g is *irregular*, i.e., maps a frobenius element at p to a scalar matrix. In that case, the (unique) p-stabilisation of g gives rise to a singular point on the eigencurve.

In both cases, the generalised eigenspace attached to g in the space of overconvergent p-adic modular forms of weight one is non-semisimple as a module over the Hecke algebra and contains non-classical elements.

The article [DLR1] formulates an *elliptic Stark conjecture* arising from a triple (f, g, h) of classical modular forms of weights $(2, 1, 1)$. This conjecture equates an analytic term – an overconvergent modular form of weight one built from f, g, and h as a kind of "p-adic iterated integral" – and an algebraic term – a regulator involving the p-adic formal group logarithms on the modular abelian variety attached to f of global points defined over the number field cut out by the Artin representation $\varrho_g \otimes \varrho_h$. In defining both sides of the conjectured equality, essential use is made in loc.cit. of the circumstance that g is non-exceptional. The purpose of this paper is to extend the conjecture of [DLR1] to the remaining cases, where g is exceptional.

This extension turns out to be far from routine, revealing genuinely new phenomena. The iterated integral in the exceptional setting is best envisaged as an overconvergent modular form of weight one in the generalised eigenspace attached to g. Its fourier coefficients are expressed as "regulators of regulators" mixing the p-adic logarithms of algebraic numbers in the field cut out by the adjoint of V_g and formal group logarithms of global points in the Mordell-Weil group of E over the field cut out by $V_g \otimes V_h$. The definition of these "regulators of regulators" rests crucially on the explicit description of the generalised eigenspace attached to f and on the representation-theoretic identity

$$\wedge^2(V_g \otimes V_h) = \mathrm{Ad}^0(V_g) \oplus \mathrm{Ad}^0(V_h)$$

between 6-dimensional Artin representations.

Section 1 introduces the set-up and briefly reviews the original elliptic Stark conjecture of [DLR1]. The extensions of this conjecture to scenarios (i) and (ii) are described in Sections 2 and 3 respectively.

1. Brief review of the elliptic Stark conjecture

Fix a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to L^\times$ of modulus $N \geq 1$ with values in a finite field extension $L/\mathbb{Q}$. Let $S_k(N, \chi)_L \subset M_k(N, \chi)_L$ denote the spaces of classical cusp forms and modular forms of weight k, level N and character χ with Fourier coefficients in L, and let $S_k^{\mathrm{oc}}(N, \chi)$ and $M_1^{\mathrm{oc}}(N, \chi)$ denote the corresponding spaces of overconvergent p-adic modular forms of tame level N and character χ with coefficients in $\mathbb{Q}_p$. The character χ is suppressed from the notation when it is trivial. The superscript $\vee$ will be used to denote the linear dual of a vector space.

Given an integer N let $\mathbb{T}_N$ denote the abstract Hecke algebra generated by the Hecke operators $\{T_\ell : \ell \nmid N, U_q : q \mid N\}$. If M is any $\mathbb{T}_N$-module and $\phi \in M$ is a simultaneous $\mathbb{T}_N$-eigenvector, let $I_\phi \subset \mathbb{T}_N$ denote the kernel of the homomorphism $\mathbb{T}_N \longrightarrow L$ taking a Hecke operator to its associated eigenvalue, and write $M[\phi]$ and $M[[\phi]]$ for the eigenspace and the generalised eigenspace attached to I_ϕ, respectively:

$$M[\phi] := M[I_\phi] = \{m \in M : Tm = 0 \text{ for all } T \in I_\phi\}, \qquad M[[\phi]] := \bigcup_{n \geq 1} M[I_\phi^n].$$

Let

$$\pi_\phi : M \longrightarrow M[[\phi]]$$

denote the canonical Hecke-equivariant projection onto the generalized eigenspace arising from the primary decomposition theorem.

For instance, if ϕ is a normalised newform in $S_k(N_\phi,\chi)_L$, and N is any multiple of N_ϕ, the space $S_k(N,\chi)_L[\phi]$ is the L-vector space spanned by $\{\phi(q^d) : d \mid \frac{N}{N_g}\}$. An element

$$\breve{\phi} = \sum_{d \mid \frac{N}{N_g}} a_d \phi(q^d) \in S_k(Np,\chi)_L[\phi]$$

is called a *test vector* of level N attached to ϕ.

Fix a triple of classical eigenforms $(f,g,h) \in S_2(N_f) \times S_1(N_g,\chi)_L \times S_1(N_h,\bar{\chi})_L$ of primitive levels (N_f,N_g,N_h), weights $(2,1,1)$ and nebentype characters $(1,\chi,\bar{\chi})$. Write $N = \text{l.c.m.}(N_f,N_g,N_h)$ and let

$$h^* := h \otimes \chi \in S_1(N_h,\chi)_L$$

denote the twist of h by χ.

Choose a prime $p \nmid N_g N_h$ such that $\mathrm{ord}_p(N_f) \leq 1$ and an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $S_k(N,\chi)^{\mathrm{ord}} \subset S_k(N,\chi)$ denote the subspace of p-ordinary modular forms with respect to the chosen embedding and let

$$e_{\mathrm{ord}} : S_k(N,\chi)_{\bar{\mathbb{Q}}_p} \longrightarrow S_k^{\mathrm{ord}}(N,\chi)_{\bar{\mathbb{Q}}_p}$$

denote Hida's ordinary idempotent.

Assume for simplicity that f has rational Fourier coefficients and hence corresponds to an elliptic curve $E/\mathbb{Q}$. Let $d = q\frac{d}{dq}$ denote the Atkin-Serre differential operator on p-adic modular forms, and let

$$F = d^{-1}(f^{[p]}) = \sum_{p \nmid n} \frac{a_n(f)}{n} q^n \in S_0^{\mathrm{oc}}(N)$$

denote the *overconvergent primitive* of f. More generally, if $\breve{f}$ is any test vector for f, the primitive $\breve{F}$ is defined exactly as in the equation above with f replaced by $\breve{f}$.

Let V_g be the Artin representation associated to it, realised as a vector space over L after enlarging L if necessary, and let

$$\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_g) \simeq \mathbf{GL}_2(L)$$

be the associated homomorphism of $G_{\mathbb{Q}}$. Note that the finite extension $L \subset \mathbb{C}$ can always be chosen to be contained in a cyclotomic field. Let H be the number field cut out by ϱ_g, i.e., the smallest field for which ϱ_g factors through $\mathrm{Gal}\,(H/\mathbb{Q})$.

Let $\wp$ be the prime ideal of H above p determined by the embedding ι_p. Since $p \nmid N_g N_h$, the arithmetic frobenius element

$$\mathrm{Frob}_p \in \mathrm{Gal}\,(H/\mathbb{Q}),$$

which is well defined up conjugation, acts on V_g, and its characteristic polynomial is equal to the Hecke polynomial

$$x^2 - a_p(g)x + \chi(p) =: (x-\alpha_g)(x-\beta_g)$$

attached to g at p. After possibly enlarging L, it may also be assumed that this coefficient field contains the roots of unity α_g and β_g, i.e., that the frobenius element is diagonalisable.

It is assumed throughout this section and the next that g is regular, i.e., that $\alpha_g \neq \beta_g$, the scenario where g is irregular being confined to Section 3. Because of this assumption, the $G_{\mathbb{Q}_p}$-module V_g decomposes naturally as a direct sum

$$V_g = V_g^{\alpha} \oplus V_g^{\beta}$$

of one-dimensional eigenspaces for Frob_p, with eigenvalues α_g and β_g respectively.

The p-stabilisations of g at p are the normalised eigenforms of weight one with Fourier coefficients in L defined by

$$g_\alpha := g(z) - \beta_g g(pz), \qquad g_\beta := g(z) - \alpha_g g(pz).$$

They are eigenvectors for the U_p-operator satisfying

$$U_p g_\alpha = \alpha_g g_\alpha, \qquad U_p g_\beta = \beta_g g_\beta.$$

Assume for simplicity that L is a subfield of $\mathbb{Q}_p$. This assumption allows ϱ_g to be viewed as a $\mathbb{Q}_p$-linear representation via the natural action of $G_{\mathbb{Q}}$ on the $\mathbb{Q}_p$-vector space $V_g \otimes_L \mathbb{Q}_p$, and is made solely to lighten the notations; it could easily be dispensed with by working with the base change of V_g to the completion of L at a prime above p.

Let

$$W_g = \mathrm{Ad}(\varrho_g) := \mathrm{End}_0(V_g)$$

denote the *adjoint representation* associated to g, equipped with the conjugation action of $G_{\mathbb{Q}}$ on the space of trace zero endomorphisms of V_g. Since complex conjugation acts on W_g with eigenvalues 1, -1 and -1, the L-vector space

$$(\mathcal{O}_H^\times \otimes W_g)^{G_{\mathbb{Q}}} = \mathrm{Hom}_{G_{\mathbb{Q}}}(W_g, \mathcal{O}_H^\times \otimes L)$$

is one-dimensional, by the Dirichlet unit theorem. Let $u = u_g$ be a basis of this 1-dimensional space. There is an isomorphism of $G_{\mathbb{Q}_p}$-modules

$$W_g = L \cdot e_{\alpha/\beta} \oplus L \cdot e_{\beta/\alpha} \oplus L \cdot e_1, \quad \text{ where } \mathrm{Frob}_p(e_\lambda) = \lambda e_\lambda.$$

Set

$$u_\lambda := u(e_\lambda) \in (\mathcal{O}_H^\times \otimes L)^{\mathrm{Frob}_p = \lambda}.$$

As explained in [DLR1], the units $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ are non-trivial and unique up to scaling by $L^\times$ *precisely* when g is non-exceptional.

Let $L(f, g, h, s)$ denote the Garrett-Rankin triple-product L-function associated to the triple (f, g, h), which in this case is also the Artin-Hasse-Weil L-series of the elliptic curve E twisted by the tensor product $\varrho_{gh} := \varrho_g \otimes \varrho_h$ of the Artin representations associated to g and h. The following hypotheses will be imposed throughout this paper:

(loc) The epsilon factors $\varepsilon_q(L(f, g, h, s))$ are $+1$ at all $q \mid N$.
(van) $L(f, g, h, 1) = 0$.

Assumption (loc) always holds when the primitive conductors of f, g and h have no prime in common. It implies that the global root number is $+1$ and hence the order of vanishing of $L(f, g, h, s)$ at the central point $s = 1$ is *even*. Hence (loc) and (van) together imply that the order of vanishing is at least 2.

When $\mathrm{ord}_{s=1} L(f, g, h, s) = 2$, the equivariant Birch and Swinnerton-Dyer conjecture predicts that the L-vector space

$$(E(H) \otimes V_{gh})^{G_{\mathbb{Q}}} = \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}, E(H) \otimes L), \qquad V_{gh} := V_g \otimes V_h$$

is spanned by two linearly independent elements P and Q. Let $\{v_1, v_2\}$ be a basis of the L-vector space $V_g^\beta \otimes V_h \subset V_{gh}$, and define the elliptic regulator

$$R_p(f, g_\alpha, h) = \det \begin{pmatrix} \log_E(P(v_1)) & \log_E(P(v_2)) \\ \log_E(Q(v_1)) & \log_E(Q(v_2)) \end{pmatrix} \tag{1}$$

where $\log_E : E(H_\wp) \otimes \mathbb{Q}_p \longrightarrow H_\wp$ is the formal group law of E over the completion of H at $\wp$.

Let $\mathrm{Tr}^N_{N_g} : S_1(Np, \chi) \to S_1(N_g p, \chi)$ denote the trace homomorphism from level N to level N_g. For any choice of test vectors $(\breve{f}, \breve{h}) \in S_2(Np)_L[f] \times S_1(Np, \bar{\chi})_L[h]$ set

$$\Phi_{\breve{f} g_\alpha \breve{h}} = \pi_{g_\alpha}\left(\mathrm{Tr}^N_{N_g} e_{\mathrm{ord}}(\breve{F} \cdot \breve{h}^*)\right) \in S_1^{\mathrm{oc}}(N, \chi)_{\mathbb{Q}_p}[[g_\alpha]]. \tag{2}$$

In addition to the regularity assumption, assume that ϱ_g is not induced from a character of a real quadratic field in which p splits. As asserted in the introduction, this implies that

$$S_1^{\mathrm{oc}}(N,\chi)_{\mathbb{Q}_p}[[g_\alpha]] = S_1(Np,\chi)_{\mathbb{Q}_p}[g_\alpha]$$

and hence that $\Phi_{\breve{f}g_\alpha\breve{h}}$ lies in $S_1(Np,\chi)_{\mathbb{Q}_p}[g_\alpha]$.

With these notations and assumptions in place, the elliptic Stark conjecture of [DLR1] can now be recalled.

Conjecture 1.1. *If* $\mathrm{ord}_{s=1}L(E,\varrho_{gh})\geq 4$, *then* $\Phi_{\breve{f}g_\alpha\breve{h}}=0$, *for any pair* $(\breve{f},\breve{h})$ *of test vectors. If* $\mathrm{ord}_{s=1}L(E,\varrho_{gh})=2$, *then*

$$\Phi_{\breve{f}g_\alpha\breve{h}} = \frac{R_p(f,g_\alpha,h)}{\log_p(u_{\beta/\alpha})}\cdot g_\alpha$$

up to a scalar in L *that is non-zero for at least one pair* $(\breve{f},\breve{h})$.

The reader is referred to [DLR2] for a discussion of the elliptic Stark conjecture at a non-exceptional g when $f\in M_2(N)$ is an Eisenstein series, and the elliptic regulator of (1) is replaced by an analogous unit regulator.

2. The elliptic Stark conjecture at smooth non-étale points

The object of this section is to extend Conjecture 1.1 to the case where the point attached to g on the eigencurve is smooth but non-étale, that is to say, to formulate a variant in scenario (i) of the introduction, where g is regular but induced from a character of a real quadratic field in which p splits.

Accordingly, let K be a real quadratic field of discriminant D and let χ_K denote the even quadratic Dirichlet character associated to it. Let

$$\psi : G_K := \mathrm{Gal}\,(\bar{K}/K)\longrightarrow\mathbb{C}^\times$$

be a ray class character (of order m, conductor $\mathfrak{f}_\psi$ and central character χ_ψ) which is of *mixed signature*, i.e., which is even at precisely one of the infinite places of K and odd at the other.

It is assumed throughout this section that $g=\theta_\psi$ is Hecke's theta series attached to ψ. It is a holomorphic newform of weight 1, level N_g and nebentype character χ with Fourier coefficients in $L:=\mathbb{Q}(\mu_m)$, where

$$N_g = D\cdot\mathrm{Norm}_{K/\mathbb{Q}}\mathfrak{f}_\psi, \qquad \chi=\chi_K\chi_\psi.$$

Assume g is *regular* at p and that the prime p splits in $K/\mathbb{Q}$ as $p=\wp\wp'$. Let g_α and g_β be the two distinct p-stabilisations of g, which are eigenvectors for the U_p operator with eigenvalues α and β respectively.

In extending Conjecture 1.1 to this setting, two difficulties arise. Firstly, the denominator in Conjecture 1.1 vanishes, since the Stark unit u_g is the fundamental unit of K, on which Frob_p acts as 1, and whose components $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ are therefore trivial. Secondly, the numerical experiments carried out in Section 2.3 suggest that one should consider the coordinate of the modular form $\Phi_{\breve{f}g_\alpha\breve{h}}$ along a Hecke module generator of the generalized eigenspace $S_1^{\mathrm{oc}}(N_g,\chi)[[g_\alpha]]$, suitably normalized. At étale points, since $S_1(N_gp)[g_\alpha]=S_1^{\mathrm{oc}}(N_g,\chi)[[g_\alpha]]$ is one-dimensional, this generator can be chosen to be the normalised eigenform g_α. At non-étale points, however, it becomes necessary to consider the coordinate along a vector in the generalised eigenspace that is not in the classical eigenspace. Stating this conjecture precisely requires a concrete description of $S_1^{\mathrm{oc}}(N_g)[[g_\alpha]]$ sufficient to put an L-rational structure on it, or at least on its quotient by $S_1(N_gp)[g_\alpha]$. This is the goal of the following section.

2.1. **The generalised eigenspace.** Assume throughout the remainder of this section that

$$S_1^{\rm oc}(N_g,\chi)[[g_\alpha]] = S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2].$$

Conditional results in this direction have been obtained by Betina in [Bet]. As explained in [DLR3], the above space is then two-dimensional and contains non-classical forms which do not lie in the image of the natural inclusion

$$S_1(N_gp,\chi)[g_\alpha] \hookrightarrow S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2].$$

A *non-classical* overconvergent form in $S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]$ is said to be *normalised* if its first Fourier coefficient is equal to zero. Let

$$S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]_0 = \{\phi\in S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2] : a_1(\phi)=0\}$$

denote the space of such forms, noting that it gives rise to a natural decomposition

$$S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2] = S_1(N_gp,\chi)[g_\alpha]\oplus S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]_0. \tag{3}$$

A non-zero normalised generalised eigenform in $S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]_0$, denoted

$$g_\alpha^\flat := \sum_{n=2}^\infty a_n(g_\alpha^\flat)q^n,$$

is uniquely determined by g_α up to scaling. The Hecke operators act on it by the rule

$$T_\ell g_\alpha^\flat = a_\ell(g_\alpha)g_\alpha^\flat + a_\ell(g_\alpha^\flat)g_\alpha, \qquad U_q g_\alpha^\flat = a_q(g_\alpha)g_\alpha^\flat + a_q(g_\alpha^\flat)g_\alpha, \tag{4}$$

for all primes $\ell\nmid N_gp$ and all $q|N_gp$.

The main theorem of [DLR3] supplies a formula for the Fourier coefficients $a_n(g_\alpha^\flat)$ for a suitable scaling of $g_\alpha^\flat$. In order to describe it explicitly, let ψ' denote the character deduced from ψ by composing it with the involution in ${\rm Gal}\,(K/\mathbb{Q})$. The ratio $\psi_\heartsuit := \psi/\psi'$ is a totally odd ring class character of K. Let H denote the ring class field of K which is fixed by the kernel of $\psi_\heartsuit$, and set $G := {\rm Gal}\,(H/K)$.

If $\ell\nmid N$ is any rational prime which is inert in $K/\mathbb{Q}$, the corresponding prime ℓ of K splits completely in H/K, and the set Σ_ℓ of primes of H above ℓ is endowed with the structure of a principal G-set. Given $\lambda\in\Sigma_\ell$, let $u(\lambda)\in\mathcal{O}_H[1/\lambda]^\times\otimes\mathbb{Q}$ be any λ-unit of H satisfying ${\rm ord}_\lambda(u(\lambda))=1$. While $u(\lambda)$ is only defined up to units in $\mathcal{O}_H^\times$, the element

$$u(\psi_\heartsuit,\lambda) = \sum_{\sigma\in G}\psi_\heartsuit^{-1}(\sigma)\otimes u(\lambda)^\sigma \quad \in \quad L\otimes\mathcal{O}_H[1/\ell]^\times$$

is independent of the choice of generator $u(\lambda)$, since there are no genuine units in $L\otimes\mathcal{O}_H^\times$ in the eigencomponents for the totally odd character $\psi_\heartsuit$. The ℓ-unit $u(\psi_\heartsuit,\lambda)$ does depend on the choice of $\lambda\in\Sigma_\ell$. In [DLR3, §2], the character ψ is used to define a function $\eta:\Sigma_\ell\longrightarrow\mu_m$ for which the element

$$u(\psi_\heartsuit,\ell) := \eta(\lambda)\otimes u(\psi_\heartsuit,\lambda) \quad \in \quad L\otimes\mathcal{O}_H[1/\ell]^\times \tag{5}$$

depends only on the inert prime ℓ and not on the choice of prime $\lambda\in\Sigma_\ell$ above it.

Recall the embeddings of L into $\mathbb{Q}_p$ and of H into $\bar{\mathbb{Q}}_p$, and let

$$\log_p : L\otimes H^\times\longrightarrow\bar{\mathbb{Q}}_p$$

be the resulting p-adic logarithm on $H^\times$, extended to $L\otimes H^\times$ by L-linearity. The main result of [DLR3] is the following.

Theorem 2.1. [DLR3] *The normalised generalised eigenform $g_\alpha^\flat$ attached to g_α can be scaled in such a way that, for all primes $\ell\nmid N_g$,*

$$a_\ell(g_\alpha^\flat) = \begin{cases} 0 & \text{if } \chi_K(\ell)=+1; \\ \log_p u(\psi_\heartsuit,\ell) & \text{if } \chi_K(\ell)=-1. \end{cases}$$

Taking $g_\alpha^\flat$ scaled as above, a global L-structure in the $\mathbb{Q}_p$-vector space $S_1^{\mathrm{oc}}(N_g,\chi)[I_{g_\alpha}^2]_0$ can be defined by setting

$$S_1^{\mathrm{oc}}(N_g,\chi)[I_{g_\alpha}^2]_{0,L} = L\cdot g_\alpha^\flat.$$

Given $\phi \in S_1^{\mathrm{oc}}(N_g,\chi)[I_{g_\alpha}^2]$, write ϕ_0 for its projection to $S_1^{\mathrm{oc}}(N_g,\chi)[I_{g_\alpha}^2]_0$.

2.2. **Statement of the conjecture.** As in (2), for any choice of test vectors $(\breve{f},\breve{h})$ in level N set

$$\Phi_{\breve{f} g_\alpha \breve{h}} = \pi_{g_\alpha}(\mathrm{Tr}^N_{N_g}\left(e_{\mathrm{ord}}(\breve{F}\cdot\breve{h}^*))\right) \in S_1^{\mathrm{oc}}(N_g,\chi)_{\mathbb{Q}_p}[I_{g_\alpha}^2]$$

and put as above

$$\Phi_{\breve{f} g_\alpha \breve{h},0} \in S_1^{\mathrm{oc}}(N_g,\chi)_{\mathbb{Q}_p}[I_{g_\alpha}^2]_0.$$

Conjecture 2.2. *There exists a period*

$$\mathcal{L}_{g_\alpha} \in \mathbb{Q}_p\otimes H_\wp^\times \quad \textit{with} \quad \mathrm{Frob}_p(\mathcal{L}_{g_\alpha}) = \frac{\beta}{\alpha}\otimes\mathcal{L}_{g_\alpha}$$

which is well-defined up to multiplication by $L^\times$ and depends only on g_α, for which the equality

$$\Phi_{\breve{f} g_\alpha \breve{h},0} = \begin{cases} \frac{R_p(f,g_\alpha,h)}{\mathcal{L}_{g_\alpha}}\cdot g_\alpha^\flat, & \textit{if } \mathrm{ord}_{s=1}L(E,\varrho_{gh}) = 2, \\ 0 & \textit{if } \mathrm{ord}_{s=1}L(E,\varrho_{gh}) \geq 4, \end{cases} \tag{6}$$

holds up to a scalar in L that is non-zero for at least one pair of test vectors $(\breve{f},\breve{h}) \in S_2(N)[f]\times S_1(Np,\bar{\chi})_L[h]$.

Note that both the numerator and denominator on the right-hand side of (6) belong to the same eigenspace for Frob_p, with eigenvalue β_g/α_g, and hence that the ratio belongs to $\mathbb{Q}_p \subset H_p$, consistent with the fact that this is clearly true of the left-hand side.

Remark 2.3. Conjecture 2.2 predicts in particular that when $\mathrm{ord}_{s=1}L(E,\varrho_{gh},s) > 2$, the overconvergent modular form $\Phi_{\breve{f} g_\alpha \breve{h}}$ is classical and thus

$$\Phi_{\breve{f} g_\alpha \breve{h}} = \mathscr{L}_p\cdot g_\alpha$$

for some $\mathscr{L}_p \in \mathbb{Q}_p$. It would be interesting to better understand the nature of the p-adic L-value $\mathscr{L}_p$. The numerical experiments reported on below show that it does not vanish in general: see for instance the case of the elliptic curve E_{145a}, whose Mordell-Weil group over H has rank 4.

2.3. **Experimental evidence.**

Example 2.4. Let χ_7 and χ_{29} be the (odd and even, respectively) quadratic characters of conductor 7 and 29, and set $\chi := \chi_7\cdot\chi_{29}$. These characters take values in $L := \mathbb{Q}$. The space $S_1(203,\chi)$ is one dimensional and spanned by the weight one form

$$g = q + q^4 - q^7 - q^9 + \cdots,$$

whose Artin representation has image isomorphic to the dihedral group D_4. The representation ϱ_g is induced from a character of the real quadratic field $\mathbb{Q}(\sqrt{29})$, and in addition from characters of each of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-203})$.

The prime $p = 5$ splits in $\mathbb{Q}(\sqrt{29})$, and the Hecke polynomial of g at p has distinct eigenvalues $\alpha_g = 1$ and $\beta_g = -1$. Hence g admits two distinct p-stabilisations g_1 and g_{-1}. The generalised eigenspace in level N_g attached to g_1 is spanned by g_1 and a second form $g_1^\flat$, normalised as in the previous section by insisting $a_1(g_1^\flat) = 0$. This form is then unique up to scaling. Section 2.1 suggests a canonical choice of $g_1^\flat$ whose ℓ-th Fourier coefficient is the logarithm of an ℓ-unit in a suitable ring class field. The more computationally convenient normalization in which the leading coefficient is 1 has been adopted here, and is distinguished from the canonically scaled form by denoting it $\tilde{g}_1^\flat$. Since $\chi_{29}(2) = -1$, it follows that $\tilde{g}_1^\flat = q^2 + \cdots$. (Note that the coefficients for $g_1^\flat$ itself differ from those for $\tilde{g}_1^\flat$ given below by the factor $\log_p u(\psi_\heartsuit, 2)$, where $u(\psi_\heartsuit, 2)$ is the ratio of the roots of x^2+x+2. This factor does not lie in $\mathbb{Q}_5$, but rather its unramified quadratic extension, so these coefficients would also be less convenient to display.)

Choosing $h = g$, one has

$$V_{gh} \;:=\; V_g \otimes V_g \;=\; L \oplus W_g \;=\; (L \oplus L(\chi_{29})) \;\oplus\; (L(\chi_7) \oplus L(\chi))\,, \tag{7}$$

where the quadratic Dirichlet characters arising on the right have been grouped according to their behaviour on the Frobenius element at 5:

$$\chi_1(5) = \chi_{29}(5) = 1, \qquad \chi_7(5) = \chi(5) = -1.$$

Consider the following elliptic curves given by their Cremona labels (as specified in the Magma Computational Algebra System), namely

$$\begin{array}{rl|rl} E_{35a}: & y^2+y = x^3+x^2+9x+1 & E_{145a}: & y^2+xy+y = x^3-x^2-3x+2 \\ E_{203b}: & y^2+xy+y = x^3+x^2-2 & E_{1015a}: & y^2+xy+y = x^3+x^2-x-22 \\ E_{1015c}: & y^2+y = x^3+2x+3. & & \end{array}$$

The ranks of the relevant isotypic parts of the Mordell-Weil groups are recorded in the following table in which the rows are indexed by elliptic curves and the columns by the four quadratic characters appearing in (7):

	χ_1	χ_{29}	χ_7	χ
E_{35a}	0	0	1	1
E_{145a}	1	1	1	1
E_{203b}	1	0	1	0
E_{1015a}	0	0	1	1
E_{1015c}	1	0	0	1

For the prime $p = 5$, the Frobenius eigenvalues are $\alpha_g = 1$ and $\beta_g = -1$, and hence

$$\alpha_g \cdot \alpha_g = 1, \qquad \alpha_g \cdot \beta_g = -1.$$

It follows from this that the elliptic regulator for E vanishes unless the total rank in each block of columns is 1 for E. The rank data in the table above therefore implies that

$$R_p(f_{35a}, g_1, g) = R_p(f_{145a}, g_1, g) = R_p(f_{1015a}, g_1, g) = 0.$$

It was indeed verified numerically, to 35 digits of 5-adic precision, that the overconvergent weight one forms

$$e_{g_1}(F_{35a} \cdot g),\ e_{g_1}(F_{145a} \cdot g),\ e_{g_1}(F_{1015a} \cdot g) \quad \in \; S_1^{\mathrm{oc}}(1015, \chi)[[g_1]]$$

are all classical (and in fact non-zero).

On the other hand, the analogous regulators attached to f_{203b} and f_{1015c} are non-zero, consistent with the calculations

$$\begin{aligned} e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f_{203b}) \times g) \quad &= \quad 118978990963679015978675 5g_1 + 1704079340765874348582088\tilde{g}_1^\flat \\ e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f_{1015c}) \times g) \quad &= \quad 2079657114322222303457220g_1 + 1107129050721161617336497\tilde{g}_1^\flat \end{aligned}$$

which were carried out to a precision of 5^{35}. Conjecture 2.2 in this case gives a formula for the coefficient of $\tilde{g}_1^\flat$ in each of these expressions. We are unable to numerically verify these predictions for either of the forms individually, for lack of an explicit description of the period

$\mathcal{L}_{g_1}$ attached to g_1. However, it was possible to verify the *ratio* of the two predictions. Namely it was checked numerically that

$$\frac{17040793407658743485820088}{11071290507211616173364497} = \frac{\frac{7}{12}\log_{E_{203b},5}(P_{203b})\log_{E_{203b},5}(Q_{203b})}{\frac{2}{5}\log_{E_{1015c},5}(P_{1015c})\log_{E_{1015c},5}(Q_{1015c})}, \tag{8}$$

where

$$P_{203b} := (2,-5) \in E_{203b}(\mathbb{Q}), \quad Q_{203b} := \big(0, -\tfrac{1}{2}(\sqrt{-7}+1)\big) \in E_{203b}(\mathbb{Q}(\sqrt{-7}))$$

$$P_{1015c} := (1,2) \in E_{1015c}(\mathbb{Q}), \quad Q_{1015c} := \big(-\tfrac{100}{7}, \tfrac{1}{98}(373\sqrt{-203}-49)\big) \in E_{1015c}(\mathbb{Q}(\sqrt{-203}))$$

and the rational numbers $\frac{7}{12}$ and $\frac{2}{5}$ appearing in numerator and denominator are the algebraic factors of [DLR1, Equation (79)], which it was helpful to include, much as in the experiments of [DLR1, Section 5.2] . The identity (8) was verified to 35 digits of 5-adic precision in perfect agreement with Conjecture 2.2.

3. The elliptic Stark conjecture at non-smooth points

This chapter turns to the setting where g is *irregular* at p, i.e., where its Hecke polynomial at p has multiple roots. This extension turns out to be the least routine and brings to light essentially new phenomena, arising from the fact that the "p-adic iterated integrals" associated to (f,g,h), which in the regular setting are classical weight one forms, need not be classical when g is irregular.

3.1. **The generalised eigenspace.** Let $g \in S_1(N_g,\chi)_L$ be an eigenform that is irregular at p, that is to say, $\alpha_g = \beta_g$. Just as in the previous section, it is assumed throughout that

$$S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2] = S_1^{\rm oc}(N_g,\chi)[[g_\alpha]].$$

This is expected to hold true in general and the reader is referred to [BDi2] for several results in this direction when g is CM.

The above space decomposes naturally as a direct sum

$$S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2] = S_1(N_gp,\chi)[g_\alpha] \oplus S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]_0,$$

where

$$S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]_0 = \{\phi \in S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2] : a_1(\phi) = a_p(\phi) = 0\}.$$

The classical space $S_1(N_gp,\chi)[g_\alpha]$ is two-dimensional and spanned by g_α and $g(q^p)$. The Hecke operators T_ℓ for $\ell \nmid N_gp$ and U_q for $q \mid N_g$ act semisimply on $S_1(N_gp,\chi)[g_\alpha]$ but U_p does not, as

$$U_pg_\alpha = \alpha g_\alpha, \quad U_pg(q^p) = \alpha g(q^p) + g_\alpha.$$

An explicit description of $S_1^{\rm oc}(N_g,\chi)[I_{g_\alpha}^2]_0$ was provided in [DLR4]. Recall that $W_g := \mathrm{Ad}(V_g)$ is equipped with the following additional structures compatible with the action of $G_{\mathbb{Q}}$:

1. the inner product $\langle A,B\rangle := \mathrm{Tr}(AB)$,
2. Lie bracket: $[A,B] = AB - BA$,
3. The determinant function: $\det(A,B,C) := \langle A,[B,C]\rangle$.

As in previous sections let H denote the field cut out by W_g, and let $G = \mathrm{Gal}\,(H/\mathbb{Q})$, which is isomorphic to a dihedral group or to one of A_4, S_4 or A_5. The irregularity hypothesis on g implies that p splits completely in H, so that $H_p := H \otimes \mathbb{Q}_p$ is isomorphic, as a $\mathbb{Q}_p$-algebra, to $d = \#G$ copies of $\mathbb{Q}_p$, on which G acts as the regular representation. Let $\theta_W \in L[G]$ denote the idempotent in the group ring of G giving rise to the projection onto the W_g-isotypic component.

Recall the unit u_g spanning $(\mathcal{O}_H^\times \otimes W_g)^{G_\mathbb{Q}}$ considered in §1. The embedding ι_p chosen at the outset restricts to a field immersion $H \hookrightarrow \mathbb{Q}_p$ and thus $\log_p(u_g)$ lies in $W_g \otimes_L \mathbb{Q}_p$. For every prime $\ell \nmid N_g p$,

$$\dim(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_\mathbb{Q}} = \begin{cases} 2 & \text{if } g \text{ is regular at } \ell \\ 4 & \text{if } g \text{ is irregular at } \ell. \end{cases} \tag{9}$$

Hence, if g is regular at ℓ, there exists a well-defined element

$$u_g(\ell) \in (\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_\mathbb{Q}}$$

up to scaling and multiples of u_g because $\dim(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_\mathbb{Q}}/(\mathcal{O}_H^\times \otimes W_g)^{G_\mathbb{Q}} = 1$.

A canonical choice of $u_g(\ell)$ is obtained by choosing a prime λ of H lying above ℓ and setting $u_g(\ell) := \theta_W(x_\lambda)$, where x_λ is a generator of the principal ideal λ^h, with h the class number of H. In this way, when W_g is regular at ℓ, one obtains what amounts to a fairly natural basis $(u_g, u_g(\ell))$ of $(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_\mathbb{Q}}$.

Theorem 3.1. *There exists an isomorphism*

$$\Phi : \frac{W_g \otimes_L \mathbb{Q}_p}{\mathbb{Q}_p \cdot \log_p(u_g)} \longrightarrow S_1^{\mathrm{oc}}(N_g, \chi)[[g_\alpha]]_0 \tag{10}$$

satisfying, for all $\ell \nmid N_g p$,

$$a_\ell(\Phi(w)) = \begin{cases} \det(w, \log_p(u_g), \log_p(u_g(\ell))) & \text{if } g \text{ is regular at } \ell; \\ 0 & \text{if } g \text{ is irregular at } \ell. \end{cases}$$

Proof. This follows from [DLR4, Th. 5.3]. □

3.2. **Statement of the conjecture.** As in the introduction, together with the irregular weight 1 eigenform $g \in S_1(N_g, \chi)$ considered above, let $f \in S_2(N_f)$ and $h \in S_1(N_h, \bar\chi)$ be classical normalised newforms and set $N := \mathrm{lcm}(N_f, N_g, N_h)$.

The object of this section is formulating an elliptic Stark conjecture describing the projection of the modular form

$$\Phi_{fg_\alpha h} \in S_1^{\mathrm{oc}}(N, \chi)[[g_\alpha]]$$

introduced in (2) onto the space $S_1^{\mathrm{oc}}(N, \chi)[[g_\alpha]]_0$. Conjecture 3.3, which is the main contribution of the present chapter, proposes an explicit formula for this non-classical p-adic overconvergent modular form, by proposing a formula for its fourier coefficients.

The following simple lemma is a key ingredient in the formulation of Conjecture 3.3 below in the irregular setting.

Lemma 3.2. *Set $V_{gh} := V_g \otimes V_h$. There is a canonical decomposition of $L[G_\mathbb{Q}]$-modules*

$$V_{gh} \wedge V_{gh} = W_g \oplus W_h.$$

Proof. Recall that the Dirichlet character χ satisfies

$$\chi = \wedge^2 V_g, \qquad \chi^{-1} = \wedge^2 V_h.$$

The Artin representation $V_{gh} \otimes V_{gh}$ therefore decomposes as

$$\begin{aligned} V_{gh} \otimes V_{gh} &= (V_g \otimes V_g) \otimes (V_h \otimes V_h) \\ &= (\chi \oplus \mathrm{Sym}^2(V_g)) \otimes (\chi^{-1} \oplus \mathrm{Sym}^2(V_h)) \\ &= 1 \oplus W_g \oplus W_h \oplus \mathrm{Sym}^2(V_g) \otimes \mathrm{Sym}^2(V_h), \end{aligned} \tag{11}$$

where the general identity $V \otimes V = \wedge^2 V \oplus \mathrm{Sym}^2(V)$ has been used in the penultimate line above, and the identities $W_g = \mathrm{Sym}^2(V_g)(\chi^{-1})$ and $W_h = \mathrm{Sym}^2(V_h)(\chi)$ have been used to reach the conclusion. The cross-terms W_g and W_h that arise in (11) are precisely those coming from the antisymmetric part $\wedge^2(V_{gh})$ of $V_{gh} \otimes V_{gh}$, while the remaining terms (which account for a $9+1=10=\binom{5}{2}$ dimensional space) come from the symmetric tensors. The lemma follows. $\square$

Let

$$p_g : V_{gh} \wedge V_{gh} \longrightarrow W_g, \qquad p_h : V_{gh} \wedge V_{gh} \longrightarrow W_h$$

denote the $G_{\mathbb{Q}}$-equivariant projections arising from Lemma 3.2.

Denoting by H_{gh} the field cut out by V_{gh}, the choice of a prime of H_{gh} above p determines an embedding of H_{gh} into $\bar{\mathbb{Q}}_p$, giving rise to a p-adic formal group logarithms

$$\log_{E,p} : E(H_{gh}) \longrightarrow \bar{\mathbb{Q}}_p, \qquad \log_{E,p}^{\otimes 2} : E(H_{gh})^{\otimes 2} \longrightarrow \bar{\mathbb{Q}}_p$$

attached to E. When $\dim(E(H_{gh}) \otimes V_{gh})^{G_{\mathbb{Q}}} = 2$, choose an L-basis (P, Q) of $(E(H_{gh}) \otimes V_{gh})^{G_{\mathbb{Q}}}$, and define the *formal regulator* $\mathbb{R}(f, g, h)$ by setting

$$\mathbb{R}(f,g,h) := P \wedge Q \quad \in \quad \bigwedge^2 ((E(H_{gh}) \otimes V_{gh})^{G_{\mathbb{Q}}}) \ \subset \ (E(H_{gh})^{\otimes 2} \otimes \wedge^2 V_{gh})^{G_{\mathbb{Q}}}, \tag{12}$$

and decreeing that $\mathbb{R}(f, g, h) = 0$ whenever $E(H_{gh}) \otimes V_{gh})^{G_{\mathbb{Q}}}$ is not two-dimensional. We then set

$$\mathbb{R}_g(E, V_{gh}) := p_g(\mathbb{R}(E, V_{gh})) \quad \in \quad (E(H_{gh})^{\otimes 2} \otimes W_g)^{G_{\mathbb{Q}}}, \tag{13}$$

and finally write

$$R_p(f,g,h) := \log_{E,p}^{\otimes 2}(\mathbb{R}_g(E, V_{gh})) \quad \in \quad W_g \otimes \bar{\mathbb{Q}}_p.$$

Recall from (10) the isomorphism

$$\Phi : \frac{W_g \otimes_L \mathbb{Q}_p}{\mathbb{Q}_p \cdot \log_p(u_g)} \xrightarrow{\sim} S_1^{\mathrm{oc}}(N_g, \chi)[[g_\alpha]]_0.$$

The elliptic Stark conjecture at irregular primes can now be stated precisely.

Conjecture 3.3. *There exists a period*

$$\mathcal{L}_{g_\alpha} \in \mathbb{Q}_p$$

which is well-defined up to multiplication by $L^\times$ and depends only on g_α, for which the equality

$$\Phi_{\breve{f} g_\alpha \breve{h},0} = \begin{cases} \frac{\Phi(R_p(f,g,h))}{\mathcal{L}_{g_\alpha}}, & \text{if } \mathrm{ord}_{s=1} L(E, \varrho_{gh}) = 2, \\ 0 & \text{if } \mathrm{ord}_{s=1} L(E, \varrho_{gh}) \geq 4, \end{cases} \tag{14}$$

holds up to a scalar in L that is non-zero for at least one pair of test vectors $(\breve{f}, \breve{h}) \in S_2(N)[f] \times S_1(Np, \bar{\chi})_L[h]$.

3.3. **Some S_3 examples.** Let K be an imaginary quadratic field of discriminant $-D$ and let χ_K denote the quadratic character associated to K.

This section attempts to make the conjecture above as precise as possible, in the setting where $g \in S_1(D, \chi_K)$ is a theta series attached to a cubic unramified class character ψ of K and p is a prime at which g is irregular. The cyclic extension H/K cut out by ψ is Galois over $\mathbb{Q}$ with Galois group $\mathrm{Gal}\,(H/\mathbb{Q}) = S_3$.

Since ψ has order 3, it takes values in $L = \mathbb{Q}(\sqrt{-3})$ and the Fourier coefficients of g lie in $\mathbb{Q}$, because

$$a_\ell(g) = \begin{cases} 0 & \text{if } \ell \text{ inert in } K, \\ \psi(\mathfrak{L}) + \bar{\psi}(\mathfrak{L}) & \text{if } \ell = \mathfrak{L}\overline{\mathfrak{L}} \text{ split in } K, \end{cases}$$

for every prime $\ell \nmid D$. In particular, $g^* = g$. Moreover, it follows that the roots of the ℓ-th Hecke polynomial $x^2 - a_\ell(g)x + \chi_K(\ell)$ are

$$\begin{cases} \alpha_{g,\ell} = 1, \quad \beta_{g,\ell} = -1 & \text{if } \ell \text{ is inert in } K, \\ \alpha_{g,\ell} = \psi(\mathfrak{L}), \quad \beta_{g,\ell} = \overline{\psi(\mathfrak{L})} & \text{if } \ell = \mathfrak{L}\overline{\mathfrak{L}} \text{ splits in } K. \end{cases}$$

Note that $\psi(\mathfrak{L}) \in \{1, \frac{-1\pm\sqrt{-3}}{2}\}$. Hence g is regular at all inert primes and at those split primes for which $\psi(\mathfrak{L}) \neq 1$.

Fix for the remainder of this section a prime $p \nmid D$ that splits in K as $p = \wp\bar{\wp}$ and such that $\psi(\wp) = 1$, so that g is irregular at p with $x^2 - a_p(g)x + 1 = (x-1)^2$.

Note that $V_{gg} := V_g \otimes V_g = 1 \oplus \chi_K \oplus V_g$. Fix a basis $\{e_1, e_K, e_\psi, e_{\bar\psi}\}$ of V_{gg} compatible with this decomposition, in such a way that $G_\mathbb{Q}$ fixes e_1, acts on e_K through χ_K and satisfies

$$\varrho_g(\sigma)e_\psi = \psi(\sigma)e_\psi, \quad \varrho_g(\sigma)e_{\bar\psi} = \bar\psi(\sigma)e_{\bar\psi}$$

for every $\sigma \in G_K$ and $\varrho(c)e_\psi = e_{\bar\psi}$, where $c \in G_\mathbb{Q} \setminus G_K$ denotes complex conjugation. Note also that

$$W_g = \chi_K \oplus V_g = \langle e_K, e_\psi, e_{\bar\psi}\rangle. \tag{15}$$

Recall the conjectural isomorphism

$$\Phi : W_g \otimes_L \mathbb{Q}_p / \langle \log_p(u)\rangle \xrightarrow{\sim} S_1^{\mathrm{oc}}(D, \chi_K)[[g_1]]_0$$

from (10). The unit $u = u_g$ generating the W_g-isotypical component of $\mathcal{O}_H^\times$ decomposes as $u = u_{\bar\psi} \otimes e_\psi + u_\psi \otimes e_{\bar\psi}$ for some $u_\psi, u_{\bar\psi}$ on which G_K acts through ψ and $\bar\psi$, respectively. Thus the coordinates of $\log_p(u)$ in the above basis of W_g are $(0, \log u_{\bar\psi}, \log u_\psi)$. Hence a basis of the domain of Φ is given by $\{w_1 = [e_K], w_2 = [ae_\psi + be_{\bar\psi}]\}$ where $a\log u_\psi - b\log u_{\bar\psi} = 1$. It follows that $S_1^{\mathrm{oc}}(D, \chi_K)[[g_1]]_0$ ought to be spanned by

$$g_1^\flat := \Phi(w_1) \quad \text{and} \quad g_2^\flat := \Phi(w_2). \tag{16}$$

The Fourier coefficients of $g_1^\flat$ (resp. $g_2^\flat$) at primes $\ell \nmid Dp$ can be computed according to the recipe in (10). Namely $a_\ell(g_i^\flat) = 0$ at all $\ell = \mathfrak{L}\bar{\mathfrak{L}}$ split in K such that $\psi(\mathfrak{L}) = 1$ – because in such case g is irregular at ℓ – and otherwise

$$a_\ell(g_1^\flat) = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & \log u_{\bar\psi} & \log u_\psi \\ \log u(\ell)_K & \log u(\ell)_{\bar\psi} & \log u(\ell)_\psi \end{pmatrix} = \log(u_{\bar\psi})\log(u(\ell)_\psi) - \log(u_\psi)\log(u(\ell)_{\bar\psi}), \tag{17}$$

and

$$a_\ell(g_2^\flat) = \det\begin{pmatrix} 0 & a & b \\ 0 & \log u_{\bar\psi} & \log u_\psi \\ \log u(\ell)_K & \log u(\ell)_{\bar\psi} & \log u(\ell)_\psi \end{pmatrix} = \log u(\ell)_K. \tag{18}$$

Here $u(\ell) = u_g(\ell) \in \mathcal{O}_H[1/\ell]^\times$ is the ℓ-unit described in §3.1 and $u(\ell)_K$, $u(\ell)_{\bar\psi}$, $u(\ell)_\psi$ denote its components at e_K, e_ψ and $e_{\bar\psi}$, respectively.

When ℓ is inert in K, $u(\ell)_K$ is trivial because $\mathcal{O}_K[1/\ell]^\times = \mathbb{Z}[1/\ell]^\times$ by Dirichlet's theorem for S-units. Hence there are no ℓ-units on which $G_\mathbb{Q}$ acts through χ_K and this means $g_2^\flat$ is supported at primes ℓ that split in K and such that $\psi(\mathfrak{L}) \neq 1$.

When ℓ splits in K, it follows from (9) and (15) that $(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_\mathbb{Q}} = \mathcal{O}_K[1/\ell]^\times[\chi_K] \oplus \mathcal{O}_H^\times[\psi]$. In particular there are no proper ℓ-units in $H^\times$ on which G_K acts through ψ (and likewise for $\bar\psi$) and hence $a_\ell(g_1^\flat) = 0$. Thus $g_1^\flat$ is supported at primes that remain inert in K.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor dividing Dp and let $f \in S_2(Dp)$ be the weight 2 eigenform associated to it.

3.3.1. *Rank patterns* $(1,0,1)$ *and* $(0,1,1)$ *over* V_{gg}. Assume that $E(K)$ has rank one and V_g also occurs with multiplicity one in $E(H)$. Up to replacing E with its twist by χ_K, it can be assumed that E has rank 1 already over $\mathbb{Q}$. Hence the rank pattern is $(1,0,1)$ with respect to the decomposition $V_{gg} = 1 \oplus \chi_K \oplus V_g$, and a similar story applies for the rank pattern $(0,1,1)$ after twisting by χ_K.

Under the running assumptions $(E(H) \otimes V_{gg})^{G_{\mathbb{Q}}}$ has a basis consisting of $P \otimes e_1$ and $Q_\psi e_{\bar\psi} + Q_{\bar\psi} e_\psi$, where P is in $E(\mathbb{Q})$ and $(Q_\psi, Q_{\bar\psi})$ generate a copy of V_g in $E(H)$.

According to the definitions in §3.2, it can be observed that

$$\mathbb{R}_g(E, V_{gg}) = P \otimes Q_{\bar\psi} \otimes e_\psi + P \otimes Q_\psi \otimes e_{\bar\psi}$$

and thus

$$R_p(f, g, h) = \log_{E,p}(P) \log_{E,p}(Q_{\bar\psi}) e_\psi + \log_{E,p}(P) \log_{E,p}(Q_\psi) e_{\bar\psi}$$

in $V_g \otimes \bar{\mathbb{Q}}_p \subset W_g \otimes \bar{\mathbb{Q}}_p$. An elementary computation shows that the class of $R_p(f, g, h)$ in $W_g \otimes_L \mathbb{Q}_p / \langle \log_p(u) \rangle$ is

$$[R_p(f, g, h)] = R_{E,\psi} \cdot w_2$$

where

$$R_{E,\psi} = \det \begin{pmatrix} \log_p u_\psi & \log_p u_{\bar\psi} \\ \log_{E,p}(P) \log_{E,p}(Q_\psi) & \log_{E,p}(P) \log_{E,p}(Q_{\bar\psi}) \end{pmatrix}.$$

Let

$$\Phi_{f,g,g,0} = e_{g_1}(Fg)_0 \in S_1^{\mathrm{oc}}(D, \chi_K)[[g_1]]_0$$

denote the overconvergent modular form attached to the triple (f, g, g) in the previous sections. Conjecture 3.3 predicts that

$$\Phi_{f,g,g,0} \stackrel{?}{=} \frac{R_{E,\psi}}{\mathcal{L}_{g_1}} \cdot g_2^\flat \tag{19}$$

up to an algebraic factor in $\mathbb{Q}(\sqrt{-3})^\times$.

3.3.2. *Rank pattern* $(1, 1, 0)$ *over* V_{gg}. Assume in this paragraph that both E and its K-twist $E \otimes \chi_K$ have rank 1 over $\mathbb{Q}$ but V_g does not occur in the Mordell-Weil group of E/H. Hence the rank pattern is $(1, 1, 0)$ with respect to the decomposition of V_{gg}.

In this case $(E(H) \otimes V_{gg})^{G_{\mathbb{Q}}} = E(K) \otimes \mathbb{Q} = P \otimes e_1 \oplus P_K \otimes e_K$. Hence

$$\mathbb{R}_g(E, V_{gg}) = P \otimes P_K \otimes e_K, \quad R_p(f, g, h) = \log_{E,p}(P) \log_{E,p}(P_K) e_K$$

and Conjecture 3.3 predicts that

$$\Phi_{f,g,g,0} \stackrel{?}{=} \frac{\log_{E,p}(P) \log_{E,p}(P_K)}{\mathcal{L}_{g_1}} \cdot g_1^\flat \tag{20}$$

up to a non-zero rational number.

3.3.3. *Rank pattern* $(1, 1)$ *over* V_{gh} *with* h *Eisenstein.* Let now $h = E(1, \chi_K)$ be the weight one Eisenstein series associated to χ_K. Note that $V_{gh} = V_g \oplus V_g$. Let $\{e_\psi, e_{\bar\psi}\}$ and $\{f_\psi, f_{\bar\psi}\}$ be bases of the two copies of V_g, as above.

Assume now the rank $(1, 1)$ scenario where V_g occurs with multiplicity 1 in $E(H)$. In that case, $(E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}$ has a basis consisting of $Q_\psi e_{\bar\psi} + Q_{\bar\psi} e_\psi$ and $Q_\psi f_{\bar\psi} + Q_{\bar\psi} f_\psi$. The regulator then becomes

$$\mathbb{R}(E, V_{gh}) = Q_\psi \otimes Q_\psi e_{\bar\psi} \wedge f_{\bar\psi} + Q_{\bar\psi} \otimes Q_{\bar\psi} e_\psi \wedge f_\psi + Q_\psi \otimes Q_{\bar\psi} (e_{\bar\psi} \wedge f_\psi + e_\psi \wedge f_{\bar\psi}) \tag{21}$$

and $\mathbb{R}_g(E, V_{gh})$ is the W_g-component of the above expression.

The W_g-component of $V_{gh} \wedge V_{gh}$ appearing in Lemma 3.2 is spanned by the vectors

$$e_{\bar\psi} \wedge f_{\bar\psi}, \; e_\psi \wedge f_\psi, \; e_{\bar\psi} \wedge f_\psi - e_\psi \wedge f_{\bar\psi}.$$

Note the different sign in the expression of the third vector above compared with the one appearing in (21): vector $e_{\bar\psi} \wedge f_\psi + e_\psi \wedge f_{\bar\psi}$ does not belong to the W_g-component of $V_{gh} \wedge V_{gh}$ but rather in the W_h-component. This implies that, in the basis of W_g chosen in (15):

$$\mathbb{R}_g(E, V_{gh}) = Q_\psi \otimes Q_\psi \otimes e_\psi + Q_{\bar\psi} \otimes Q_{\bar\psi} \otimes e_{\bar\psi},$$

and thus

$$R_p(f,g,h) = \log^2_{E,p}(Q_\psi) \otimes e_\psi + \log^2_{E,p}(Q_{\bar\psi}) \otimes e_{\bar\psi}.$$

A similar computation as above yields that Conjecture 3.3 predicts in this case that

$$\Phi_{f,g,h,0} \stackrel{?}{=} B g_2^\flat \tag{22}$$

where, up to an algebraic factor in $\mathbb{Q}(\sqrt{-3})^\times$:

$$B = \frac{1}{\mathcal{L}_{g_1}} \times \det \begin{pmatrix} \log_p u_\psi & \log_p u_{\bar\psi} \\ \log_{E,p}(Q_{\bar\psi})\log_{E,p}(Q_{\bar\psi}) & \log_{E,p}(Q_\psi)\log_{E,p}(Q_\psi) \end{pmatrix}.$$

The next example gave the first evidence in the S_3 setting in support of Conjecture 3.3.

Example 3.4. Let χ be the quadratic character of conductor 83. The space $S_1(83,\chi)$ is one dimensional and spanned by the S_3-form

$$g = q - q^3 + q^4 - q^7 - q^{11} - q^{12} + q^{16} - q^{17} + \cdots.$$

Let

$$h = 3/2 + q + 2q^3 + q^4 + 2q^7 + 3q^9 + 2q^{11} + 2q^{12} + q^{16} + \cdots$$

be the Eisenstein series in $M_1(83,\chi)$. Choose $p = 23$, which is split in $K = \mathbb{Q}(\sqrt{-83})$, and note $a_{23}(g) = 2$. This corresponds to an irregular case, in which there is a unique p-stabilisation g_1.

Consider two curves of conductor dividing $23 \cdot 83 = 1909$ namely

$$E_{83a} : y^2 + xy + y = x^3 + x^2 + x$$
$$E_{1909a} : y^2 + y = x^3 - 4x + 2$$

labelled $83a$ and $1909a$ in Cremona's table, with associated newforms f_{83a} and f_{1909a}.

On the analytic side, for $f = f_{83a}$ and f_{1909a}, consider the projections

$$\begin{aligned} \Phi_{f,g_1,g} &= e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times g) &&= \alpha g_1 + \tilde\beta \tilde g_1^\flat + \tilde\gamma \tilde g_2^\flat + \delta g(q^p) \\ \Phi_{f,g_1,h} &= e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times h) &&= \alpha' g_1 + \tilde\beta' \tilde g_1^\flat + \tilde\gamma' \tilde g_2^\flat + \delta' g(q^p). \end{aligned}$$

Here $\tilde g_i^\flat$ denotes the canonical flat form $g_i^\flat$ from (16), but scaled to have leading coefficient 1. This is computationally more convenient, and in any case only a ratio which cancels leading terms shall be considered. Thus $\tilde g_1^\flat = q^2 + \cdots$ and $\tilde g_2^\flat = q^3 + \cdots$. (See Example 3.5 for more discussion on this point.)

The coefficients were computed to precision 23^{15}, as shown in the following tables. Write α_{83a} for the top left entry in the first table, and likewise for the remaining entries.

Curve	α	$\tilde\beta$	$\tilde\gamma$	δ
$83a$	48760277293702435198	0	−76690635484322354011	93085274895986171577
$1909a$	−6919003185063442830	0	0	0

Curve	α'	$\tilde\beta'$	$\tilde\gamma'$	δ'
$83a$	−97234278703633451870	0	40444443783855159045	−60119850903882168619
$1909a$	−62665548622385483459	0	−116101535509698118782	74624323060871198940

On the algebraic side, let H be the Hilbert class field of K, which is given explicitly as $H = \mathbb{Q}(a)$ where

$$a^6 - 6a^4 + 9a^2 + 17107628 = 0.$$

Take the elliptic unit

$$u = (41a^4 - 16921a^2 + 2201900)/3252456,$$

a root of $x^3 - 2x^2 - 2x - 1 = 0$ in H. The ranks with which the relevant representations occur in $E(H)$ for each elliptic curve are as follows:

Curve	1	χ	V_g
$83a$	1	0	1
$1909a$	2	1	1

Recall again that $V_{gg} = 1 \oplus \chi \oplus V_g$ and $V_{gh} = V_g \oplus V_g$.

Let us first focus just on the curve E_{83a}, which shall be denoted by E. The curve E is of rank 1 over $\mathbb{Q}$ with generator $P = (0, -1)$. Write Q for the Heegner point of discriminant -83 on E, namely

$$Q = \left((59a^4 + 2093a^2 + 1079612)/3252456, (-41a^4 + 16921a^2 - 5454356)/3252456\right).$$

Let ω be a primitive cube root of unity in the unramified extension of $\mathbb{Q}_{23}$ of degree 2, and σ denote a generator of $\mathrm{Gal}\,(H/K)$. Define

$$\begin{aligned}
\log_{E,p}(Q_\psi) &= \log_{E,p}(Q) + \omega \log_{E,p}(Q^\sigma) + \omega^2 \log_{E,p}(Q^{\sigma^2}) \\
\log_{E,p}(Q_{\overline{\psi}}) &= \log_{E,p}(Q) + \omega^2 \log_{E,p}(Q^\sigma) + \omega \log_{E,p}(Q^{\sigma^2}) \\
\log_p(u_\psi) &= \log_p(u) + \omega \log_p(u^\sigma) + \omega^2 \log_p(u^{\sigma^2}) \\
\log_p(u_{\overline{\psi}}) &= \log_p(u) + \omega^2 \log_p(u^\sigma) + \omega \log_p(u^{\sigma^2}).
\end{aligned}$$

One finds

$$\frac{\log_p(u_\psi) \log_{E,p}(P) \log_{E,p}(Q_{\overline{\psi}}) - \log_p(u_{\overline{\psi}}) \log_{E,p}(P) \log_{E,p}(Q_\psi)}{\log_p(u_\psi)(\log_{E,p}(Q_\psi))^2 - \log_p(u_{\overline{\psi}})(\log_{E,p}(Q_{\overline{\psi}}))^2} = \frac{1}{2} \cdot \frac{\tilde{\gamma}_{83a}}{\tilde{\gamma}'_{83a}}$$

to 15 digits of 23-adic precision. This is in perfect agreement with Conjecture 3.3. (See the coefficients of $g_2^\flat$ in (19) and (22), and recall the scaling coefficient between $\tilde{g}_2^\flat$ and $g_2^\flat$ cancels.) Note that by taking a ratio the unknown period $\mathcal{L}_{g_1}$, which depends only upon the form g_1, has been cancelled out.

For the curve $E' = E_{1909a}$, let Q' be the Heegner point

$$((-5683a^4 + 1525691a^2 - 159135172)/269953848,$$
$$(-6646a^5 + 1067831a^3 - 431437291a - 1867180782)/3734361564).$$

With definitions as above, we find

$$\frac{\log_p(u_\psi)(\log_{E,p}(Q_\psi))^2 - \log_p(u_{\overline{\psi}})(\log_{E,p}(Q_{\overline{\psi}}))^2}{\log_p(u_\psi)(\log_{E',p}(Q'_\psi))^2 - \log_p(u_{\overline{\psi}})(\log_{E',p}(Q'_{\overline{\psi}}))^2} = \frac{11^2}{2^2 \cdot 7^2} \cdot \frac{\tilde{\gamma}'_{83a}}{\tilde{\gamma}'_{1909a}}.$$

Again here the unknown period $\mathcal{L}_{g_1}$ has been cancelled. Finally notice that the representation V_{gg} occurs with multiplicity $2+1+1 = 4$ in the Mordell-Weil group of E_{1909a}. So in agreement with Conjecture 3.3, the form $\Phi_{f_{1909a}, g_1, g, 0}$ here is zero, but intriguingly the projection to the classical subspace of the generalised eigenspace is non-zero. This rank 4 non-vanishing phenomenon shall be revisited in Example 3.8.

The following is another S_3 example, giving further evidence of a similar nature to that in Example 3.4 for Conjecture 3.3, but in addition illustrates a different aspect of it.

Example 3.5. Let χ be the quadratic character of conductor 59. The space $S_1(59, \chi)$ is one dimensional and spanned by the S_3-form

$$g = q - q^3 + q^4 - q^5 - q^7 - q^{12} + q^{15} + q^{16} + 2q^{17} - q^{19} + \cdots.$$

Let

$$h = 3/2 + q + 2q^3 + q^4 + 2q^5 + 2q^7 + 3q^9 + 2q^{12} + 4q^{15} + q^{16} + 2q^{17} + 2q^{19} + \cdots$$

be the Eisenstein series in $M_1(59, \chi)$. Choose $p = 17$ which is split in $K = \mathbb{Q}(\sqrt{-59})$, and note that $a_{23}(g) = 2$. The p-stabilisation g_1 is then unique.

We consider four curves of conductor $1003 = 59 \cdot 17$.

$$\begin{aligned}
E_a &: y^2 + y = x^3 - x^2 + x + 1 \\
E_b &: y^2 + xy + y = x^3 - 8x - 11 \\
E_c &: y^2 + xy + y = x^3 - x^2 + 63x - 332 \\
E_d &: y^2 + y = x^3 - 41x + 135,
\end{aligned}$$

labelled $1003a, 1003b, 1003c$ and $1003d$ in Cremona's tables. Let f_a, f_b, f_c, f_d be the associated newforms.

On the analytic side, consider for $f = f_a, f_b, f_c$ and f_d the projections

$$\begin{aligned}
\Phi_{f,g_1,g} &= e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times g) &= \alpha g_1 + \tilde{\beta}\tilde{g}_1^\flat + \tilde{\gamma}\tilde{g}_2^\flat + \delta g(q^p) \\
\Phi_{f,g_1,h} &= e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times h) &= \alpha' g_1 + \tilde{\beta}'\tilde{g}_1^\flat + \tilde{\gamma}'\tilde{g}_2^\flat + \delta' g(q^p).
\end{aligned}$$

Here the forms $\tilde{g}_i^\flat$ are as defined in (16), but we take the computationally convenient scaling in which the leading coefficients are 1. So in this case $\tilde{g}_1^\flat = q^2 + \cdots$ and $\tilde{g}_2^\flat = q^3 + \cdots$. Note from (18) that $g_2^\flat = \log_{17}(u(3)_K) \cdot \tilde{g}_2^\flat$ where $u(3)_K = (7\sqrt{-59} - 5)/54 \in \mathbb{Q}_{17}$ and so with the canonical scaling the coefficient is $\gamma = \tilde{\gamma}/\log_{17}(u(3)_K)$. We shall likewise discuss the scaling of $g_1^\flat$ later.

These coefficients were computed to precision 17^{40}, and are displayed to a precision of 17^{20} in the following two tables. (The $*$ indicates a 17-adic unit which has been suppressed to save space.) Here we shall write α_a for the top left entry of the first table, and so on.

Curve	α	$\tilde{\beta}$	$\tilde{\gamma}$	δ
$1003a$	$*/17^2$	0	$181419707557488881222715032/17$	$*/17$
$1003b$	$*/17$	0	$-523847743247977448668851186 \cdot 17$	$*$
$1003c$	$*/17$	0	$-251265137798087771136751941/17$	$*/17$
$1003d$	$*/17$	$10625252200361504978696209/17$	0	0

Curve	α'	$\tilde{\beta}'$	$\tilde{\gamma}'$	δ'
$1003a$	$*/17$	0	$-477989696282588760904328152$	$*$
$1003b$	$*/17^2$	0	$582090391597267739281836759/17$	$*/17$
$1003c$	$*/17$	0	$-379264218879673945263387983$	$*$
$1003d$	$*/17$	$57161456039491177003705817/17$	0	0

Let H be the Hilbert class field of $K = \mathbb{Q}(\sqrt{-59})$. On the algebraic side, the ranks with which the relevant representations occur in $E(H)$ for each elliptic curve are as follows:

Curve	1	χ	V_g
$1003a$	1	0	1
$1003b$	0	1	1
$1003c$	0	1	1
$1003d$	1	1	0

Following the notation in Example 3.4, for each of the curves $1003a, 1003b$ and $1003c$ there is a point P, which generates $E(\mathbb{Q})$ for $1003a$ and which generates the χ-component of $E(K)$ for $1003b$ and $1003c$. For example

$$P_a = (1, -2), P_b = (-167/16, (-269\sqrt{-59} + 302)/64).$$

We do not write down P_c as it is of very large height. Likewise we have points Q_ψ and $Q_{\bar{\psi}}$ which generate the V_g-component for each of these three curves. (Again we shall not write these large points down.) We return to the curve $1003d$ shortly.

Using the three curves we are now able to test Conjecture 3.3 in two different ways. Namely we can take a ratio considering $\Phi_{f,g_1,g,0}$ and $\Phi_{f,g_1,h,0}$ for a fixed form $f = f_a, f_b$ or f_c. Second, we can consider a ratio in which the two weight one forms are now fixed but f varies; for example, by considering $\Phi_{f_a,g_1,g,0}$ and $\Phi_{f_b,g_1,g,0}$. In each case the ratio cancels the unknown period $\mathcal{L}_{g_1}$.

In terms of the coefficients computed in the tables, using the first ratio test Conjecture 3.3 predicts, as in Example 3.4, that

$$\frac{\log_p(u_\psi)\log_{E,p}(P)\log_{E,p}(Q_{\bar{\psi}}) - \log_p(u_{\bar{\psi}})\log_{E,p}(P)\log_{E,p}(Q_\psi)}{\log_p(u_\psi)(\log_{E,p}(Q_\psi))^2 - \log_p(u_{\bar{\psi}})(\log_{E,p}(Q_{\bar{\psi}}))^2} = C\cdot\frac{\tilde{\gamma}}{\tilde{\gamma}'}$$

for some $C \in L$. Note that the righthand side also equals $C\cdot\gamma/\gamma'$ since the $\log_{17}(u(3)_K)$ cancels. In these experiments, and in agreement with Conjecture 3.3, we found that $C = -\frac{1}{2}, -\frac{1}{2}$ and -2 for the three curves $1003a, 1003b$ and $1003c$, respectively, to 40 digits of 17-adic precision.

For the second ratio test we shall first look at $\Phi_{f_a,g_1,g,0}$ and $\Phi_{f_b,g_1,g,0}$. Conjecture 3.3 then predicts that

$$\frac{\log_p(u_\psi)\log_{E_b,p}(P_b)\log_{E_b,p}(Q_{b,\bar{\psi}}) - \log_p(u_{\bar{\psi}})\log_{E_b,p}(P_b)\log_{E_b,p}(Q_{b,\psi})}{\log_p(u_\psi)\log_{E_a,p}(P_a)\log_{E_a,p}(Q_{a,\bar{\psi}}) - \log_p(u_{\bar{\psi}})\log_{E_a,p}(P_a)\log_{E_a,p}(Q_{a,\psi})} = C_{a,b}\cdot\frac{\tilde{\gamma}_b}{\tilde{\gamma}_a}$$

for some $C_{a,b} \in L$. Note again the righthand side also equals $C_{a,b}\cdot\gamma_b/\gamma_a$. Since the curve E is varying, we have adorned the notation above with subscripts a and b to distinguish between points on the two curves. We find that indeed

$$C_{a,b} = \frac{3^3}{2^6}$$

to 40-digits of 17-adic precision, in complete agreement with Conjecture 3.3.

Performing the same test but now with $1003a$ and $1003c$ we find that

$$C_{a,c} = \frac{3^4}{2^4}$$

to 40-digits of 17-adic precision.

Let us now bring the fourth curve $1003d$ into play. Note here that $V_{gh} = V_g \oplus V_g$ occurs with multiplicity zero in the Mordell-Weil group $E(H)$ and so Conjecture 3.3 makes no prediction at all on the coefficients $\alpha'_d, \tilde{\beta}'_d, \tilde{\gamma}'_d$ and δ'_d. However, once again V_{gg} has multiplicity two, but in this case by (20) we expect up to non-zero scaling in L that

$$\Phi_{f_d,g,g,0} = \frac{\log_{E_d,p}(P_d)\log_{E_d,p}(R_d)}{\mathcal{L}_{g_1}} g_1^\flat.$$

Here

$$P_d = (9/4, -63/8) \in E_d(\mathbb{Q}),\ R_d = (-201/25, (241\sqrt{-59} - 125)/250) \in E_d(K).$$

We now consider the ratio of $\Phi_{f_a,g_1,g,0}$ and $\Phi_{f_d,g_1,g,0}$. Conjecture 3.3 predicts that

$$\frac{\log_{E_d,p}(P_d)\log_{E_d,p}(R_d)}{\log_p(u_\psi)\log_{E_a,p}(P_a)\log_{E_a,p}(Q_{a,\bar{\psi}}) - \log_p(u_{\bar{\psi}})\log_{E_a,p}(P_a)\log_{E_a,p}(Q_{a,\psi})} = C_{a,d}\cdot\frac{\beta_d}{\gamma_a}$$

for some $C_{a,d} \in L$. Note in this case we have from (17) and (18) that

$$\frac{\beta_d}{\gamma_a} = \frac{\tilde{\beta}_d}{\tilde{\gamma}_a}\cdot\frac{\log_p(u(3)_K)}{\log_p(u_{\bar{\psi}})\log_p(u(2)_\psi) - \log_p(u_\psi)\log_p(u(2)_{\bar{\psi}})}.$$

Here $u(2)_\psi$ and $u(2)_{\bar{\psi}}$ are constructed by starting with a root of $x^3 + x^2 - x - 2$ in H and following a similar recipe to that described for u_ψ and $u_{\bar{\psi}}$ in Example 3.4. (Note also that as in Example 2.4 the coefficient β_d itself does not lie in $\mathbb{Q}_{17}$ so would be less convenient to display.) Experimentally, we find to 40 digits of 17-adic precision that

$$C_{a,d} = -\frac{3^2}{2^7},$$

in complete agreement with Conjecture 3.3.

3.4. D_4 **examples.** Let K be an imaginary quadratic field as before and $g \in S_1(D, \chi_K)$ be a theta series attached to an unramified class character $\psi : G_K \longrightarrow \mathbb{Q}(\sqrt{-1})^\times$ of order 4.

Similar to the S_3-case, the roots of the ℓ-th Hecke polynomial $x^2 - a_\ell(g)x + \chi_K(\ell)$ at a prime $\ell \nmid D$ are

$$\begin{cases} \alpha_{g,\ell} = 1, \beta_{g,\ell} = -1 & \text{if } \ell \text{ is inert in } K, \\ \alpha_{g,\ell} = \psi(\mathfrak{L}), \beta_{g,\ell} = \overline{\psi(\mathfrak{L})} & \text{if } \ell = \mathfrak{L}\overline{\mathfrak{L}} \text{ splits in } K. \end{cases}$$

Let $p \nmid D$ be a prime that splits in K as $p = \wp\bar{\wp}$ and is such that $\psi(\wp) = \pm 1$, so that g is irregular at p. Again we have $g^* = g$ but the main difference with the previous example is that now $V_{gg} = 1 \oplus \chi_K \oplus \chi_{K'} \oplus \chi_F$ where K' (resp. F) is an imaginary (resp. real) quadratic field. Hence V_{gg} decomposes completely as the direct sum of four quadratic characters and we may fix a basis $\{e_1, e_{\chi_K}, e_{\chi_{K'}}, e_{\chi_F}\}$ of V_{gg} compatible with this decomposition.

The adjoint representation W_g is the quotient of V_{gg} by the trivial character and thus $W_g = \langle e_{\chi_K}, e_{\chi_{K'}}, e_{\chi_F}\rangle$. Set $H = KF$, the field cut out by ψ^2. We have $\mathrm{Gal}\,(H/\mathbb{Q}) = D_4$. Since there are no non-torsion units in the ring of integers of an imaginary quadratic field, $(\mathcal{O}_H^\times \otimes W_g)^{G_\mathbb{Q}}$ is spanned by $u = u_F \otimes e_{\chi_F}$ where u_F is the fundamental unit in F.

Hence the coordinates of $\log_p(u)$ in the above basis of W_g are $(0, 0, \log u_F)$ and a basis of the domain of Φ may be taken to be

$$\left\{ w_K = \frac{[e_{\chi_{K'}}]}{\log_p(u_F)}, w_{K'} = \frac{[e_{\chi_K}]}{\log_p(u_F)} \right\}.$$

According to (10), $S_1^{\mathrm{oc}}(D, \chi_K)[[g_1]]_0$ is expected to be spanned by

$$g_K^\flat := \Phi(w_K) \quad \text{and} \quad g_{K'}^\flat := \Phi(w_{K'}).$$

Moreover, we should have $a_\ell(g_K^\flat) = a_\ell(g_{K'}^\flat) = 0$ at all $\ell = \mathfrak{L}\overline{\mathfrak{L}}$ split in K such that $\psi(\mathfrak{L}) = \pm 1$, while at the remaining primes:

$$a_\ell(g_{K'}^\flat) = \det \begin{pmatrix} 1/\log_p(u_F) & 0 & 0 \\ 0 & 0 & \log_p(u_F) \\ \log_p u(\ell)_K & \log_p u(\ell)_{K'} & \log_p u(\ell)_F \end{pmatrix} = -\log_p(u(\ell)_{K'}),$$

$$a_\ell(g_K^\flat) = \det \begin{pmatrix} 0 & 1/\log_p(u_F) & 0 \\ 0 & 0 & \log_p u_F \\ \log_p u(\ell)_K & \log_p u(\ell)_{K'} & \log_p u(\ell)_F \end{pmatrix} = \log_p(u(\ell)_K).$$

As explained in the previous section, $u(\ell)_K$ is trivial when ℓ is inert in K, and likewise $u(\ell)_{K'}$ is trivial when ℓ remains inert in K'. It follows that $a_\ell(g_K^\flat) = 0$ whenever ℓ is inert in K and $a_\ell(g_{K'}^\flat) = 0$ at primes ℓ inert in K'.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor dividing Dp and for any quadratic field M let E_M denote the twist of E by χ_M. Set

$$(r_\mathbb{Q}, r_K, r_{K'}, r_F) := (\mathrm{rank}\, E(\mathbb{Q}), \mathrm{rank}\, E_K(\mathbb{Q}), \mathrm{rank}\, E_{K'}(\mathbb{Q}), \mathrm{rank}\, E_F(\mathbb{Q}))$$

and assume throughout that $r_\mathbb{Q} + r_K + r_{K'} + r_F = 2$. We further assume for simplicity that there are exactly two fields M_1, M_2 among $\{\mathbb{Q}, K, K', F\}$ such that $r_{M_1} = 1$ and $r_{M_2} = 1$. Then $(E(H) \otimes V_{gg})^{G_\mathbb{Q}}$ has a basis consisting of $P \otimes e_{\chi_1}$ and $Q \otimes e_{\chi_2}$. The regulator introduced in §3.2 is then

$$\mathbb{R}_g(E, V_{gh}) = P \otimes Q \otimes e_{\chi_1\chi_2}$$

so that

$$R_p(f, g, h) = \log_{E,p}(P) \log_{E,p}(Q) \otimes e_{\chi_1\chi_2}. \tag{23}$$

Note that $\chi_1\chi_2$ is always one of the characters χ_K, $\chi_{K'}$ or χ_F.

Write $e_{g_1}(F \times g)$ as

$$e_{g_1}(F \times g) = \alpha g_1 + \beta g_K^\flat + \gamma g_{K'}^\flat + \delta g(q^p), \tag{24}$$

so that its projection to $S_1^{\mathrm{oc}}(D, \chi_K)[[g_1]]_0$ is $e_{g_1}(F \times g)_0 = \beta g_K^\flat + \gamma g_{K'}^\flat$.

In light of (23), Conjecture 3.3 predicts that

$$\begin{cases} \beta = \gamma = 0 & \text{if } \chi_1\chi_2 = \chi_F \\ \beta = \frac{\log_p(u_F)\log_{E,p}(P)\log_{E,p}(Q)}{\mathcal{L}_{g_1}}, \quad \gamma = 0 & \text{if } \chi_1\chi_2 = \chi_{K'} \\ \beta = 0, \quad \gamma = \frac{\log_p(u_F)\log_{E,p}(P)\log_{E,p}(Q)}{\mathcal{L}_{g_1}} & \text{if } \chi_1\chi_2 = \chi_K \end{cases}$$

up to a non-zero algebraic factor in $\mathbb{Q}(\sqrt{-1})$.

The numerical examples below provide evidence for this conjecture and even give a hint to what the mysterious denominator $\mathcal{L}_{g_1}$ should be in this case:

Conjecture 3.6. *Let K be an imaginary quadratic field and $g \in S_1(D, \chi_K)$ be the theta series attached to an unramified class character $\psi : G_K \longrightarrow \mathbb{Q}(\sqrt{-1})^\times$ of order 4. Let $u_K(p) \in \mathcal{O}_K[1/p]^\times$ and $u_{K'}(p) \in \mathcal{O}_{K'}[1/p]^\times$ be fundamental p-units. Then*

$$\mathcal{L}_{g_1} = \log_p^2(u_F)(\log_p(u_K(p)) - \log_p(u_{K'}(p)))$$

up to a non-zero algebraic factor in $\mathbb{Q}(\sqrt{-1})$.

The numerical examples below also illustrate an intriguing phenomenon that goes beyond Conjecture 3.3. Namely, when $\chi_1\chi_2 = \chi_F$, we not only verify that $\beta = \gamma = 0$ as predicted above, but it also hints at what the coefficient δ along $g(q^p)$ should be. Namely:

Conjecture 3.7. *Assume $\chi_1\chi_2 = \chi_F$. Then*

$$\delta = \frac{\log_{E,p}(P)\log_{E,p}(Q)}{\log_p(u_F)}$$

up to a non-zero algebraic factor in $\mathbb{Q}(\sqrt{-1})$.

Example 3.8. Let χ_8 and χ_{-7} be the (even and odd, respectively) quadratic characters of conductors 8 and -7 and let $\chi := \chi_8 \cdot \chi_{-7}$. The space $S_1(56, \chi)$ is one-dimensional and spanned by the form

$$g = q - q^2 + q^4 - q^7 - q^8 - q^9 + \cdots.$$

Let $p = 23$, an irregular prime for g. We have $a_{23}(g) = 2$, with $\chi_8(23) = \chi_{-7}(23) = \chi(23) = 1$.

We consider all curves of conductor dividing 23×56 in the next table. Here the second to fifth columns give the ranks $r(E), r(E, \chi_8), r(E, \chi_{-7})$ and $r(E, \chi_{-56})$ and the sixth to ninth columns the coefficients in (24) to precision 23^{10}.

Curve	id	χ_8	χ_{-7}	χ_{-56}	α	β	γ	δ
$14a$	0	0	0	0	0	0	0	0
$46a$	0	0	1	1	-3975097185284	0	0	-945005819843
$56a$	0	1	1	0	14352457709431	5640666171804/23	0	0
$56b$	0	0	0	0	0	0	0	0
$92a$	0	1	1	0	2618172201698	12672525684729/23	0	0
$92b$	1	0	0	1	-19716303118943	5063646764719/23	0	0
$161a$	0	2	0	0	0	0	0	0
$184a$	1	0	0	1	-8849640277357	$-12034743090295/23$	0	0
$184b$	1	1	0	0	2488846330016	0	0	11712106557302
$184c$	0	0	1	1	12767670057052	0	0	13912542397730
$184d$	0	0	1	1	20082611393598	0	0	17675998850758
$322a$	1	1	1	1	0	0	0	0
$322b$	0	0	0	0	0	0	0	0
$322c$	0	1	0	1	0	0	$-14074337071266/23$	0
$322d$	1	0	0	1	-14031025892117	$-14899260693889/23$	0	0
$644a$	1	0	0	1	13192495681964	$-1819657478174/23$	0	0
$644b$	1	0	1	0	0	0	$-17882538474414/23$	0
$1288a$	1	1	0	0	-12887806466128	0	0	-9022365673563
$1288b$	0	2	1	1	14163609502103	0	0	0
$1288c$	0	0	1	1	15725566502785	0	0	-4411481895818
$1288d$	0	2	0	0	0	0	0	0
$1288e$	1	1	1	1	0	0	0	0
$1288f$	1	1	1	1	0	0	0	0
$1288g$	0	1	1	0	-17462205584266	14360194422860/23	0	0
$1288h$	1	0	0	1	5344148518790	$-9657587156908/23$	0	0
$1288i$	0	1	0	1	0	0	19841263299919/23	0

The fundamental 23-units in $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-56})$ and fundamental unit in $\mathbb{Q}(\sqrt{8})$ are as follows: Let $u_{-7} \in \mathbb{Q}_{23}$ be the unit root of $x^2 - 8x + 23$, $u_{-56} \in \mathbb{Q}_{23}$ the unit root of $x^2 - 6x + 23$, and $u_8 \in \mathbb{Q}_{23}$ the ratio of the roots of $x^2 - 2x - 1$.

We examine curves with each possible rank pattern. Note that the equalities stated below were checked to precision 23^{10}.

Let E be the elliptic curve $46a$, which has rank pattern 0011. We take

$$P = \left((-\sqrt{-7} - 3)/2, -2\right), \, Q = \left(1177/800, (42891\sqrt{-14} - 23540)/32000\right)$$

and find

$$\delta = \frac{2 \cdot 11}{23} \times \frac{\log_{E,p}(P) \cdot \log_{E,p}(Q)}{\log_p(u_8)}.$$

Let E be the elliptic curve $184b$, which has rank pattern 1100. We take

$$P = (2, -1), \, Q = (3/2, \sqrt{2}/4)$$

and find

$$\delta = \frac{2^7 \cdot 3^2}{11 \cdot 23} \times \frac{\log_{E,p}(P) \cdot \log_{E,p}(Q)}{\log_p(u_8)}.$$

Let E be the elliptic curve $92a$, which has rank pattern 0110. We take

$$P = \left(2\sqrt{2} + 4, 6\sqrt{2} + 11\right), \, Q := \left(-(\sqrt{-7} + 1)/2, 1\right)$$

and find

$$\beta = -\frac{2^5 \cdot 3^2}{11 \cdot 23} \times \frac{\log_{E,p}(P) \log_{E,p}(Q)}{\log_p(u_8)(\log_p(u_{-7}) - \log_p(u_{-56}))}.$$

Let E be the elliptic curve $322d$, which has rank pattern 1001. We take

$$P = (0, 2), \, Q = \left(-380/63, (-3904\sqrt{-14} + 3990)/1323\right)$$

and find

$$\beta = -\frac{2 \cdot 5 \cdot 11}{23} \times \frac{\log_{E,p}(P) \log_{E,p}(Q)}{\log_p(u_8)(\log_p(u_{-7}) - \log_p(u_{-56}))}.$$

Let E be the elliptic curve $322c$, which has rank pattern 0101. We take

$$P = \left(\sqrt{2} - 1, -\sqrt{2} + 1\right), \, Q = \left(19/28, (51\sqrt{-14} - 329)/392\right)$$

and find
$$\gamma = -\frac{2\cdot 11}{23} \times \frac{\log_{E,p}(P)\log_{E,p}(Q)}{\log_p(u_8)(\log_p(u_{-7})-\log_p(u_{-56}))}.$$

Let E be the elliptic curve $644b$, which has rank pattern 1010. We take
$$P=(4,7),\, Q=\left(-301/9, -2005\sqrt{-7}/27\right)$$
and find
$$\gamma = -\frac{2^5\cdot 3^2}{11\cdot 23} \times \frac{\log_{E,p}(P)\log_{E,p}(Q)}{\log_p(u_8)(\log_p(u_{-7})-\log_p(u_{-56}))}.$$

The rational numbers which appear seem closely related to a modified version of the algebraic factor in [DLR1, Equation (79)]. Namely, if one removes the first factor from the denominator of that expression (as it vanishes) then what remains gives for the six curves above, respectively, the rational numbers
$$\frac{2\cdot 11}{23}, \frac{2^5\cdot 3^2}{11\cdot 23}, \frac{2^5\cdot 3^2}{11\cdot 23}, \frac{2\cdot 11}{23}, \frac{2\cdot 11}{23}, \frac{2^5\cdot 3^2}{11\cdot 23}.$$

To conclude notice that the curves $1288e$ and $1288f$ have rank $4=1+1+1+1$, and here the projection to the generalised eigenspace appears to be zero. However, for $1288b$ the rank patter is $4=0+2+1+1$ and the projection is now non-zero and supported on the classical form $g(q)$. This non-vanishing phenomenon in rank 4 is similar to that for curve $1909a$ in Example 3.4. This suggests that in the rank 4 setting using irregular weight one forms one might be able to construct non-vanishing Selmer classes, in the same way that non-vanishing classes are constructed in the rank 2 setting using regular weight one forms in [DR2].

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