# STARK-HEEGNER POINTS FOR ASAI REPRESENTATIONS

HENRI DARMON, ALAN LAUDER AND VICTOR ROTGER

Abstract. A conjectural construction of global points on modular abelian varieties is proposed. These points are defined over the field cut out by the tensor induction (or Asai representation) of a totally odd two-dimensional Galois representation of a real quadratic field.

Contents



## Introduction

Let $E$ be an elliptic curve over $\mathbb{Q}$ and let $V$ be an Artin representation of $\mathbb{Q}$, that is to say, a finite-dimensional vector space over a finite extension $\kappa \subset \mathbb{C}$ of $\mathbb{Q}$, called the *field of coefficients* of $V$, endowed with a continuous linear action of $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The action of $G_{\mathbb{Q}}$ factors through the quotient $\mathrm{Gal}(H/\mathbb{Q})$ for a finite extension $H$ of $\mathbb{Q}$, referred to as the field *cut out* by $V$, giving rise to a homomorphism

$$\varrho : \mathrm{Gal}(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V).$$

For each prime $\ell$, denote by $\kappa_\ell$ a completion of $\kappa$ at a prime above $\ell$. The pair $(E, V)$ gives rise to a continuous $\kappa_\ell$-linear representation of $G_{\mathbb{Q}}$ whose underlying vector space is the tensor product of the Tate module of $E$ with $V$:

$$(1) \qquad W_{E,V,\ell} := T_\ell(E) \otimes_{\kappa_\ell} (V \otimes_\kappa \kappa_\ell), \quad \text{where} \quad T_\ell(E) := (\lim_{\substack{\leftarrow \\ n}} E[\ell^n]) \otimes_{\mathbb{Z}_\ell} \kappa_\ell.$$

It is unramified at any rational prime $p$ not dividing $\ell ND$, where $N$ and $D$ are the conductors of $E$ and $V$ respectively. The frobenius element at such a $p$, denoted $\sigma_p$, is well-defined up to conjugation in $\mathrm{Aut}(W_{E,V,\ell})$. Its characteristic polynomial has coefficients in $\kappa \subset \kappa_\ell$ and does not depend on the choice of $\ell \neq p$.

The *equivariant Birch and Swinnerton-Dyer conjecture* relates the order of vanishing at $s = 1$ of the Hasse-Weil-Artin $L$-series

$$L(E, V, s) := \prod_{p \nmid ND} \det(1 - \sigma_p \cdot p^{-s})^{-1} \tag{2}$$

to the $V$-rank of $E$, defined as

$$\mathrm{rank}(E, V) := \dim_\kappa E(H)^V, \quad \text{where} \quad E(H)^V := \hom_{G_{\mathbb{Q}}}(V, E(H) \otimes \kappa). \tag{3}$$

**Conjecture BSD(E,V)**: $\mathrm{ord}_{s=1} L(E, V, s) = \mathrm{rank}(E, V)$.

Progress on this conjecture has been painstaking and hard-won. In the analytic rank 0 situation where $L(E, V, 1) \neq 0$, it is known when $V$ is one-dimensional [Kato], when $V$ is induced from certain ring class characters of an imaginary quadratic field [BD1] or a real quadratic field [DR2], or when $V$ is the tensor product of two odd irreducible two-dimensional Artin representations [DR2].

Results in the analytic rank one case, where $L(E, V, s)$ has a simple zero at $s = 1$, are even more fragmentary and largely confined to scenarios where the vanishing of $L(E, V, s)$ is forced by parity considerations. This happens when $V$ is isomorphic to its contragredient—so that the functional equation relates the values of $L(E, V, s)$ at $s$ and $2 - s$—and when the sign in this functional equation, denoted $\mathrm{sign}(E, V)$, is equal to $-1$. The $L$-function $L(E, V, s)$ is then essentially an odd function of $s - 1$, and hence vanishes to odd order at the central point.

Assuming for simplicity that $N$ and $D$ are relatively prime to each other, the most basic instances of this occur when the representation $V$ is induced from a ring class character of a quadratic field $F$. In that case, $V$ is always self-dual and

$$\mathrm{sign}(E, V) = \varepsilon_F(-N), \tag{4}$$

where $\varepsilon_F$ is the quadratic Dirichlet character attached to $F$.

(a) When $F$ is imaginary and $\varepsilon_F(-N) = -1$, the systematic vanishing of $L(E, V, 1)$ is accounted for by a plentiful supply of complex multiplication points defined over all ring class fields of $F$ of conductor prime to $N$, lying on suitable modular or Shimura curves which uniformise $E$. The significance of the resulting *Heegner points* in $E(F^{\mathrm{ab}})$ for the study of the Birch and Swinnerton Dyer conjecture can hardly be overstated. These points are the basis for the important results of Gross-Zagier [GZ] and Kolyvagin [Ko], and provide essentially the only instance where a simple zero of $L(E, V, s)$ at $s = 1$ can be parlayed into the construction of a non-trivial element of $E(H)^V$.

(b) When $F$ is real and $\varepsilon_F(N) = -1$, the role of Heegner points is ostensibly played by the *Stark-Heegner points* introduced in [Dar], whose conjectural nature prevents their being used to prove $\mathrm{BSD}(E, V)$ in this case.

The article [DR2] does prove some non-trivial instances of $\mathrm{BSD}(E, V)$ in the analytic rank zero variant of scenario (b), where $\varepsilon_F(N) = 1$. The proof in this case is somewhat roundabout and makes no use of Stark-Heegner points: rather it proceeds by realising $V$ as a direct summand of a tensor product $V_1 \otimes V_2$ of two odd two-dimensional Artin representations in order to reduce $\mathrm{BSD}(E, V)$ to $\mathrm{BSD}(E, V_1 \otimes V_2)$, which is proved using global cohomology classes in $H^1(\mathbb{Q}, W_{E,V_1\otimes V_2,\ell})$ arising from $p$-adic families of étale Abel-Jacobi images of diagonal cycles in triple products of modular curves.

A motivation for singling out scenario (b) for special attention is that it has revealed tantalising possibilities for constructing global points on elliptic curves in settings that lie squarely beyond the scope of the theory of complex multiplication. It is natural to ask whether other instances of self-dual $V$ for which $\mathrm{sign}(E, V) = -1$ might lead to analogous—conjectural, but entirely explicit —constructions of global points on $E$.

The setting where $V = V_1 \otimes V_2$ is a self-dual tensor product of two odd irreducible two-dimensional Artin representations may seem promising at first glance in light of the progress initiated in [DR2]. However, one always has

$$\mathrm{sign}(E, V_1 \otimes V_2) = 1 \quad \text{when} \quad \gcd(N, D) = 1, \tag{5}$$

which makes this setting somewhat unpropitious for suggesting new Stark-Heegner point constructions. (See however [DLR1] where $p$-adic regulators attached to $L(E, V_1 \otimes V_2, s)$ are related to appropriate "Stark points" in favorable "analytic rank two" scenarios, and the work of Dall'Ava and Horawa [DaH] which focuses on situations of analytic rank one, where the conductors of $E$ and $V_1 \otimes V_2$ are no longer assumed to be coprime. An ongoing work in progress of Andreatta, Bertolini, Seveso and Venerucci [ABSV] approaches a similar setting through the study of endoscopic lifts to unitary groups.)

Motivated by these considerations, this paper explores the setting where $V$ is the *tensor induction*, or *Asai representation*, of a two-dimensional Artin representation of a real quadratic field $F$. The definition and key properties of this tensor induction are recalled in Section 2. Let $\mathrm{Ver}_F^{\mathbb{Q}}$ denote the *transfer* (*Verlagerung*) of a character of $G_F$ to a character of $G_{\mathbb{Q}}$, which is dual to the transfer map $G_{\mathbb{Q}}^{\mathrm{ab}} \longrightarrow G_F^{\mathrm{ab}}$ of group theory.

**Sign Formula** ([Pra, Thms B & D, Rk. 4.1.1]): *Let $V_0$ be a two-dimensional Artin representation of a real quadratic field $F$ of conductor $\mathfrak{D} \subset \mathcal{O}_F$ with $\gcd(N, \mathfrak{D}) = 1$ which is odd at the two real places of $F$, and let*

$$V = \mathrm{Ind}^{\otimes}(V_0) \tag{6}$$

*be the tensor induction of $V_0$ from $G_F$ to $G_{\mathbb{Q}}$. If*

$$\mathrm{Ver}_F^{\mathbb{Q}}(\det V_0) = 1, \tag{7}$$

*then $V$ is self-dual and*

$$\mathrm{sign}(E, V) = \varepsilon_F(N). \tag{8}$$

This strikingly simple recipe for $\mathrm{sign}(E,V)$, which is identical to (4), suggests that $E(H)^V$ is non-trivial for an explicit, systematic class of Asai representations, and it is natural to seek a construction of the non-trivial elements of $E(H)^V$ whose existence is predicted by the equivariant Birch and Swinnerton-Dyer conjecture, guided by the ideas in the proof of the Gross-Zagier formula. Proposing such an analytic expression is one of the main goals of this paper.

Assume from now on that $V_0$ and $V$ are as in the statement of the Sign Formula above, and that $\mathrm{sign}(E,V)=-1$. It follows that $\varepsilon_F(N)=-1$, which implies that there is a prime $p$ dividing $N$ with odd multiplicity for which $\varepsilon_F(p)=-1$. Fix such a $p>3$ and assume for simplicity that it divides $N$ with multiplicity one.

In fact, the rest of the paper will focus solely on the case where the conductor $N$ of $E$ satisfies the following somewhat stronger *generalised Heegner hypothesis*:

$$N = pM, \text{ where } \varepsilon_F(p) = -1, \text{ and } \varepsilon_F(\ell) = 1 \text{ for all } \ell | M. \tag{9}$$

This hypothesis, although stronger than the more natural condition $\varepsilon_F(N)=-1$, simplifies the setting without eliding any of its essential features. It implies that there is an ideal $\mathfrak{M} \subset \mathcal{O}_F$ for which $\mathcal{O}_F/\mathfrak{M} = \mathbb{Z}/M\mathbb{Z}$, which shall be fixed from now on.

The Artin representation $V_0$ is associated to a holomorphic Hilbert modular form over $F$ of parallel weight one, level $\mathfrak{D}$ and character $\chi := \det V_0$. Such a Hilbert modular form is attached to a function

$$G : I(F) \longrightarrow \kappa \tag{10}$$

on the semigroup $I(F)$ of integral ideals of $F$, which is defined by extending by multiplicativity the following definition on prime powers:

$$G(\lambda) = \mathrm{Trace}\left(\sigma_\lambda|_{V_0^{I_\lambda}}\right), \qquad G(\lambda^{n+1}) = \begin{cases} G(\lambda)G(\lambda^n) - \chi(\lambda)G(\lambda^{n-1}) & \text{if } \lambda \nmid \mathfrak{D}, \\ G(\lambda)^n & \text{if } \lambda | \mathfrak{D}, \end{cases}$$

where $I_\lambda$ is the inertia group at $\lambda$ and $V_0^{I_\lambda}$ is the space of $I_\lambda$-invariants. Fix an ordering of the real embeddings of $F$ and for $\nu \in F$, write $\nu_1, \nu_2 \in \mathbb{R}$ for the images of $\nu$ under these two embeddings. To the function $G$ and to the ideal $\mathfrak{M}$ of $\mathcal{O}_F$ is associated a Hilbert modular generating series

$$G_{\mathfrak{M}}(\tau_1,\tau_2) := \sum_{\nu \in (\mathfrak{M}\mathfrak{d}^{-1})_+} G((\nu)\mathfrak{d})q^\nu, \qquad q^\nu := e^{2\pi i(\nu_1\tau_1+\nu_2\tau_2)}, \tag{11}$$

where the sum runs over all the totally positive elements of $\mathfrak{M}\mathfrak{d}^{-1}$, and $\mathfrak{d}$ denotes the different ideal of $\mathcal{O}_F$. This generating series is a Hilbert modular form of parallel weight one and character $\chi$ on the group

$$\Gamma_0(\mathfrak{D};\mathfrak{M}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, d \in \mathcal{O}_F, \quad b \in \mathfrak{M}^{-1}, \quad c \in \mathfrak{M}\mathfrak{D} \right\}. \tag{12}$$

After choosing an embedding of $F$ into a completion $F_p$ at $p$, and a $p$-adic logarithm $\log_p : F^\times \longrightarrow F_p$, we consider the following $p$-adic generating series deforming (11):

$$G'_{\mathfrak{M}}(\tau_1,\tau_2) := \sum_{\substack{\nu\in(\mathfrak{M}\mathfrak{d}^{-1})_+,\\ p\nmid\nu}} \log_p(\nu) G((\nu)\mathfrak{d}) q^\nu. \tag{13}$$

Let

$$\Phi^{(p)}_{DM} := G'_{\mathfrak{M}}(\tau,\tau), \qquad \Phi^{(p)}_{M} := \mathrm{Trace}^{DM}_M(\Phi^{(p)}_{DM}) \tag{14}$$

denote the diagonal restriction of $G'_{\mathfrak{M}}$ and its formal trace from level $DM$ to level $M$. Here $D \geq 1$ is such that $\mathfrak{D}\cap\mathbb{Z} = (D)$.

In Section 3, it is shown that $\Phi^{(p)}_M$ is a $p$-adic modular form of weight two and (tame) level $M$. Let $e_{\mathrm{ord}}$ be the ordinary projection from this space of $p$-adic modular forms to the space of classical modular forms of weight two and level $N$ that are fixed by $U_p^2$ (and hence new at $p$). The modular form attached to the Asai representation $V$, defined by

$$\Phi_V := e_{\mathrm{ord}}(\Phi^{(p)}_M) \in M_2(\Gamma_0(N)) \tag{15}$$

is a classical modular form of weight two on $\Gamma_0(N)$ that is new at $p$.

Let $J_0(N)$ denote the Jacobian of $X_0(N)$ and define

$$J_0(N)(H)^V = \hom_{G_{\mathbb{Q}}}(V, J_0(N)(H)\otimes\kappa)$$

as in (3). The main conjecture of this paper expresses it in terms of the $p$-adic logarithm of a global point arising from an element of $J_0(N)(H)^V$.

The $p$-new part of $J_0(N)$, denoted $J_0^{(p)}(N)$ has purely toric reduction, and there is a natural exact sequence arising from the Tate-Morikawa uniformisation of $J_0^{(p)}(N)$,

$$1\longrightarrow X \xrightarrow{\ j\ } T \longrightarrow J_0^{(p)}(N)(\bar{\mathbb{Q}}_p)\longrightarrow 1,$$

where $T$ is a $p$-adic torus, $X := \hom(T,\mathbb{G}_m)$ is the character group of $T$ (a free $\mathbb{Z}$-module of rank $g = \dim J_0^{(p)}(N)$), and $j$ arises from the monodromy pairing $X\times X\longrightarrow\mathbb{G}_m$. A differential on $T$ is said to be *toric* if it is of the form $\eta^*(\frac{dt}{t})$ for some $\eta\in X$. Such a differential is invariant under translation by $j(X)$, and theorefore descends to a differential on $J_0^{(p)}(N)$ . The space $\Omega^1_{\mathrm{tor}}$ of toric differentials on $J_0^{(p)}(N)$ is a free $\mathbb{Z}$-module of rank $g$ by definition, and is endowed with a natural linear action of the Hecke operators $T_n$ acting as correspondences on $J_0^{(p)}(N)$. A toric differential is said to be *generic* if its translates under the Hecke operators $T_n$ span the $\mathbb{Q}$-vector space $\Omega^1_{\mathrm{tor}}\otimes\mathbb{Q}$.

**Main Conjecture**. *There is a point $P_V \in J_0^{(p)}(N)(H)\otimes\kappa$ belonging to the image of an element of $J_0^{(p)}(N)(H)^V$, and a generic toric differential $\omega$ on $J_0^{(p)}(N)$ for which*

$$\Phi_V = \sum_{n=1}^{\infty} \log_\omega(T_n(P_V)) q^n. \tag{16}$$

One of the simplest cases of this conjecture is considered in [DPV], where $V_0$ is taken to be the direct sum of the trivial character and an odd (unramified) ring class character $\psi$

of $F$, and $G$ is the Hilbert Eisenstein series over $F$ of parallel weight one attached to this pair of characters. The Asai representation of $V_0$ is then equal to

$$\mathrm{Asai}(V_0) =: V = 1 \oplus 1 \oplus \mathrm{Ind}_F^{\mathbb{Q}}(\psi).$$

The main result of [DPV] expresses the $n$th fourier coefficients of the modular form $\Phi_V$ (where $\mathfrak{M} = 1$ and $N = p$) as a linear combination of RM values of the *winding cocycle*

$$J_w \in H^1(\mathbf{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times), \tag{17}$$

where $\mathcal{A}^\times$ is the multiplicative group of rigid analytic functions on the Drinfeld $p$-adic upper half-plane $\mathcal{H}_p$. More precisely, the set of RM points on $\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash\mathcal{H}_p$ of (prime-to-$p$) discriminant $D_F = \mathrm{Disc}(F)$ is endowed with an action of the class group of $F$, and the main result of loc.cit. is the equality,

$$a_n(\Phi_V) = \sum_{\sigma \in \mathrm{Gal}(H/F)} \psi(\sigma)^{-1} \log_p \mathrm{Nrm} J_w[T_n \tau^\sigma], \tag{18}$$

where the value $J_w[\tau]$ of the cocycle $J_w$ at an RM point has been extended to RM divisors by multiplicativity, and Nrm denotes the norm from $\mathbb{Q}_{p^2}$ to $\mathbb{Q}_p$. The general theory of rigid analytic and meromorphic cocycles predicts that the RM values $J_w[T_n\tau^\sigma]$ map to points in $J_0(p)(H)$ under the Tate-Morikawa uniformisation of $J_0(p)$. The Main Conjecture of this paper can thus be envisaged as an extension of (18) to a setting where Hilbert Eisenstein series are replaced by general Hilbert modular forms of weight one (satisfying the self-duality assumption).

For an earlier discussion that also aims to place (18) in a more general framework, see [FLPSW].

## 1. Refinement of the main conjecture

Write

$$\Phi_V = \sum_f \lambda_{V,f} \cdot f + \Phi_V^{\mathrm{old}}, \tag{19}$$

where the sum is taken over a basis of normalised newforms in $S_2(\Gamma_0(N))$ and $\Phi_V^{\mathrm{old}}$ is an oldform. Our goal is to reformulate the main conjecture of the introduction as a description of the coefficients $\lambda_{V,f}$ in this decomposition.

The $\mathbb{Q}$-algebra $\mathbb{T}_N$ generated by the Hecke operators $T_m$ with $\gcd(m, N) = 1$ acting on $S_2(\Gamma_0(N))$ is an étale algebra, isomorphic to a product of totally real fields. Assume that $\kappa$ is large enough to split $\mathbb{T}_N$, i.e., that

$$\mathbb{T}_N \otimes \kappa \simeq \bigoplus_f \kappa,$$

where the sum is taken over pairs $(f, N_f)$ with $f$ a normalised newform of level $N_f|N$ with coefficients in $\kappa$. The idempotents $\pi_f \in \mathbb{T}_N \otimes \kappa$ attached to this decomposition lead to a direct sum decomposition

$$J_0(N)(H) \otimes \kappa = \bigoplus_f J_0(N)(H)_f,$$

where $J_0(N)(H)_f := \pi_f(J_0(N)(H) \otimes \kappa)$. The "Stark-Heegner point" $P_V$ of the Main Conjecture of the introduction is the sum of its $f$-isotypic components:

$$P_V := \sum_f P_{V,f} + P_{V,\mathrm{old}} \in J_0(N) \otimes \kappa, \qquad \text{where } P_{V,f} := \pi_f(P_V) \in J_0(N)(H)_f^V.$$

The main conjecture of the introduction can be reformulated as follows:

**Conjecture 1.1.** *The coefficient* $\lambda_{V,f}$ *arising in the spectral expansion* (19) *of* $\Phi_V$ *is given by*

$$\lambda_{V,f} = \log_\omega(P_{V,f}) \tag{20}$$

*up to a multiplicative constant in* $\kappa$*.*

To further lighten the notations, we will focus henceforth on the case where $f$ is a weight two newform of level $N$ with *rational* fourier coefficients, i.e., a form that corresponds to an elliptic curve $E/\mathbb{Q}$ of conductor $N$. In that case, the field $\kappa$ can simply be taken to be the field of coefficients of the Artin representation $V_0$.

The running assumption that $p$ divides $N$ implies that $E$ has multiplicative reduction at $p$, and hence that

$$a_p := a_p(E) = \begin{cases} +1 & \text{if } E \text{ has split multiplicative reduction at } p, \\ -1 & \text{if } E \text{ has non-split multiplicative reduction at } p. \end{cases}$$

The base change of $E$ to the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$ admits a rigid-analytic uniformization according to Tate's theory. Namely, there is a $p$-adic period $q_E \in p\mathbb{Z}_p$ and an isomorphism of rigid-analytic varieties over $\bar{\mathbb{Q}}_p$

$$\Phi_{\mathrm{Tate}} : \bar{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}} \xrightarrow{\sim} E(\bar{\mathbb{Q}}_p) \tag{21}$$

which is defined over $\mathbb{Q}_p$ when $a_p = +1$ and over $\mathbb{Q}_{p^2}$ when $a_p = -1$.

Let $\log_p : \bar{\mathbb{Q}}_p^\times \longrightarrow \bar{\mathbb{Q}}_p$ denote the branch of the $p$-adic logarithm such that $\log_p(q_E) = 0$, and define

$$\log_{\mathrm{Tate}} : E(\bar{\mathbb{Q}}_p) \longrightarrow \bar{\mathbb{Q}}_p, \quad \log_{\mathrm{Tate}}(x) := \log_p \Phi_{\mathrm{Tate}}^{-1}(x).$$

Note that $\log_{\mathrm{Tate}}$ is defined only at primes of multiplicative reduction, and is different from that with respect to the canonical differential on $E$ itself (denoted $\log_{E,p}$ in [DLR1]). Up to a non-zero rational factor, we then have

$$\log_\omega(P_{V,f}) = \log_{\mathrm{Tate}}(P_{V,f}).$$

The main goal of this section is to further refine Conjecture 1.1 by predicting when the coordinate $\lambda_{V,f}$ in (19) gives rise to a non-trivial element of the Mordell-Weil group $J_0(N)(H)_f$. More precisely, the global sign of the Hasse-Weil Artin $L$-function $L(E,V,s)$ is $-1$ and hence this $L$-function vanishes to odd order. If $\mathrm{ord}_{s=1} L(E,V,s) = 1$, then there is a unique irreducible constituent $W_1$ of $V = W_0 \oplus W_1$ for which

$$L(E, W_0, 1) \neq 0, \qquad \mathrm{ord}_{s=1} L(E, W_1, s) = 1.$$

Letting $\alpha_G$ and $\beta_G = \alpha_G^{-1}$ denote the eigenvalues of the frobenius element at $p$ in $G_F$ acting on the Artin representation $V_0$ attached to $G$, Proposition 2.2 below asserts that the eigenvalues of $\sigma_p$ in $G_{\mathbb{Q}}$ on its tensor induction $V$ are $1, -1, \alpha_G$ and $\beta_G$.

**Conjecture 1.2.** *The scalar $\lambda_{V,f}$ is non-trivial if and only if the following two conditions are satisfied:*

(I) $\mathrm{ord}_{s=1}L(E,V,s)=1$*;*

(II) *The invariant $a_p \in \{\pm 1\}$ has multiplicity* 1 *as an eigenvalue of the frobenius element $\sigma_p$ acting on $V$, and its associated eigenvector belongs to $W_1$.*

*When the above conditions hold, then $\lambda_{V,f} = \log_{\mathrm{Tate}}(P_{V,f})$ and the global point $P_{V,f}$ lies in the image of $E(H)^{W_1}$. More precisely, this point forms a basis of the one dimensional subspace of this image on which $\sigma_p$ acts with the eigenvalue $a_p$.*

The remainder of this paper will be devoted to describing evidence for Conjecture 1.2. Theorem 4.5 provides theoretical evidence in the special case where $G$ is associated to the theta series of a character of a biquadratic field, and Section 5 documents a series of numerical experiments in support of this conjecture which also illuminate some of its more subtle features.

## 2. Asai representations

This section collects a few simple facts about Asai representations, in the level of generality that is relevant to the constructions of this article.

Let $\mathfrak{G}$ be a finite group and let $G$ be a subgroup of index two in $\mathfrak{G}$. The induction to $\mathfrak{G}$ of a finite-dimensional representation $V_0$ of $G$ is the space of $G$-invariant functions

$$\mathrm{Ind}_G^{\mathfrak{G}}(V_0) := \{\eta : \mathfrak{G} \longrightarrow V_0 \text{ satisfying } \eta(gx) = g\eta(x), \text{ for all } g \in G\}$$

endowed with the action of $\gamma \in \mathfrak{G}$ given by

$$(\gamma \cdot \eta)(x) = \eta(x\gamma).$$

Since a vector in $\mathrm{Ind}_F^{\mathbb{Q}}(V_0)$ is completely determined by its values on a system of coset representatives for $G\backslash\mathfrak{G}$, the representation $\mathrm{Ind}_G^{\mathfrak{G}}(V_0)$ is identified with $V_0 \oplus V_0$ after fixing an element $\tau_0 \in \mathfrak{G} - G$, and sending $\eta$ to $(\eta(1), \eta(\tau_0))$. Conjugation by this element $\tau_0$ determines an automorphism of $G$ which will be written as $g \mapsto g' := \tau_0 g \tau_0^{-1}$. Under the identification of $\mathrm{Ind}_G^{\mathfrak{G}}(V_0)$ with $V_0 \oplus V_0$ determined by $\tau_0$, the action of $\mathfrak{G}$ on the latter space is given by

$$g((v_1, v_2)) = \begin{cases} (gv_1, g'v_2) & \text{if } g \in G, \\ (g\tau_0^{-1}v_2, \tau_0 g v_1) & \text{if } g \in \mathfrak{G} - G. \end{cases} \tag{22}$$

The *tensor induction* of $V_0$ to $\mathfrak{G}$, denoted $\mathrm{Ind}^{\otimes}(V_0)$, is defined by replacing direct sums with tensor products in (22). Its underlying vector space is $V := V_0 \otimes V_0$, with action of $\mathfrak{G}$ defined by

$$g(v_1 \otimes v_2) = \begin{cases} gv_1 \otimes g'v_2 & \text{if } g \in G, \\ g\tau_0^{-1}v_2 \otimes \tau_0 g v_1 & \text{if } g \in \mathfrak{G} - G. \end{cases} \tag{23}$$

**Lemma 2.1.** *The character $\chi_V$ attached to $V = \mathrm{Ind}^{\otimes}(V_0)$ is given by*

$$\chi_V(g) = \begin{cases} \chi_{V_0}(g)\chi_{V_0}(g'), & \textit{if } g \in G, \\ \chi_{V_0}(g^2) & \textit{if } g \in \mathfrak{G} - G. \end{cases}$$

*Proof.* For $g \in G$, the formula for $\chi_V(g)$ follows directly from (23). Consider the second case, where $g$ belongs to $\mathfrak{G} - G$, and assume for notational simplicity that $V_0$ is two-dimensional, although the general case can be treated in the same way. Since the isomorphism class of the Asai induction does not depend on the choice of $\tau_0 \in \mathfrak{G} - G$ that was made to define it, we may set $\tau_0 = g$. Every element of $G$ acts semisimply on $V_0$ since $G$ is finite; let $v_\alpha, v_\beta \in V_0$ be an eigenbasis for $g^2$ acting on $V_0$, with eigenvalues $\alpha$ and $\beta$ respectively. Relative to the basis $(v_\alpha \otimes v_\alpha, v_\beta \otimes v_\beta, v_\alpha \otimes v_\beta, v_\beta \otimes v_\alpha)$ for $V$, the action of $g$ is given by

$$\begin{aligned} g(v_\alpha \otimes v_\alpha) &= \alpha \cdot v_\alpha \otimes v_\alpha, & g(v_\beta \otimes v_\beta) &= \beta \cdot v_\beta \otimes v_\beta, \\ g(v_\alpha \otimes v_\beta) &= \alpha \cdot v_\beta \otimes v_\alpha, & g(v_\beta \otimes v_\alpha) &= \beta \cdot v_\alpha \otimes v_\beta. \end{aligned}$$

The matrix of $g$ in this basis is therefore given by

$$M_g := \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \alpha & 0 \end{pmatrix}. \tag{24}$$

The result follows directly. □

If $V_0$ is a one-dimensional representation of $G$, associated to a character $\chi : G \longrightarrow \mathbb{C}^\times$, the tensor induction of $V_0$ coincides with the transfer of $\chi$.

Of special interest for the constructions of this article are two-dimensional representations $V_0$ of $G$ satisfying the condition

$$\mathrm{Ind}^{\otimes} \det(V_0) = 1. \tag{25}$$

**Proposition 2.2.** *If $V_0$ is a two dimensional representation of $G$ satisfying* (25)*, then $V := \mathrm{Ind}^{\otimes} V_0$ is a self-dual Artin representation. The eigenvalues of any $g \in \mathfrak{G} - G$ acting on $V$ are equal to* 1, −1*, $\alpha$, and $\beta = \alpha^{-1}$, where $\alpha$ and $\beta$ are the eigenvalues of $g^2 \in G$ acting on $V_0$.*

*Proof.* If $g$ belongs to $G$, let $\alpha$ and $\beta$ be the eigenvalues of $g$ acting on $V_0$, and let $\alpha'$ and $\beta'$ denote the eigenvalues of $g'$. Assumption (25) implies that $\alpha\beta\alpha'\beta' = 1$, while the eigenvalues for $g$ on $V$ are equal to $\alpha\alpha'$, $\alpha\beta'$, $\beta\alpha'$, and $\beta\beta'$. This set of eigenvalues is therefore closed under the map $\zeta \mapsto \bar{\zeta}$, and hence the trace of $g$ acting on $V$ is real. If $g$ belongs to $\mathfrak{G} - G$, then letting $\alpha$ and $\beta$ denote the eigenvalues of $g^2$ acting on $V_0$, condition (25) implies that

$$\alpha\beta = \det(V_0)(g^2) = \mathrm{Ind}^{\otimes} \det(V_0)(g) = 1. \tag{26}$$

Since the matrix of $g$ acting on $V$ relative to a suitable basis is given by (24), its trace $\alpha + \beta$ is real, and the self-duality of $V$ follows. The second assertion in Proposition 2.2 follows likewise from (24) and (26). □

## 3. Reinterpretation via $p$-adic $L$-functions

The purpose of this section is to relate the conjecturally global point $P_{V,f}$ of Conjecture 1.2 to derivatives of suitable $p$-adic $L$-functions. This will lead to some theoretical evidence for Conjecture 1.2 when $V_0$ is induced from a character of a quadratic extension of $F$ which is biquadratic over $\mathbb{Q}$ (Theorem 4.5).

Following the notations of earlier sections, recall that $G$ is the holomorphic Hilbert eigenform of conductor $\mathfrak{D}$, nebentype $\chi$ and parallel weight 1 over a real quadratic field $F$ whose Asai representation is isomorphic to $V$, and that $f \in S_2(N)$ is a normalised newform associated to an elliptic curve $E$ of conductor $N = pM$ by modularity. Here as above $p > 3$ is a prime and $M \geq 1$ an integer relatively prime to $p$. It is assumed that $(N, D_F\mathfrak{D}) = 1$ and $(F, N)$ satisfy the generalised Heegner hypothesis of (9).

Because $U_p f = \pm f$, the weight two newform $f$ is ordinary at $p$. It can therefore be realized as the weight two specialization of a *Hida family* of modular forms, denoted $\mathbf{f}$. This Hida family is a formal $q$-expansion with coefficients in a finite flat extension $\Lambda_{\mathbf{f}}$ of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[(1+p\mathbb{Z}_p)^\times]] \simeq \mathbb{Z}_p[[T]]$, as defined in [DR1, Def. 2.16], whose specialisations at a dense set of *classical* points are classical eigenforms of tame level $N$ and varying weights. More precisely, the spaces

$$\mathcal{X}_{\mathbf{f}} := \hom_{\mathrm{alg}}(\Lambda_{\mathbf{f}}, \mathbb{C}_p), \qquad \mathcal{X} := \hom_{\mathrm{alg}}(\Lambda, \mathbb{C}_p) = \hom_{\mathrm{grp}}((1+p\mathbb{Z}_p)^\times, \mathbb{C}_p^\times)$$

are endowed with a natural structure of $p$-adic analytic spaces. The *weight space* $\mathcal{X}$ contains the integers as a countable dense subset via the inclusion $k \mapsto (x \mapsto x^{k-2})$, and restriction gives a finite flat morphism

$$\mathrm{w} : \mathcal{X}_{\mathbf{f}} \longrightarrow \mathcal{X},$$

called the *weight map*. By definition, the specialisation $f_x := x(\mathbf{f})$ of $\mathbf{f}$ at a point $x \in \mathcal{X}_{\mathbf{f}}$ is an ordinary overconvergent $p$-adic eigenform of level $N$ and weight $k := \mathrm{w}(x)$. When $k := \mathrm{w}(x)$ belongs to $\mathbb{Z}^{>2}$, Hida's classicality theorem implies that $f_x$ is a classical modular form of weight $k$ on $\Gamma_0(N)$ which is new at the primes dividing $M$ and is a simultaneous eigenvector for the Hecke operators, satisfying

$$U_p f_x = \alpha_x f_x, \qquad \text{with } \alpha_x \in \mathcal{O}_{\mathbb{C}_p}^\times.$$

The Hida family $\mathbf{f}$ is uniquely characterised by these properties along with the condition that $f_{x_0} = f$, where $x_0 \in \mathcal{X}_{\mathbf{f}}$ is a suitable classical point of weight $\mathrm{w}(x_0) = 2$.

The ordinarity of $f_x$ when $\mathrm{w}(x) > 2$ implies that $f_x$ cannot be new at $p$: rather, it is the ordinary $p$-stabilisation of a classical normalised newform of level $M$, denoted $f_x^\circ$.

Let $L(f_x^\circ, V, s)$ denote the Hecke-Artin $L$-series attached to $f_x^\circ$ and to $V$, i.e., to the tensor product of $V$ with the compatible system of two-dimensional Galois representations attached to $f_x^\circ$. A twisted variant of Rankin's method, which will be detailed below, implies that $L(f_x^\circ, V, s)$ admits analytic continuation and a functional equation relating its values at $s$ and $k - s$. Denoting by $\mathrm{sign}(f_x, V)$ the sign that arises in this functional equation, the generalised Heegner hypothesis (9) implies that

$$\mathrm{sign}(f_x, V) = \begin{cases} -1 & \text{if } x = x_0; \\ 1 & \text{if } k := \mathrm{w}(x) > 2. \end{cases}$$

In particular, the central critical values $L(f_x^\circ, V, k/2)$ need not vanish when $k > 2$.

It is therefore natural to attempt to interpolate these values $p$-adically to a function on $\mathcal{X}_{\mathbf{f}}$. Starting with the fundamental work of Ichino [Ich] and following the principles of [DR1], this task has been undertaken recently in the literature in various degrees of generality thanks to the contributions of Hsieh [Hs], Blanco-Fornea [BF], Chen-Cheng [CC], Ishikawa [Ish], Kazi [Kazi] and Kazi-Loeffler [KL].

In order to describe the results, recall the Hilbert generating series $G_{\mathfrak{M}}(\tau_1, \tau_2)$ attached to $G$ and $\mathfrak{M}$ in (11). Let

$$\delta_{k_1} = \frac{1}{2\pi i}\left(\frac{d}{d\tau_1} - i\frac{k_1}{2y_1}\right)$$

be the partial Shimura-Maass derivative operator mapping holomorphic Hilbert modular forms of weight $(k_1, k_2)$ to *nearly holomorphic* Hilbert modular forms of weight $(k_1+2, k_2)$, and write

$$\delta_{k_1}^t := \delta_{k_1+2t-2} \circ \cdots \circ \delta_{k_1+2} \circ \delta_{k_1}$$

for its $t$-fold iterate. The nearly holomorphic modular form

$$\delta_1^t G_{\mathfrak{M}}(\tau_1, \tau_2) \in M_{1+2t,1}^{\mathrm{nh}}(\Gamma_0(\mathfrak{D}, \mathfrak{M}))$$

transforms like a holomorphic Hilbert modular form of weight $(1+2t, 1)$ under the action of the congruence subgroup of (12). For any even weight $k = 2t+2 > 2$, the diagonal restriction

$$H_k^{\mathrm{nh}}(q) := \delta_1^t G_{\mathfrak{M}}(\tau, \tau) \in M_k^{\mathrm{nh}}(\Gamma_0(DM))$$

is a nearly holomorphic modular form of weight $k$ and level $DM$. Let

$$H_k := e_{\mathrm{hol}} \mathrm{Tr}_M^{DM} H_k^{\mathrm{nh}} \in M_k(\Gamma_0(M)) \tag{27}$$

denote the holomorphic projection of its trace to level $M$. Then for all $x \in \mathcal{X}_f$ with $\mathrm{w}(x) = k > 2$, let

$$I(f_x^\circ, G) := \frac{\langle H_k, f_x^\circ\rangle}{\langle f_x^\circ, f_x^\circ\rangle}, \tag{28}$$

where $\langle\ ,\ \rangle$ denotes the Petersson scalar product on weight $k$ modular forms.

Let $d_1 := q_1\frac{d}{dq_1}$ be the partial $d$ operator which maps Hilbert modular forms of weight $(k_1, k_2)$ to nearly overconvergent Hilbert modular forms of weight $(k_1+2, k_2)$. The generating series

$$d_1^t G_{\mathfrak{M}}(q_1, q_2) := \sum_{\nu \in (\mathfrak{M}\mathfrak{d}^{-1})_+} \nu_1^t \cdot G((\nu)\mathfrak{d}) q_1^{\nu_1} q_2^{\nu_2} \ \in\ M_{1+2t,1}^{\mathrm{no}}(\Gamma_0(\mathfrak{D}, \mathfrak{M})) \tag{29}$$

is the $q$-expansion of a *nearly overconvergent* $p$-adic Hilbert modular form of weight $(1+2t, 1)$. Its diagonal restriction

$$H_k^{\mathrm{no}} := d_1^t G_{\mathfrak{M}}(q, q) \in M_k^{\mathrm{no}}(\Gamma_0(DM))$$

is a nearly overconvergent modular form of weight $k$ and tame level $DM$. Let

$$H_k^\flat := e_{\mathrm{ord}} \mathrm{Tr}_M^{DM} H_k^{\mathrm{no}} \in M_k(\Gamma_0(N))$$

be the classical modular form of weight $k$ and level $N$ obtained by applying to $H_k^{\mathrm{no}}$:

(1) the trace map $\mathrm{Tr}_M^{DM}$ from overconvergent forms of level $DM$ to level $M$;
(2) the ordinary projection $e_{\mathrm{ord}}$ from overconvergent modular forms of tame level $M$ to classical ordinary modular forms of level $N$.

The geometric principles evoked in the proof of [DR1, Prop. 2.8] imply that

$$H_k^\flat = e_{\mathrm{ord}} H_k,$$

and hence $H_k^\flat$ can be envisaged as a $p$-adic avatar of the modular form $H_k$ of (27). Replacing $H_k$ by $H_k^\flat$ in (28) leads to an alternate expression for $I(f_x^\circ, G)$ in terms of the period $\langle H_k^\flat, f_x\rangle\langle f_x, f_x\rangle^{-1}$.

Although more genuinely $p$-adic in nature, this quantity does not interpolate $p$-adically to an analytic or even continuous function of $x \in \mathcal{X}_{\mathbf{f}}$, because the quantity $\nu^t$ arising in the fourier coefficients of $d_1^t G_{\mathfrak{M}}$ are only analytic functions of $t$ when $p \nmid \nu$. This motivates replacing the form $d_1^t G_{\mathfrak{M}}(q_1, q_2)$ by its $p$-depletion in (29):

$$d_1^t G_{\mathfrak{M}}^{[p]}(q_1, q_2) := \sum_{\substack{\nu \in (\mathfrak{M}\mathfrak{d}^{-1})_+ \\ p \nmid \nu}} \nu_1^t G((\nu)\mathfrak{d}) q_1^{\nu_1} q_2^{\nu_2}, \tag{30}$$

a modular generating series whose fourier coefficients interpolate to $p$-adic analytic functions of $t \in (\mathbb{Z}/(p^2-1)\mathbb{Z}) \times \mathbb{Z}_p$. Let

$$H_k^{[p]} := e_{\mathrm{ord}} \mathrm{Tr}_M^{DM} \left( d_1^t G_{\mathfrak{M}}^{[p]}(q, q) \right) \in M_k(\Gamma_0(N))$$

be the ordinary projection of the trace to level $M$ of the diagonal restriction of this nearly overconvergent Hilbert modular form, and set

$$\mathscr{L}_p(\mathbf{f}, V)(x) := \frac{\langle H_k^{[p]}, f_x\rangle}{\langle f_x, f_x\rangle}, \qquad k := \mathrm{w}(x). \tag{31}$$

This quantity interpolates to a $p$-adic meromorphic function of $x$, denoted $\mathscr{L}_p(\mathbf{f}, V)$, and commonly referred to as the *twisted triple-product $p$-adic $L$-function.* Since it suffices for our purposes, here we have limited ourselves to introduce this $p$-adic $L$-function as a single-variable function on the weight variable of $\mathbf{f}$, but the reader is invited to consult the references above for a three-variable version of it.

As anticipated before, $\mathscr{L}_p(\mathbf{f}, V)$ interpolates the square-root of the algebraic part of central critical classical $L$-values, which we now describe more precisely. Here we state a version of this result proved by Ishikawa in [Ish, Theorem 1.5.1] although the above references show it holds in greater generality and under more relaxed assumptions.

Recall that $\alpha_G$ and $\beta_G$ are the eigenvalues of the frobenius element at $p$ in $G_F$ acting on the Artin representation $V_0$ attached to $G$, and the eigenvalues of $\sigma_p$ in $G_{\mathbb{Q}}$ on its tensor induction $V$ are $1, -1, \alpha_G$ and $\beta_G$, by Proposition 2.2.

Given two functions $A(x)$ and $B(x)$ on a subset of classical points of $\mathcal{X}_{\mathbf{f}}$ that is dense for the rigid-analytic topology, we will write

$$A(x) \sim B(x) \tag{32}$$

if they differ by a quantity that interpolates to a $p$-adic meromorphic function on $\mathcal{X}_{\mathbf{f}}$ that is regular at $x_0$ and whose value at $x_0$ belongs to $\kappa^\times$. We call such functions *admissible.*

**Theorem 3.1** (Ishikawa). *Assume*

($H_p$) *The mod $p$ residual Galois representation $\bar{\varrho}_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_p)$ is absolutely irreducible and the restriction of $\bar{\varrho}_f$ to the decomposition group at $p$ is the sum of two* distinct *characters.*

($H_M$) *The local admissible representation associated to $f$ at a prime $q \mid M$ is not supercuspidal (which is automatically fulfilled when $q \mid\mid M$).*

*For all classical points $x \in \mathcal{X}_f$ of weight $k = 2t+2$ with $t \geq 0$ we have*

$$\mathscr{L}_p(\mathbf{f}, V)(x)^2 \sim \frac{\mathcal{E}(f_x, V)}{\mathcal{E}_0(f_x)} L_{\mathrm{alg}}(f_x^\circ, V, k/2)$$

*where*

$$\mathcal{E}(f_x, V) = \left(1 - \frac{p^t \alpha_G}{\alpha_x}\right)^2 \left(1 - \frac{p^t \beta_G}{\alpha_x}\right)^2, \qquad \mathcal{E}_0(f_x) = \left(1 - \frac{p^{k-1}}{\alpha_x^2}\right)^2 \tag{33}$$

*and*

$$L_{\mathrm{alg}}(f_x^\circ, V, k/2) = \frac{(\frac{k-2}{2}!)^4}{\pi^{2k} \langle f_x^\circ, f_x^\circ \rangle^2} L(f_x^\circ, V, k/2). \tag{34}$$

*Proof.* This follows from [Ish, Theorem 1.5.1]. Indeed, observe first that all assumptions in loc. cit. are fulfilled, since our running hypothesis that $(N, D_F\mathfrak{D}) = 1$ implies Ishikawa's set $\Sigma^-$ is empty. The formula in loc. cit. asserts that

$$\mathscr{L}_p(\mathbf{f}, V)(x)^2 = \Gamma_{f_x,g}(0) \frac{L(f_x^\circ, V, k/2)}{(-1)^{\alpha_g} D^t \, \Omega_{f_x}^2} \mathcal{E}^\dagger(f_x, g) \prod_{\ell \in \Sigma^{\mathrm{dist}}} (1 + \ell^{-1})^2 \prod_{\ell \in \Sigma^{\mathrm{excep}}} ((\ell+1)^2 - \alpha_x^2)^2$$

where

(i) $\Gamma_{f_x,g}(s)$ is the Gamma function introduced in [Ish, (1.2.1)], whose value at $s = 0$ is $(\frac{k-2}{2}!)^4 \pi^{-2k}$ up to a power of 2 (whose exponent is a polynomial in $k$), and thus interpolates to an admissible function;

(ii) $\Sigma^{\mathrm{dist}}$ and $\Sigma^{\mathrm{excep}}$ are finite sets of primes distinct from $p$ introduced in [Ish, §1.5]; note that the functions $((\ell+1)^2 - \alpha_x^2)^2$ interpolate to admissible $p$-adic analytic functions on $\mathcal{X}_{\mathbf{f}}$;

(iii) the period $\Omega_{f_x}$ is defined at the end of [Ish, §1.4] and is precisely the Petersson scalar product $\langle f_x^\circ, f_x^\circ \rangle$ quoted above, up to an Euler-like factor $\mathcal{E}_p(f_x, \mathrm{Ad})$ introduced in loc.cit, a power of 2 that again interpolates to an admissible function of $k$, and a function denoted $x(\eta_{\mathbf{f}})$ given by a choice of generator of the congruence ideal of $f_x$, which also interpolates to an admissible $p$-adic analytic function on $\mathcal{X}_{\mathbf{f}}$.

Since the ratio $\mathcal{E}^\dagger(f_x, g)/\mathcal{E}_p(f_x, \mathrm{Ad})^2$ is readily seen to be equal to the ratio $\mathcal{E}(f_x, V)/\mathcal{E}_0(f_x)$, it follows that Ishikawa's formula quoted above is equal to the one recorded in the main statement of the theorem after removing quantities that visibly interpolate to an admissible $p$-adic meromorphic function on $\mathcal{X}_{\mathbf{f}}$. □

*Remark* 3.2. It is instructive to compare the "$p$-adic multiplier" $\mathcal{E}(f_x, V)\mathcal{E}_0(f_x)^{-1}$ of Theorem 3.1 with the ostensibly more complicated expression in [DR1, Theorem 4.7], where $G$ is replaced by a pair $(g, h)$ of elliptic modular forms of weight one, and $V$ is replaced by the tensor product $V_{gh}$ of the odd two-dimensional Artin representations attached to $g$ and $h$. In that setting, the factor that enters into the $p$-adic interpolation is given by

$$\mathcal{E}_+(f_x, V_{gh}) \cdot \mathcal{E}_0(f_x)^{-1} \cdot \mathcal{E}_1(f_x)^{-1}, \tag{35}$$

where

$$\mathcal{E}_+(f_x, V_{gh}) = \prod_{i=1}^{4} \left(1 - \frac{p^t \gamma_i}{\alpha_x}\right), \tag{36}$$

with $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ the eigenvalues of $\sigma_p$ acting on $V_{gh}$, and

$$\mathcal{E}_0(f_x) = \left(1 - \frac{p^{k-1}}{\alpha_x^2}\right), \qquad \mathcal{E}_1(f_x) = \left(1 - \frac{p^{k-2}}{\alpha_x^2}\right).$$

Replacing $V_{gh}$ by $V$ in (36) and noting that 1 and $-1$ occur among the $\sigma_p$-eigenvalues on $V$, we find

$$\frac{\mathcal{E}_+(f_x, V)}{\mathcal{E}_0(f_x)\mathcal{E}_1(f_x)} = \frac{(1-\alpha_G p^t)(1-\beta_G p^t)\mathcal{E}_1(f_x)}{\mathcal{E}_0(f_x)\mathcal{E}_1(f_x)} = \frac{\mathcal{E}(f_x, V)}{\mathcal{E}_0(f_x)}, \tag{37}$$

and hence the $p$-adic multipliers in [BF] and [DR1] are consistent.

The next result can be viewed as a Gross-Zagier formula for $\mathscr{L}_p(\mathbf{f}, V)$ since it expresses its first derivative at $x = x_0$ in terms of the $p$-adic logarithm of the point $P_{V,f}$ of Conjecture 1.2. The weight map w is étale at $x_0$ and the standard local parameter at $k = 2$ in $\mathcal{X}$ induces a local parameter on $\mathcal{X}_f$ at $x_0$, with respect to which the derivatives evoked in the following theorem are to be taken.

**Theorem 3.3.** *We have*

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}, V)_{|x=x_0} = C\lambda_{V,f} \tag{38}$$

*for some constant* $C \in \kappa^\times$.

*Proof.* The generating series $G'_{\mathfrak{M}}(q_1, q_2)$ of (13) can be written as

$$G'_{\mathfrak{M}} = \frac{d}{dt}\left(d_1^t G_{\mathfrak{M}}^{[p]}\right)_{t=0}.$$

The reader is cautioned that the letter $d$ appears in this equation in two different guises, whose distinct meanings should nonetheless be apparent from the context. It follows that the ordinary modular form $\Phi_V$ of (15) is given by

$$\Phi_V = \frac{d}{dk}(H_k^{[p]})_{k=2}. \tag{39}$$

The modular forms $H_k^{[p]}$ are the classical specialisations of a $\Lambda$-adic family of ordinary modular forms. By (31), the component of this $\Lambda$-adic form along the eigenform $\mathbf{f}$ is equal to $\mathscr{L}_p(\mathbf{f}, V) \cdot \mathbf{f}$. Since $\mathscr{L}_p(\mathbf{f}, V)(x_0) = 0$, projecting the equality (39) to the $f$-isotypic component we obtain

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}, V)_{x=x_0} = \lambda_{V,f}$$

up to a constant in $\kappa^\times$, as claimed. □

In view of (20) we may thus recast Conjecture 1.1 in the following equivalent way.

**Conjecture 3.4.** *The derivative of the twisted triple-product $p$-adic $L$-function at $x = x_0$ is given by*

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}, V)_{|x=x_0} = \log_\omega(P_{V,f}) \tag{40}$$

*up to a multiplicative constant in $\kappa$.*

## 4. Hilbert theta series

Let $K/F$ be a quadratic extension of the real quadratic field $F$, and let

$$\psi : G_K \longrightarrow \kappa^\times$$

be a finite order character of $K$. The case where $G = \theta_\psi$ is the Hilbert theta series over $F$ attached to $\psi$ provides a particularly enticing setting for Conjecture 1.2, since the associated Asai representation then factors through a finite abelian extension of (the Galois closure of) $K$. The possibility of varying $\psi$ suggests the construction of a systematic collection of global points on elliptic curves defined over such abelian extensions.

To remain within the scope of the main conjecture in the introduction, it will be assumed throughout that

1. $\psi$ is of mixed signature at any pair of real places of $K$ that lie over a common real place of $F$. This ensures that the Artin representation $V_0$ is odd at both real places of $F$, and that $G = \theta_\psi$ is a holomorphic Hilbert modular form of parallel weight one.
2. The induced representation $V_0$ of $G_F$ satisfies the assumption in equation (7) of the introduction, implying that $V$ is self-dual.

We begin by spelling out what this second condition on the self-duality of $V$ implies about $\psi$. Let $\chi_{K/F}$ be the quadratic character of $G_F$ attached to $K$.

**Lemma 4.1.** *Let $V_0$ be the induced representation of $G_F$ attached to $\psi$, and let $V$ denote its tensor induction to $\mathbb{Q}$. Then $V$ is self-dual if and only if*

$$\mathrm{Ver}_F^{\mathbb{Q}}(\chi_{K/F}) \cdot \mathrm{Ver}_K^{\mathbb{Q}}(\psi) = 1.$$

*Proof.* Since

$$\det(V_0) = \chi_{K/F} \cdot \mathrm{Ver}_K^F(\psi),$$

it follows from the transitivity of the transfer map that

$$\mathrm{Ver}_F^{\mathbb{Q}} \det(V_0) = \mathrm{Ver}_F^{\mathbb{Q}}(\chi_{K/F}) \cdot \mathrm{Ver}_K^{\mathbb{Q}}(\psi).$$

Lemma 4.1 now follows from (7). ☐

The quadratic character $\mathrm{Ver}_F^{\mathbb{Q}}(\chi_{K/F})$ of $G_{\mathbb{Q}}$ that arises in Lemma 4.1 can be described explicitly, according to the following analysis in three cases.

**Case 1**. If $K/\mathbb{Q}$ is a biquadratic field, the transfer map from $\mathrm{Gal}(K/\mathbb{Q})$ to $\mathrm{Gal}(K/F)$ is the trivial homomorphism, and hence

$$\mathrm{Ver}_F^{\mathbb{Q}}(\chi_{K/F}) = 1. \tag{41}$$

**Case 2**. If $K/\mathbb{Q}$ is a cyclic quartic extension, the transfer map from $\mathrm{Gal}(K/\mathbb{Q})$ to $\mathrm{Gal}(K/F)$ has kernel equal to $\mathrm{Gal}(K/F)$ and induces an isomorphism from $\mathrm{Gal}(F/\mathbb{Q})$ to $\mathrm{Gal}(K/F)$ by passing to the quotient. Therefore

$$\mathrm{Ver}_F^{\mathbb{Q}}(\chi_{K/F}) = \chi_{F/\mathbb{Q}}. \tag{42}$$

**Case 3**. If $K/\mathbb{Q}$ is a non-Galois quartic extension, then the normal closure $\tilde{K}$ of $K$ has Galois group isomorphic to the dihedral group $D_8$ of order 8. It is a biquadratic extension of $F$, and hence $F = \tilde{K}^{\Pi}$ where $\Pi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a Klein four-group in $D_8$. There are exactly two other subgroups of order 4 in $D_8$: the cyclic group of order 4 and a second Klein four-group, denoted $\Pi_\star$. Let $F_\star = \tilde{K}^{\Pi_\star}$ be the quadratic extension of $\mathbb{Q}$ associated to $\Pi^\star$ under the Galois correspondence.

A direct group-theoretic calculation reveals that the transfer map $\phi_\Pi : D_8 \longrightarrow \Pi$ has kernel $\Pi_\star$ and identifies $D_8/\Pi_\star$ with the center $\{\pm 1\}$ of $D_8$ (viewed as a subgroup of $\Pi$), i.e.,

$$\phi_\Pi(g) = \begin{cases} 1 & \text{if } g \in \Pi_\star, \\ -1 & \text{if } g \notin \Pi_\star. \end{cases}$$

It follows that

$$\mathrm{Ver}_F^{\mathbb{Q}}(\chi_{K/F}) = \chi_{F_\star/\mathbb{Q}}. \tag{43}$$

**Corollary 4.2.** *The Asai representation arising from the two-dimensional representation* $V_0 = \mathrm{Ind}_K^F \psi$ *is self-dual if and only if*

$$\begin{cases} \mathrm{Ver}_K^{\mathbb{Q}}(\psi) = 1, & \textit{when } K/\mathbb{Q} \textit{ is bi-quadratic}, \\ \mathrm{Ver}_K^{\mathbb{Q}}(\psi) = \chi_{F/\mathbb{Q}}, & \textit{when } K/\mathbb{Q} \textit{ is cyclic}, \\ \mathrm{Ver}_K^{\mathbb{Q}}(\psi) = \chi_{F_\star/\mathbb{Q}}, & \textit{when } K/\mathbb{Q} \textit{ is not normal.} \end{cases}$$

*Proof.* This follows by combining (41), (42), and (43), with Lemma 4.1. □

4.1. **Biquadratic extensions.** We now focus further on the scenario where $K$ is a quadratic extension of $F$ that is biquadratic over $\mathbb{Q}$. It then contains two further quadratic extensions of $\mathbb{Q}$, denoted $K_1$ and $K_2$. Because $F$ is a real quadratic field, the fields $K_1$ and $K_2$ are either both real, or both imaginary, depending on whether $K$ is a totally real or CM extension of $F$.

Let

$$\psi_1 = \mathrm{Ver}_K^{K_1}(\psi), \qquad \psi_2 = \mathrm{Ver}_K^{K_2}(\psi)$$

denote the transfers of $\psi$ to $G_{K_1}$ and $G_{K_2}$ respectively.

**Lemma 4.3.** *The characters $\psi_1$ and $\psi_2$ are ring class characters of their respective quadratic fields. If $K_1$ and $K_2$ are real, then $\psi_1$ and $\psi_2$ have opposite parity, i.e., $\psi_2$ is totally odd when $\psi_1$ is totally even, and vice-versa.*

*Proof.* Corollary 4.2 implies that $\mathrm{Ver}^{\mathbb{Q}}_{K_1}(\psi_1) = \mathrm{Ver}^{\mathbb{Q}}_{K_2}(\psi_2) = 1$, and this implies the first assertion. The second assertion follows from the assumption that $\psi$ has mixed signature at any pair of real places of $K$ that lie above a common real place of $F$. □

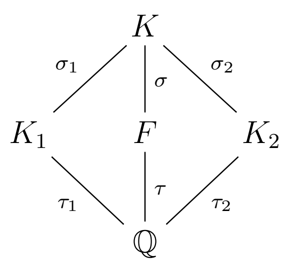
Recall that $V_0 := \mathrm{Ind}^F_K(\psi)$ is the representation of $G_F$ induced from $\psi$, and that $V = \mathrm{Ind}^{\otimes}(V_0)$ is the associated Asai representation of $G_{\mathbb{Q}}$.

**Proposition 4.4.** *The Artin representation $V$ decomposes as*

$$V = V_1 \oplus V_2, \qquad \textit{where} \quad V_1 := \mathrm{Ind}^{\mathbb{Q}}_{K_1}(\psi_1), \quad V_2 := \mathrm{Ind}^{\mathbb{Q}}_{K_2}(\psi_2).$$

*Proof.* Let $\sigma$, $\sigma_1$, $\sigma_2$, $\tau$, $\tau_1$ and $\tau_2$ denote the non-trivial elements of $\mathrm{Gal}(K/F)$, $\mathrm{Gal}(K/K_1)$, $\mathrm{Gal}(K/K_2)$, $\mathrm{Gal}(F/\mathbb{Q})$, $\mathrm{Gal}(K_1/\mathbb{Q})$, and $\mathrm{Gal}(K_2/\mathbb{Q})$ respectively, as summarized in the following field diagram:

(44)




The character of the induced representation $V_0$ is given by the formula

$$\chi_{V_0}(g) = \begin{cases} \psi(g) + \psi^{\sigma}(g) & \text{if } g \in G_K; \\ 0 & \text{if } g \in G_F - G_K, \end{cases}$$

and the character of $V$ is given by

$$\chi_V(g) = \begin{cases} \chi_{V_0}(g)\chi_{V_0}(g^{\tau}) & \text{if } g \in G_F; \\ \chi_{V_0}(g^2) & \text{if } g \in G_{\mathbb{Q}} - G_F. \end{cases}$$

These two formulae can be used to show that

$$\chi_V(g) = \chi_{V_1}(g) + \chi_{V_2}(g), \tag{45}$$

according to the following division into three cases:

**Case 1**: $g \in G_K$. Then

$$\begin{aligned} \chi_V(g) &= \chi_{V_0}(g)\chi_{V_0}(g^{\tau}) \\ &= (\psi(g) + \psi^{\sigma}(g))(\psi^{\sigma_1}(g) + \psi^{\sigma_2}(g)) \\ &= (\psi\psi^{\sigma_1} + \psi^{\sigma}\psi^{\sigma_2})(g) + (\psi\psi^{\sigma_2} + \psi^{\sigma}\psi^{\sigma_1})(g) \\ &= (\psi_1 + \psi_1^{\tau_1})(g) + (\psi_2 + \psi_2^{\tau_2})(g) \\ &= \chi_{V_1}(g) + \chi_{V_2}(g). \end{aligned}$$

**Case 2**: $g \in G_F - G_K = (G_{\mathbb{Q}} - G_{K_1}) \cap (G_{\mathbb{Q}} - G_{K_2})$. Then

$$\chi_V(g) = \chi_{V_0}(g)\chi_{V_0}(g^{\tau}) = 0.$$

Since $g$ belongs to neither $G_{K_1}$ or $G_{K_2}$, we also have

$$\chi_{V_1}(g) = \chi_{V_2}(g) = 0,$$

and (45) follows.

**Case 3**. $g \in G_{\mathbb{Q}} - G_F = (G_{K_1} - G_{K_2}) \sqcup (G_{K_2} - G_{K_1})$. Then $g^2$ belongs to $G_K$ and hence

$$\begin{aligned} \chi_V(g) &= \chi_{V_0}(g^2) = \psi(g^2) + \psi^\sigma(g^2) \\ &= \begin{cases} (\psi\psi^{\sigma_1} + \psi^\sigma\psi^{\sigma_2})(g) = (\psi_1 + \psi_1^{\tau_1})(g) & \text{if } g \in G_{K_1} - G_{K_2} \\ (\psi\psi^{\sigma_2} + \psi^\sigma\psi^{\sigma_1})(g) = (\psi_2 + \psi_2^{\tau_2})(g) & \text{if } g \in G_{K_2} - G_{K_1} \end{cases} \\ &= \begin{cases} \chi_{V_1}(g) + 0 & \text{if } g \in G_{K_1} - G_{K_2} \\ 0 + \chi_{V_2}(g) & \text{if } g \in G_{K_2} - G_{K_1} \end{cases} \\ &= \chi_{V_1}(g) + \chi_{V_2}(g). \end{aligned}$$

The proposition follows. □

As in Section 1, let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N = pM$ where $p$ is a prime that remains inert in $F$ and $M \geq 1$ is a positive number all whose prime divisors split in $F$. Letting $D_{\psi_1}$, $D_{\psi_2}$ denote the conductors of the central characters of $\psi_1$, $\psi_2$, we further assume that $(N, D_F D_{\psi_1} D_{\psi_2}) = 1$. Finally, let us also assume throughout this section that hypotheses $(H_p)$ and $(H_M)$ in Theorem 3.1 are in place for the eigenform $f$ associated to $E$.

When combined with the Artin formalism for Hasse-Weil-Artin $L$-series, Proposition 4.4 implies the factorisation

$$\begin{aligned} L(f_x^\circ, V, k/2) &= L(f_x^\circ, V_1, k/2) \cdot L(f_x^\circ, V_2, k/2) \\ &= L(f_x^\circ/K_1, \psi_1, k/2) \cdot L(f_x^\circ/K_2, \psi_2, k/2) \end{aligned} \tag{46}$$

for all classical $x \in \mathcal{X}_{\mathbf{f}}$, and suggests that a similar principle might apply to $p$-adic $L$-series interpolating these special values.

In order to state this more precisely, let $\Omega_x^+$ and $\Omega_x^-$ be the real and imaginary periods attached to $f_x^\circ$, as introduced e.g. in [Hi2, p. 488]. As explained in loc. cit., [Ish, §1.4] and [BD2, §1.1], these periods can be chosen in such a way that

$$\Omega_x^+ \Omega_x^- = \langle f_x^\circ, f_x^\circ \rangle. \tag{47}$$

Let $D_1$ and $D_2$ denote the discriminants of $K_1$ and $K_2$ respectively. Following [BD2, (93)] and [BD3, Theorem 3.5], the algebraic parts of the special values attached to $f_x^\circ$ and the ring class characters $\psi_i$ of $K_i$ $(i = 1, 2)$ are obtained by setting

$$L^{\mathrm{alg}}(f_x^\circ/K_i, \psi_i, k/2) = \frac{(\frac{k-2}{2}!)^2 D_i^{\frac{k-1}{2}}}{(2\pi)^k\, \Omega_{x,\psi_i}} L(f_x^\circ/K_i, \psi_i, k/2), \tag{48}$$

where the period $\Omega_{x,\psi_i}$ is given as in [BD3, Definition 3.4] by

$$\Omega_{x,\psi_i} = \begin{cases} \Omega_x^+\Omega_x^- & \text{if } D_i < 0; \\ (\Omega_x^+)^2 & \text{if } D_i > 0 \text{ and } \psi_i \text{ is even;} \\ (\Omega_x^-)^2 & \text{if } D_i > 0 \text{ and } \psi_i \text{ is odd.} \end{cases}$$

Note that in our statement of (48) we have formulated a different power of $\pi$ with respect to [BD2] and [BD3], which is due to the different normalizations in the definition of the Petersson scalar product adopted in loc. cit. versus [Ish].

Comparing these definitions with the one in Theorem 3.1, we see that for all classical $x \in \mathcal{X}_{\mathbf{f}}$ of weight $k > 2$:

$$L_{\mathrm{alg}}(f_x^\circ, V, k/2) \sim L_{\mathrm{alg}}(f_x^\circ/K_1, \psi_1, k/2) \cdot L_{\mathrm{alg}}(f_x^\circ/K_2, \psi_2, k/2). \tag{49}$$

The generalised Heegner hypothesis (9) for $V$ implies that all the primes $\ell | M$ have the same splitting behaviours in $K_1$ and $K_2$, since they are split in $F/\mathbb{Q}$. Let

$$\Sigma := \{\ell | M\infty \text{ such that } \varepsilon_{K_i}(\ell) = -1\}$$

be the set of places of $\mathbb{Q}$ dividing $M\infty$ at which the quadratic field $K_1$ (and hence also $K_2$) is inert.

The sign in the functional equations for $L(f_x^\circ, V_1, s)$ and $L(f_x^\circ, V_2, s)$ are controlled by the parity of the cardinality of $\Sigma$:

$$\mathrm{sign}(f_x^\circ, V_1) = \mathrm{sign}(f_x^\circ, V_2) = (-1)^{\#\Sigma}.$$

When $\#\Sigma$ is odd, the central critical values $L(f_x^\circ/K_i, \psi_i, k/2)$ vanish identically, and one has no resort but to set

$$\mathscr{L}_p(\mathbf{f}, V_i) = 0. \tag{50}$$

When $\#\Sigma$ is even, the special values $L(f_x^\circ, V_i, k/2)$ can be expressed in terms of elementary quantities attached to optimal embeddings of orders in $K_i$ (of suitable conductor, equal to the conductor of the character $\psi_i$) in the quaternion algebra $B$ ramified at $\Sigma$. This explicit expression forms the basis for the construction of $p$-adic $L$-functions $\mathscr{L}_p(\mathbf{f}, V_i)$ ($i = 1, 2$) on $\mathcal{X}_f$.

This $p$-adic $L$-function is constructed in [BD2] and [BD3] when $K_i$ is imaginary and real respectively, where they are also denoted $\mathscr{L}_p(\mathbf{f}/K_i, \psi_i)$. They satisfy the interpolation property

$$\mathscr{L}_p(\mathbf{f}, V_i)(x)^2 \sim \frac{\mathcal{E}_+(f_x^\circ, V_i)^2}{\mathcal{E}_0(f_x^\circ)\mathcal{E}_1(f_x^\circ)} L_{\mathrm{alg}}(f_x^\circ, V_i, k/2), \qquad i = 1, 2, \tag{51}$$

where $\mathcal{E}_+(f_x^\circ, V_i)$ is the $p$-adic multiplier attached to $f_x^\circ$ and $V_i$, as defined in (36) after replacing $V_{gh}$ with $V_i$. Note that

$$\mathcal{E}_+(f_x, V_1 \oplus V_2) = \mathcal{E}_+(f_x, V_1) \cdot \mathcal{E}_+(f_x, V_2). \tag{52}$$

For imaginary quadratic fields, the construction is detailed in [BD2, §3.2], and generalised in [Mok, §3]. For real quadratic fields, see [BD3, §3.2] and its further refinements given in [LV, §4.4, 4.5]. We also refer to the work of Hernández and Molina [HM] for a recent construction of these $p$-adic $L$-functions that applies to the general setting considered here in all cases.

**Theorem 4.5.** *There is a factorization of $p$-adic $L$-functions*

$$\mathscr{L}_p(\mathbf{f}, V) \sim \mathscr{L}_p(\mathbf{f}/K_1, \psi_1) \cdot \mathscr{L}_p(\mathbf{f}/K_2, \psi_2),$$

*where $\sim$ is defined in* (32).

*Proof.* Note first that Theorem 3.1 is in force thanks to our running hypotheses. If $\#\Sigma$ is odd, the right-hand side is identically 0, and so is the left-hand side in light of Theorem 3.1 and (46). When $\#\Sigma$ is even, it follows from Theorem 3.1 and equations (37), (49), (51) and (52) that both sides of the factorization claimed in the statement are rigid-analytic functions taking the same values in a Zariski dense subset of $\mathcal{X}_f$, up to an admissible function. This yields the theorem. □

Since the prime $p||N$ is inert in $F/\mathbb{Q}$ and $\mathrm{Gal}\,(K/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it follows that $p\mathcal{O}_F$ necessarily splits in $K/F$. In particular, $p$ has different splitting behaviours in the quadratic fields $K_1$ and $K_2$. From now on in this section, these two fields will be ordered in such a way that $p$ remains inert in $K_1$ and splits in $K_2$. (Note that this convention is not followed in Section 5.) This implies that $\sigma_p$ acts on $V_1$ with eigenvalues $\pm 1$, and on $V_2$ with eigenvalues $\alpha_G$ and $\beta_G = \alpha_G^{-1}$, and that

$$\mathrm{sign}(E, V_1) = -\mathrm{sign}(f_x^\circ, V_1), \qquad \mathrm{sign}(E, V_2) = \mathrm{sign}(f_x^\circ, V_2) \tag{53}$$

for all classical $x \in \mathcal{X}_{\mathbf{f}}$ of weight $k > 2$.

The following theorem is the main result of this section:

**Theorem 4.6.** *Assume that either*

(1) *$K_1$ and $K_2$ are imaginary quadratic fields, or*
(2) *$K_1$ and $K_2$ are real quadratic fields, and the conjectures of* [Dar, §5] *on Stark-Heegner points hold for $(K_1, \psi_1)$.*

*Then Conjecture 1.2 is true.*

*Proof.* By Theorem 3.3, the scalar $\lambda_{V,f}$ is equal to the first derivative of $\mathscr{L}_p(\mathbf{f}, V)$ at $x = x_0$ up to a multiplicative constant in $\kappa^\times$.

Assume first $\#\Sigma$ is odd. As already argued above, in this case $\mathscr{L}_p(\mathbf{f}, V)$ vanishes identically on $\mathcal{X}_{\mathbf{f}}$ and hence $\lambda_{V,f} = 0$. This is aligned with Conjecture 1.2, as conditions (I) and (II) of that conjecture are not both fulfilled. Indeed, if condition (I) were true, namely $\mathrm{ord}_{s=1} L(E, V, s) = 1$, it would follow from (53) that

$$\mathrm{sign}(E, V_1) = 1, \qquad \mathrm{sign}(E, V_2) = -1,$$

and therefore

$$L(E, V_1, s) \neq 0, \qquad \mathrm{ord}_{s=1} L(E, V_2, s) = 1,$$

so that $W_0 \supseteq V_1$ and $W_1 \subseteq V_2$ in the notation of Conjecture 1.2. But $\sigma_p$ acts with eigenvalues 1 and $-1$ on $V_1$, and therefore condition (II) would not be met.

Assume now that $\#\Sigma$ is even. By Theorem 4.5, $\mathscr{L}_p(\mathbf{f}, V)$ has a simple zero at $x = x_0$ if and only if

$$\mathrm{ord}_{x=x_0} \mathscr{L}_p(\mathbf{f}, V_1) + \mathrm{ord}_{x=x_0} \mathscr{L}_p(\mathbf{f}, V_2) = 1. \tag{54}$$

Since

$$\mathrm{sign}(E, V_1) = -1, \qquad \mathrm{sign}(E, V_2) = 1,$$

equation (51) implies the $p$-adic $L$-function $\mathscr{L}_p(\mathbf{f}, V_1)$ vanishes at $x_0$ and hence (54) is equivalent to

$$\mathrm{ord}_{x=x_0} \mathscr{L}_p(\mathbf{f}, V_1) = 1, \qquad \mathscr{L}_p(\mathbf{f}, V_2)(x_0) \neq 0. \tag{55}$$

Note that $\mathcal{E}_+(f_x^\circ, V_2) = (1 - \frac{\alpha_G}{\alpha_x})(1 - \frac{\beta_G}{\alpha_x})$. Hence it follows again from (51) that $\mathscr{L}_p(\mathbf{f}, V_2)(x_0) \neq 0$ if and only if

$$L(E/K, \psi_2, 1) \neq 0 \quad \text{and } a_p \neq \alpha_G, \beta_G. \tag{56}$$

Besides, by [BD2, §4.4], [Mok, §3], [BD3, §4], [LV, Theorem 4.31], [HM, Theorem 8.2], we have

$$\frac{d}{dx}\mathscr{L}_p(\mathbf{f}, V_1)_{x=x_0} = C_1 \log_{\mathrm{Tate}}(x_{E,V_1} + a_p x_{E,V_1}^{\sigma_p}), \quad C_1 \in \kappa^\times \tag{57}$$

where $x_{E,V_1}$ is a local point in $E(\mathbb{Q}_{p^2})$, with coordinates in the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$. Note $x_{E,V_1} + a_p x_{E,V_1}^{\sigma_p}$ lies in the $a_p$-eigenspace for the action of $\sigma_p$.

Moreover, $x_{E,V_1}$ is

- the image of a global point $Q_{E,V_1} \in E(H)^{V_1}$ under the natural embedding $E(H) \hookrightarrow E(\mathbb{Q}_{p^2})$, given by a Heegner point on a Shimura curve attached to the odd set $\Sigma \cup \{p\}$ of places, when $K_1$ is an imaginary quadratic field;
- a Stark-Heegner point attached to the $p$-adic uniformisation of $E$ via rigid analytic cocycles, which is predicted to arise from a global point $Q_{E,V_1} \in E(H)^{V_1}$ similarly as above, by the conjectures of [Dar, §5].

The Gross-Zagier formula in the scenario where $K$ is imaginary ([GZ]) and its conjectural extension to real quadratic fields (cf. [Dar, §5]) imply that $L(f, V_1, s)$ has a simple zero at $s = 1$ if and only if $Q_{E,V_1}$ is not trivial in $E(H) \otimes \kappa$.

If $V_1$ is irreducible (which is precisely the case when $\psi_1^2 \neq 1$), then the image of $E(H)^{V_1}$ in $E(\mathbb{Q}_{p^2}) \otimes \kappa$ is a two-dimensional $\kappa$-vector space spanned by $x_{E,V_1}$ and $x_{E,V_1}^{\sigma_p}$, and hence $Q_{E,V_1}$ is non-trivial if and only if $P_{E,V_1} := Q_{E,V_1} + a_p Q_{E,V_1}^{\sigma_p}$ is non-trivial, and this in turn is equivalent to the non-vanishing of (57). Note that the irreducible component $W_1$ of $V$ introduced in Conjecture 1.2 is $V_1$ in this case.

If $V_1$ decomposes as the sum of a pair of 1-dimensional sub-representations $V_1 = W_1 \oplus W_1'$, then we may order them accordingly to the notations in Conjecture 1.2, so that $\mathrm{ord}_{s=1} L(E, W_1, s) = 1$ and $L(E, W_1', 1) \neq 0$. Let $\epsilon = \pm 1$ denote the eigenvalue of $\sigma_p$ acting on $W_1$. In the favorable case where $\epsilon = a_p$, then $P_{E,V_1} := Q_{E,V_1} + a_p Q_{E,V_1}^{\sigma_p}$ is a generator of $E(H)^{W_1}$, and its image in $E(\mathbb{Q}_{p^2})$ is the local point appearing in the right-hand side of (57). In particular (57) again does not vanish. In the unfortunate case that $\epsilon = -a_p$, the global point $Q_{E,V_1} + a_p Q_{E,V_1}^{\sigma_p}$ lies in $E(H)^{W_1'} = 0$ and hence (57) vanishes.

Summing up, we conclude that $\lambda_{V,f} \neq 0$ if and only if

(i) $L(E/K, \psi_2, 1) \neq 0$ and $\mathrm{ord}_{s=1} L(E/K, \psi_1, s) = 1$,
(ii) $a_p \neq \alpha_G, \beta_G$,
(iii) the eigenvector of eigenvalue $a_p$ lies in $W_1 \subseteq V_1$.

Since $L(E, V, s) = L(E/K, \psi_1, s) L(E/K, \psi_2, s)$, we have proved (subject to the conjectures in [Dar, §5] in the case of real quadratic fields) that $\lambda_{V,f} \neq 0$ if and only if conditions (I) in (II) in Conjecture 1.2 hold, and in that case $\lambda_{V,f} = \log_{\mathrm{Tate}}(P_{V,f})$ for a point $P_{V,f}$ that forms a basis of the one-dimensional subspace of the image of $E(H)^{W_1}$ on which $\sigma_p$ acts with eigenvalue $a_p$. □

## 5. Numerical experiments

In this section we present some numerical experiments which both illustrate various aspects of Theorem 4.6 and give further evidence in support of the broader Conjecture 1.2. We begin by considering examples illuminating Theorem 4.6 in the case where $K$ is a biquadratic field. These examples were computed prior to the formulation of Conjecture 1.2 and the proof of Theorem 4.6, and were key to suggesting their finer detail. They also suggest an analogous but subtly different conjecture when the nearly overconvergent family interpolating $d^t G_{\mathfrak{M}}$ is replaced by a Hida family of parallel weight specialising to $G$ in weight $(1, 1)$.

### 5.1. **Biquadratic extensions: the CM setting.**

5.1.1. *The basic set-up.* Let $D_1$ and $D_2$ be negative coprime fundamental discriminants and let $D_F := D_1 \cdot D_2 > 0$. Assume that these discriminants are odd and that $D_1, D_2 < -3$. Define as in (44)

$$F := \mathbb{Q}(\sqrt{D_F}), \quad K_1 := \mathbb{Q}(\sqrt{D_1}), \quad K_2 := \mathbb{Q}(\sqrt{D_2}), \quad \text{and} \quad K := \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}).$$

The biquadratic field $K$ is an odd *genus field* of the real quadratic field $F$: an unramified CM quadratic extension of $F$ that is also abelian over $\mathbb{Q}$, with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

We consider the simplest non-trivial setting, in which the quadratic fields $K_1$, $K_2$ and $F$ have class numbers 3, 1 and 1 respectively. The narrow class number of $F$ is then equal to 2, and $K$ is the narrow Hilbert class field of $F$. The extension $K/F$ has (relative) class number 3.

The Hilbert class field $H$ of the quartic field $K$ is a cyclic cubic extension of $K$, and a degree 12 Galois extension of $\mathbb{Q}$ which admits six irreducible Artin representations: the four one-dimensional representations $\mathbf{1}$, $\varepsilon_{D_1}$, $\varepsilon_{D_2}$, and $\varepsilon_{D_F}$, attached to the quadratic Dirichlet characters of conductors 1, $D_1$, $D_2$ and $D_F$ respectively, and two irreducible two-dimensional representations

$$W = \mathrm{Ind}_{K_1}^{\mathbb{Q}} \chi, \qquad W \otimes \varepsilon_{D_2},$$

where $\chi$ is any of the two cubic unramified characters of $K_1$.

Let

$$\psi : \mathrm{Cl}_{K/F} \to L^\times, \qquad L := \mathbb{Q}(\zeta_3)$$

be a non-trivial cubic character of the relative class group of $K/F$. The induced representation $V_0 := \mathrm{Ind}_K^F(\psi)$ is a two-dimensional representation of $G_F$, whose tensor induction to $\mathbb{Q}$ is given by

$$V := \mathrm{Ind}^{\otimes}(V_0) = \mathbf{1} \oplus \varepsilon_{D_2} \oplus W.$$

The Artin representation $V_0$ induced by $\psi$ corresponds to a Hilbert theta series $G = G_\psi$ which is described as a function on the ideals of $F$ by the formula

$$G(I) = \sum_{\substack{J \in I(K), \\ \mathrm{Norm}(J) = I}} \psi(J).$$

Recall that the construction of a weight $(1,1)$ modular generating series $\theta_\psi$ attached to $G$ also depends on the choice of an ideal $\mathfrak{M} := (\eta)$ in $O_F$. In the notation of the introduction, the theta series $\theta_\psi = G_{\mathfrak{M}}(\tau_1, \tau_2)$ is given by the formula in (11). Moreover, the $p$-depletion of $\theta_\psi$ is the weight $(1,1)$ specialisation of a natural $p$-adic *theta family*, given by $d_1^t G_{\mathfrak{M}}^{[p]}$ in (30). (Beware the two different uses of the term "theta" here.) If $\mathfrak{M} = (\nu)$ belongs to the trivial narrow ideal class of $F$, i.e., if $\nu$ is totally positive, then the class of $(\nu)\mathfrak{d}$ in the class group $\mathrm{Cl}_F$ is not a norm from $\mathrm{Cl}_K$ and hence the sum defining $G((\nu)\mathfrak{d})$ is empty, so that the $p$-adic theta family $d^t G_{\mathfrak{M}}^{[p]}$ vanishes identically. So it is necessary to choose an ideal $(\eta)$ whose associated narrow ideal class is non-trivial, i.e., for which $\mathrm{Norm}(\eta) < 0$. Note there is no element in $\mathcal{O}_F$ of norm $-1$, and hence that the integer $|\mathrm{Norm}(\eta)|$ is strictly greater than 1. Since $K/F$ is unramified, the diagonal restriction of the theta series $\theta_\psi$ lies in $S_2(|\mathrm{Norm}(\eta)|)$. In the notation of the introduction, we have

$$M = |\mathrm{Norm}(\eta)|, \qquad D = 1.$$

Note that all the primes dividing $M$ must necessarily split in $F$.

Let $p$ be a prime which is inert in $F$. We consider elliptic curves $E$ of square-free conductor $pM$ with associated eigenforms $f_E \in S_2(pM)$. Since $pM$ satisfies condition (9) of the introduction, the Asai representation $V$ is expected to occur with odd multiplicity in the Mordell-Weil group of $E$.

Let us assume now that this multiplicity is one. This implies that the rank patterns

$$(\ \mathrm{rank}(E(\mathbb{Q})),\ \ \mathrm{rank}(E^{D_2}(\mathbb{Q})),\ \ \dim \hom_{G_\mathbb{Q}}(W, E(H)_L)\ )$$

are equal to either $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$. The parities of ranks on each component which occur are correlated to the eigenvalues of $\sigma_p$ on the Asai representation, and in the rank one setting the following three scenarios are the only ones that can occur:

| Frobenius eigenvalues | Asai ranks |
|---|---|
| $(1,-1,(\zeta_3,\zeta_3^{-1}))$ | $(1,0,0)$ or $(0,1,0)$ |
| $(1,-1,(1,1))$ | $(1,0,0)$ or $(0,1,0)$ |
| $(1,1,(1,-1))$ | $(0,0,1)$ |

Write

$$\lambda_\theta = \lambda_\theta(\psi, (\eta), p, f_E)$$

for the coefficient along $f_E$ of the diagonal restriction of the derivative ($G'_{\mathfrak{M}}(\tau_1,\tau_2)$ in (13)) of the $p$-adic theta family through $\theta_\psi$. Note that, in the notation of the introduction, the diagonal restriction of the derivative is given by $\Phi_M^{(p)}$ in (14). Here $D = 1$ and so it is not necessary to lower the level of $\Phi_M$ by taking its trace from level $DM$ to level $M$.

Conjecture 1.2 makes the predictions shown in Table 1, which are proved in Theorem 4.6. In this table, the global points $P^+$, $P^-$, and $P_W^-$ are generators of $E(\mathbb{Q}) \otimes \mathbb{Q}$, of $E(K_2) \otimes \mathbb{Q}$, and of the image of a generator of $E(H)_L^W$ on which $\sigma_p$ acts as $-1$. In the cases arising in the fifth and sixth lines of the table, $E$ has rank 1 over any cubic subfield of the Hilbert class field of $K_1$, and after choosing an embedding of one of these fields into $\mathbb{Q}_{p^2}$, there are global points $P_W^+$ and $P_W^-$ upon which $\sigma_p$ acts by $+1$ and $-1$, respectively. In line 5, note that $\lambda_\theta = 0$ when $a_p(E) = 1$, because $+1$ does not have multiplicity one in the Asai representation itself.

| | Frobenius eigenvalues | Ranks | $a_p(E)$ | Coefficient $\lambda_\theta$ along $f_E$. |
|---|---|---|---|---|
| 1. | $(1,-1,(\zeta_3,\zeta_3^{-1}))$ | $(1,0,0)$ | $+1$ | $\log_{\mathrm{Tate}}(P^+)$ |
| 2. | $(1,-1,(\zeta_3,\zeta_3^{-1}))$ | $(1,0,0)$ | $-1$ | $0$ |
| 3. | $(1,-1,(\zeta_3,\zeta_3^{-1}))$ | $(0,1,0)$ | $+1$ | $0$ |
| 4. | $(1,-1,(\zeta_3,\zeta_3^{-1}))$ | $(0,1,0)$ | $-1$ | $\log_{\mathrm{Tate}}(P^-)$ |
| 5. | $(1,1,(1,-1))$ | $(0,0,1)$ | $+1$ | $0$ |
| 6. | $(1,1,(1,-1))$ | $(0,0,1)$ | $-1$ | $\log_{\mathrm{Tate}}(P_W^-)$ |
| 7. | $(1,-1,(1,1))$ | $(1,0,0)$ | $\pm1$ | $0$ |
| 8. | $(1,-1,(1,1))$ | $(0,1,0)$ | $+1$ | $0$ |
| 9. | $(1,-1,(1,1))$ | $(0,1,0)$ | $-1$ | $\log_{\mathrm{Tate}}(P^-)$ |

TABLE 1. Prediction for $\lambda_\theta$ in the CM biquadratic setting.

5.1.2. *Hida deformations of eigenforms.* We now take a small detour. This paper has as its focus *theta deformations* of Hilbert eigenforms. But one can also study the analogous question for Hida deformations of eigenforms. That is, the first infinitesimal deformation of an eigenform in the $p$-adic Hida family in parallel weight through that eigenform (this family is unique, except in certain exceptional cases). Studying this setting numerically and formulating a conjecture here was one of the starting points for this paper.

The Hida setting has the practical disadvantage that currently one only disposes of explicit Hida deformations in the case of CM forms. The experimental evidence in this setting suggests though that letting

$$\lambda_{\mathrm{Hida}} = \lambda_{\mathrm{Hida}}(\psi,(\eta),p,f_E)$$

denote the coefficient along $f_E$ of the diagonal restriction of the derivative of the $p$-adic Hida family in parallel weight through an ordinary stabilisation of $\theta_\psi$, the behaviour is exactly the same as for $\lambda_\theta$ *except* one only need require that $a_p(E)$ has multiplicity one in the component of the Asai representation which itself occurs with multiplicity one in the Mordell-Weil group. For theta deformations, multiplicity one is required in the full Asai representation. As a consequence, if one replaces $\lambda_\theta$ by $\lambda_{\mathrm{Hida}}$ in Table 1 there is no change, *except* in line 5 where

$$\lambda_{\mathrm{Hida}} = \log_{\mathrm{Tate}}(P_W^+),$$

and in line 7 in which for rank $(1,0,0)$ and $a_p(E)=+1$ we get $\lambda_{\mathrm{Hida}} = \log_{\mathrm{Tate}}(P^+)$.

In the even more special case of the dihedral quartic setting, there is also a *third* natural family to consider. To explain to the origin of this, note that the Hida family in parallel weight can be constructed explicitly by using grossencharacters associated to $\psi$. The infinity type in this grossencharacter arises from a power of the norm map from $K$ down to the imaginary quadratic field ($K_1$ or $K_2$) in which $p$ splits. Replacing this norm map by the one down to the imaginary quadratic field in which $p$ is inert yields another natural and entirely explicit family.

Curiously, in all examples that have been calculated, the coefficient "$\lambda_{\mathrm{inert}}$" resulting from this new family is $\log_{\mathrm{Tate}}(P^\pm)$ in the $\sigma_p$-eigenvalues $(1,-1,(\zeta_3,\zeta_3^{-1}))$ and ranks $(1,0,0)$

and $(0,1,0)$ scenarios, regardless of the splitting type of $E$ at $p$. It would be interesting to formulate a systematic description of the behaviour of the coefficient "$\lambda_{\mathrm{inert}}$" in the remaining settings.

The same notations $\lambda_\theta$ and (when computable) $\lambda_{\mathrm{Hida}}$ shall be used beyond the CM biquadratic setting, in the examples in Sections 5.2, 5.3 and 5.4.

5.1.3. *Biquadratic extensions: numerical examples in the CM setting.* We were able to numerically compute with all the class number 3 and 1 imaginary quadratic fields (satisfying our mild hypothesis) and reasonably small values of $p$. We focussed on the relatively small primes $p = 5, 7, 11$, which already gave rise to a wealth of examples. The computations illustrated both the vanishing in rank 3 settings, and the proposed interpretation of $\lambda_\theta, \lambda_{\mathrm{Hida}}$ (and sometimes $\lambda_{\mathrm{inert}}$) in rank 1 settings. Without describing all the experiments in detail, the discussion below presents some illustrative examples covering almost all the essential different cases.

More precisely, the phenomena predicted in rows 1 to 4 of Table 1 are illustrated in Examples 5.1, 5.2, 5.3 and 5.4; rows 5 and 6 in Example 5.5; and most cases in rows 7 to 9 in Example 5.6. These examples also illustrate the unproven predictions for $\lambda_{\mathrm{Hida}}$ and partial predictions for $\lambda_{\mathrm{inert}}$ which are evoked in Section 5.1.2.

We begin by considering two Asai rank $(1,0,0)$ examples in which the $\sigma_p$-eigenvalues are $(1,-1,(\zeta_3,\zeta_3^{-1}))$, one where the elliptic curve has split multiplicative reduction at $p$ and the other where it has non-split reduction.

*Example* 5.1. Let $D_1 := -31$ and $D_2 := -67$. The field $K_1 = \mathbb{Q}(\sqrt{-31})$ has class number 3, while $K_2 = \mathbb{Q}(\sqrt{-67})$ has class number 1, and $F = \mathbb{Q}(\sqrt{31 \cdot 67})$ has class number 1 but narrow class number 2. Let

$$\psi : \mathrm{Cl}_{K/F} \to L^\times, \qquad L := \mathbb{Q}(\zeta_3)$$

be a character of order 3 of the biquadratic field $K$.

Let

$$\eta := 22 + \left(\frac{1+\sqrt{31\cdot 67}}{2}\right) \in O_F, \qquad \mathrm{Norm}(\eta) = -13, \qquad M = 13.$$

Choose $p := 7$, which is inert in $F$. Note that $S_2(7) = S_2(13) = \{0\}$, and in particular there are no elliptic curves of conductor 7 or 13.

Let $E$ be the elliptic curve of conductor $91 = 7 \cdot 13$ labelled $91b$ in Cremona's tables, which is given by the equation

$$E : y^2 + y = x^3 + x^2 - 7x + 5.$$

It corresponds to a newform $f_E \in S_2(91)$. Letting $H$ denote the Hilbert class field of $K$, the multiplicities with which the Asai representation $V$ occur within the Mordell-Weil group $E(H)_L$ are $(1,0,0)$. That is to say, $E$ has rank 1 over $\mathbb{Q}$, and this rank does not increase over $\mathbb{Q}(\sqrt{-67})$ or over any of the cubic subfields of $H$ of discriminant $-31$.

Furthermore, $E$ has split multiplicative reduction at 7, so $a_7(E) = +1$. One finds

$$\lambda_\theta = (1-\zeta_3)\cdot\lambda_{\mathrm{Hida}} = \lambda_{\mathrm{inert}} = -4184884843330974 \cdot 7 \pmod{7^{20}}.$$

Let $P^+ = (3, -5)$ be a generator for $E(\mathbb{Q})$. Then one checks that

$$\lambda_\theta = \log_{\mathrm{Tate}}(P^+) \pmod{7^{20}}.$$

*Example* 5.2. Set $D_1 := -83$ and $D_2 := -67$, and take

$$\eta := 1585843 - 41969 \left( \frac{1 + \sqrt{83 \cdot 67}}{2} \right) \in O_F, \qquad \mathrm{Norm}(\eta) = -8, \qquad M = 8.$$

The prime $p := 11$ is inert in $F$, and there is an elliptic curve $E$ of conductor 88, labelled $88a$ in Cremona's tables,

$$E : y^2 = x^3 - 4x + 4.$$

This elliptic curve now has non-split multiplicative reduction at $p$, i.e., $a_p(E) = -1$. It has rank one over $\mathbb{Q}$, with Mordell-Weil group generated by $P^+ = (2, -2)$, and just as in the preceding example this rank stays the same over $\mathbb{Q}(\sqrt{-67})$ or any of the cubic subfields of $H$ of discriminant $-83$.

A numerical calculation shows that

$$\lambda_\theta = \lambda_{\mathrm{Hida}} = 0 \pmod{11^{20}}$$

exactly as predicted by the theory, and a further calculation reveals the tantalising numerical identity

$$\lambda_{\mathrm{inert}} = -5 \cdot \sqrt{-3} \cdot \log_{\mathrm{Tate}}(P^+) \pmod{11^{20}}.$$

The next two examples examine settings where the Mordell-Weil ranks are $(0, 1, 0)$, the $\sigma_p$-eigenvalues are $(1, -1, (\zeta_3, \zeta_3^{-1}))$, and $E$ has either split or non-split multiplicative reduction at $p$.

*Example* 5.3. Let $D_1 := -83$ and $D_2 := -11$ and pick

$$\eta := 745 + 51 \left( \frac{1 + \sqrt{83 \cdot 11}}{2} \right) \in O_F, \qquad \mathrm{Norm}(\eta) = -8, \qquad M = 8.$$

The prime $p := 7$ is inert in $F$ and also in $K_2$.

Let $E$ the elliptic curve of conductor $56 = 7 \cdot 8$ labelled $56b$ in Cremona's tables:

$$E : y^2 = x^3 - x^2 - 4.$$

It has rank 0 over $\mathbb{Q}$, but rank 1 over $K_2 = \mathbb{Q}(\sqrt{-11})$, and again rank 0 over any the cubic subfields of $H$ of discriminant $-83$. Furthermore $E$ has split reduction at $p$. As predicted, a calculation shows that

$$\lambda_\theta = \lambda_{\mathrm{Hida}} = 0 \pmod{7^{20}}.$$

The relevant point now to consider is $P^- := (-7, 6\sqrt{-11}) \in E(K_2)$. We compute

$$\lambda_{\mathrm{inert}} = (5563318767325300c + 5291061116602757) \cdot 7 \pmod{7^{20}},$$

where $\mathbb{Q}_{p^2} = \mathbb{Q}_p(c)$ with $c^2 + 6c + 3 = 0$, and find

$$\lambda_{\mathrm{inert}} = \frac{1}{2} \cdot \sqrt{-3} \cdot \log_{\mathrm{Tate}}(P^-) \pmod{7^{20}}.$$

*Example* 5.4. Let $D_1 := -31$ and $D_2 := -7$ and choose

$$\eta := -1888 - 275\left(\frac{1+\sqrt{31\cdot 7}}{2}\right) \in O_F, \qquad \mathrm{Norm}(\eta) = -6, \qquad M = 6.$$

Let $p := 5$, which is inert in $F$ and also in $K_2 = \mathbb{Q}(\sqrt{-7})$.

We take $E$ the elliptic curve labelled $30a$ in Cremona's tables,

$$E : y^2 + xy + y = x^3 + x + 2.$$

This curve has non-split multiplicative reduction at $p$. Just as in Example 5.3 we have Asai ranks $(0,1,0)$ and the relevant point to consider is

$$P^- := ((-3\sqrt{-7} - 9)/8, (-3\sqrt{-7} + 31)/16) \in E(K_2).$$

We verify that

$$\lambda_\theta = (1 + \zeta_3) \cdot \lambda_{\mathrm{Hida}} = \lambda_{\mathrm{inert}} = 5248359978986 \cdot 5 \pmod{5^{20}},$$

and that

$$\lambda_\theta = \log_{\mathrm{Tate}}(P^-) \pmod{5^{20}}.$$

The following experiments consider two Asai rank $(0,0,1)$ examples, which necessarily have $\sigma_p$-eigenvalues $(1,1,(1,-1)))$, when $a_p(E) = 1$ and $-1$. These are the most interesting since the irreducible component of the Asai representation which occurs with multiplicity one in the Mordell-Weil group is two-dimensional, cuts out a cubic abelian extension of $K_1$, and supports points in both the plus and minus eigenspace for $\sigma_p$.

*Example* 5.5. Let $D_1 := -23$ and $D_2 := -43$ and

$$\eta := 16 - \left(\frac{1+\sqrt{23\cdot 43}}{2}\right) \in O_F, \qquad \mathrm{Norm}(\eta) = -7, \qquad M = 7.$$

Let $p := 11$, which is inert in $F$.

Although $S_2(M) = \{0\}$, there are three elliptic curves of conductor $77 = pM$, which are labelled $77a, 77b, 77c$ in the tables of Cremona:

$$\begin{aligned} E_a &: y^2 + y = x^3 + 2x \\ E_b &: y^2 + y = x^3 + x^2 - 49x + 600 \\ E_c &: y^2 + xy = x^3 + x^2 + 4x + 11. \end{aligned}$$

These curves non-split, non-split and split multiplicative reduction at $p = 11$, respectively.

Recall that $H$ is the Hilbert class field of the quartic field $K = F(\sqrt{-23})$, and that the subfields cut out by the Asai representation $V$ are $K_2 = \mathbb{Q}(\sqrt{-43})$ (of class number one), and the Galois closure of any of the cubic subfields of $H$ of conductor $-23$. (These are also subfields of the Hilbert class field $H_1$ of the class number three field $K_1 = \mathbb{Q}(\sqrt{-23})$.) The cubic subfield is isomorphic to $\mathbb{Q}(u)$ where $u^3 - 3u^2 - 23 = 0$.

The Asai ranks for the elliptic curve $E_a$ are $(1,1,1)$, i.e., $E_a$ has rank 1 over $\mathbb{Q}$, and its rank jumps to 2 over both $K_2$ and the cubic subfield $\mathbb{Q}(u)$ of conductor $-23$. As expected, we observe that

$$\lambda_{\mathrm{theta}} = \lambda_{\mathrm{Hida}} = \lambda_{\mathrm{inert}} = 0 \pmod{11^{20}},$$

so that all three invariants vanish up to the computed precision.

Turning next to $E_b$, which has non-split reduction at 11, the Asai ranks are equal to $(0,0,1)$. Let

$$Q_W = ((-u^2+6u+10)/3, (7u^2+35u-71)/9) \in E_b(\mathbb{Q}(u)).$$

The global point $P_W^- \in E(\mathbb{Q}_{p^2})$ in the *negative* eigenspace for $\sigma_p$, is obtained as follows. Let $u_1$ and $u_2$ be the two roots of $x^3-3x^2-23=0$ in $\mathbb{Q}_{p^2}$ which are not in $\mathbb{Q}_p$ and are conjugate to each other. Replacing $u$ in the point $Q_W$ above by $u_1$ and $u_2$ yields two points $Q_1$ and $Q_2$ in $E(\mathbb{Q}_{p^2})$ which are conjugate under $\sigma_p$. Define

$$P_W^- = Q_1 - Q_2.$$

Numerically, one finds

$$2\cdot\lambda_\theta = \lambda_{\mathrm{Hida}} = 146469997806978632\,02\cdot 11 \pmod{11^{20}},$$

and

$$2\cdot\lambda_\theta = \lambda_{\mathrm{Hida}} = -\frac{1}{5}\cdot\log_{\mathrm{Tate}}(P_W^-) \pmod{11^{20}}.$$

Finally, the elliptic curve $E_c$ has split reduction at 11 and Asai ranks $(0,0,1)$. The global point on $E_c$ over the cubic subfield of the Hilbert class field of $K_1$ is equal to

$$Q_W = ((2u^2-5u-7)/9, (-u^2-u+9)/3) \in E_c(\mathbb{Q}(u)).$$

The point $P_W^+ \in E(\mathbb{Q}_p)$ in the *positive* eigenspace for $\sigma_p$ is obtained from $Q_W$ by mapping $u$ to the unique root of $x^3-3x^2-23$ that lies in $\mathbb{Q}_p$.

Numerically we verify that

$$\lambda_\theta = 0 \pmod{11^{20}},$$

as predicted since $a_p(E_c)=+1$ occurs with multiplicity 3 as an eigenvalue for $\sigma_p$ on $V$. On the other hand,

$$\lambda_{\mathrm{Hida}} = (-109454178437585506\,51a - 772941722894765413\,3)\cdot 11 \pmod{11^{20}},$$

where $a^2+7a+2=0$, and

$$\lambda_{\mathrm{Hida}} = \frac{1}{2}\cdot\sqrt{-3}\cdot\log_{\mathrm{Tate}}(P_W^+) \pmod{11^{20}},$$

as expected since $a_p(E_c)$ has multiplicity 1 in the component $W$ of the Asai representation which occurs with rank 1 in the Mordell-Weil group. (For $E_a$ and $E_b$ the value $\lambda_{\mathrm{inert}}$ is non-zero, but we have no interpretation of it to offer.)

Finally we consider Asai rank $(1,0,0)$ and $(0,1,0)$ examples, but where the $\sigma_p$-eigenvalues are $(1,-1,(1,1))$. The smallest prime occurring in any example in this setting is $p=17$.

*Example* 5.6. Let $D_1 := -59$ and $D_2 := -11$ and

$$\eta := 35786444535052398935 - 2703365279120236257\left(\frac{1+\sqrt{59\cdot 11}}{2}\right) \in O_F,$$

so that $M = -\mathrm{Norm}(\eta) = 8$.

Letting $p := 17$ we find the $\sigma_p$-eigenvalues on the Asai components are $(1,-1,(1,1))$.

First consider the curve of conductor $Mp$ labelled $136a$:

$$E_a : y^2 = x^3 + x^2 - 4x.$$

It has ranks $(1,0,0)$ and split reduction at $p$. But since $+1$ also occurs as an eigenvalue in the dimension 2 component of the Asai representation, Theorem 4.6 predicts that $\lambda_\theta = 0$ and we indeed observe this numerically (to precision $17^{10}$). However, since $+1$ occurs with multiplicity one in the component of the Asai representation which itself has rank one in the Mordell-Weil group, we expect $\lambda_{\mathrm{Hida}}$ to be more interesting, and indeed find that

$$\lambda_{\mathrm{Hida}} = (-8218050809a + 52843470025) \cdot 17 \pmod{17^{10}}$$

where $a^2 + 16a + 3 = 0$, and that

$$\lambda_{\mathrm{Hida}} = \frac{2\zeta_3 + 1}{2} \cdot \log_{\mathrm{Tate}}(P^+) \pmod{17^{10}}$$

where $P^+ = (2, -2)$ generates $E_a(\mathbb{Q})$ modulo torsion. On the other hand we also find that $\lambda_{\mathrm{inert}} = 0 \pmod{17^{10}}$.

Consider next the curve labelled $136b$:

$$E_b : y^2 = x^3 - x^2 - 8x - 4.$$

This has ranks $(0,1,0)$ and non-split reduction at $p$. We find as expected that

$$\lambda_\theta := -55227507250 \cdot 17 = \lambda_{\mathrm{inert}} = 2 \cdot \lambda_{\mathrm{Hida}} = \frac{1}{2} \cdot \log_{\mathrm{Tate}}(P^-) \pmod{17^{10}},$$

where $P^- = (-430/891, -18512\sqrt{-11}/88209) \in E_b(K_2)$ is a generator modulo torsion.

5.1.4. *Further examples for CM biquadratic extensions.* We computed some further examples in settings very close to that outlined in Section 5.1.3. Namely, when $K_1$ had class number 5 and $K_2$ class number 1, and when $K_1$ had class number 5 and $K_2$ class number 3. We spare the reader any details, beyond saying that they were in agreement with our theorem for the theta derivation coefficient $\lambda_\theta$ and in line with our expectation (from Section 5.1.2) for the Hida derivation coefficient $\lambda_{\mathrm{Hida}}$.

5.2. **Biquadratic extensions: the RM setting.** Let $D_1 = \ell \equiv 1 \bmod 4$ and $D_2 = q \times r$ where $\ell, q, r$ are distinct primes with $q, r \equiv 3 \bmod 4$. We assume $\mathbb{Q}(\sqrt{D_1})$ has narrow class number 1, and $\mathbb{Q}(\sqrt{D_2})$ has class number 1 but narrow class number 2.

Define

$$F = \mathbb{Q}(\sqrt{D_1 D_2}) \text{ and } K = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}).$$

Then $K$ is an unramified totally real extension of $F$ which is Galois over $\mathbb{Q}$. The field $F$ has class number 2 and narrow class number 4.

There is a mixed signature Hecke character $\psi$ on $K$ of order 2 with trivial finite conductor $O_F$. By induction one obtains as before a (totally odd) 2-dimensional Galois representation $V_0$ of $F$. The four dimensional Asai induction of $V_0$ decomposes as

$$\mathrm{Asai}(V_0) = \mathbf{1} \oplus \varepsilon_\ell \oplus \varepsilon_q \oplus \varepsilon_r$$

where $\varepsilon_\ell, \varepsilon_q, \varepsilon_r$ are quadratic characters attached to the fields $\mathbb{Q}(\sqrt{\ell}), \mathbb{Q}(\sqrt{-q})$ and $\mathbb{Q}(\sqrt{-r})$ respectively.

After choosing an ideal $(\eta) \subset \mathcal{O}_F$ which is trivial in the class group but non-trivial in the narrow class group, i.e. an element $\eta$ of negative norm, one obtains the Fourier expansion of a Hilbert modular form $\theta_\psi$ on $F$ of weight $(1,1)$. Note that here as $K$ is the only unramified extension of $F$, there cannot exist a CM extension of $F$ and Hecke character on that CM extension which defines a Hilbert Modular Form of trivial level $O_F$. So in particular, the theta series $\theta_\psi$ has RM but no CM.

Let $p$ be a prime which is inert in $F$. We observe that always then $pO_K = PP'$ for distinct prime ideals in $O_K$. Let $E$ be an elliptic curve of conductor $pM$, where $M := |\mathrm{Norm}(\eta)|$. Just as in the CM dihedral examples of Section 5.1, the possible patterns of ranks of the Asai components in the Mordell-Weil group and corresponding eigenvalues of $\sigma_p$ on these components are correlated with the pair $(\psi(P), a_p(E))$. Without going into too much detail, observe that Conjecture 1.2 only predicts the non-vanishing of $\lambda_\theta$ in the 25% of times in which $(\psi(P), a_p(E)) = (+1, -1)$. We give below a selection of examples in that setting (our coefficients vanished experimentally in the other settings, as expected).

Note that these examples give evidence for our conjecture beyond what can be proved in our main theorem (Theorem 4.6). Our theorem only predicts "$\lambda_\theta = \log_{\mathrm{Tate}}(P)$" for a Stark-Heegner point $P$, and these are only known in this setting to be local points. Our examples show $P$ is a global point.

*Example* 5.7. Let

$$D_1 := 13, \quad D_2 := 57 = 3 \times 19, \qquad \eta = -14 - \left(\frac{1+\sqrt{13 \cdot 57}}{2}\right) \in O_F, \quad M = 3,$$

and choose $p := 7$, which is inert in $F$. We take

$$E : y^2 + xy = x^3 - 4x - 1$$

to be the elliptic curve labelled $21a$ in the tables of Cremona. This curve has non-split multiplicative reduction at $p$. Let

$$P^- = ((-\sqrt{13}+3)/2, \sqrt{13}-3) \in E(\mathbb{Q}(\sqrt{13})).$$

Then

$$\lambda_\theta = 903550508804407 \times 7 = -4 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{7^{20}}.$$

*Example* 5.8. Let

$$D_1 := 13, \quad D_2 := 93 = 3 \times 31, \qquad \eta = 8983 + 532\left(\frac{1+\sqrt{13 \cdot 93}}{2}\right) \in O_F,$$

so that $M = 3$. Choose $p := 7$. We again take $E$ to be the elliptic curve $21a$ and $P^-$ as in the preceding example. Then we have

$$\lambda_\theta = -903550508804407 \times 7 = 4 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{7^{20}}.$$

*Example* 5.9. Let

$$D_1 := 5, \quad D_2 := 93 = 3 \times 31,$$

and choose

$$\eta = 10 + \left(\frac{1+\sqrt{5\cdot 93}}{2}\right) \in O_F, \qquad M = 6, \quad p := 7.$$

Let

$$E : y^2 + xy + y = x^3 + x^2 - 4x + 5$$

be the elliptic curve $42a$, which has non-split multiplicative reduction at $p$. After defining

$$P^- = (-\sqrt{5}, 2\sqrt{5} - 1) \in E(\mathbb{Q}(\sqrt{5})),$$

we observe that

$$\lambda_\theta = 3617170189375121 \times 7 = \lambda_\theta = 2 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{7^{20}}.$$

*Example* 5.10. Let

$$D_1 := 193, \quad D_2 = 33 = 3 \times 11,$$

and choose

$$\eta = -2707 + 67\left(\frac{1+\sqrt{193\cdot 33}}{2}\right) \in O_F, \qquad M = 8, \qquad p := 7.$$

We take

$$E : y^2 = x^3 + x + 2$$

to be the elliptic curve $56a$, which has non-split multiplicative reduction at $p$, and

$$P^- = (-26, 2\sqrt{-11}) \in E(\mathbb{Q}(\sqrt{-11})).$$

Then

$$\lambda_\theta = 3274774510128699 \times 7 = -2 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{7^{20}}.$$

*Example* 5.11. Let

$$D_1 := 181, \qquad D_2 = 21 = 3 \times 7,$$

and choose

$$\eta := -5 - 1441\left(\frac{1+\sqrt{181\cdot 21}}{2}\right) \in O_F, \qquad M = 5, \qquad p := 11.$$

We take

$$E : y^2 + xy = x^3 - x^2 - 4x + 3$$

to be the elliptic curve labelled $55a$ in the tables of Cremona. This curve has non-split multiplicative reduction at $p$. Define

$$P^- = (1, (-\sqrt{-3} - 1)/2) \in E(\mathbb{Q}(\sqrt{-3})).$$

Then

$$\lambda_\theta = 834444252005378646 \times 11 = 2 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{11^{20}}.$$

Observe that in the RM case the Hida coefficients $\lambda_{\mathrm{Hida}}$ from Section 5.1.2 may still be defined, but we do not know how to efficiently compute them.

5.3. **A cyclic quartic extension.** We now give an example which is beyond the reach of our current theorem and the machinery of Heegner and Stark-Heegner points (though still within the purview of CM extensions).

*Example* 5.12. Let $F := \mathbb{Q}(\sqrt{17})$ and

$$\alpha := -8 - \frac{1+\sqrt{17}}{2}.$$

Then $K := F(\sqrt{\alpha})$ is a CM extension of $F$ which is a cyclic quartic extension of the rational field. We have $17 \cdot O_K = P^4$ for a prime ideal $P$. The group of Hecke characters of $K$ of conductor $P$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and we take $\psi$ to be an element of order 4 in this group.

The diagonal restriction of the theta series $\theta_\psi$ attached to $\psi$ lies in $S_2(4 \times 17)$, a space of dimension 7. (Here $F$ has narrow class number 1 and the Fourier expansion for $\theta_\psi$ is found using the trivial ideal $O_F$.) Note that the diagonal restriction of $\theta_\psi$ is killed by the Hecke operators $U_2(U_2 - I)$ and $U_{17} + I$.

Let $p := 5$, which is inert in $F$. Using our computational methods, it is possible then to consider all elliptic curves of conductor divisibly by 5 and dividing $5 \times 68$. In the Magma computer algebra package these have Cremona labels

$$20a, 85a, 170a, 170b, 170c, 170d, 170e, 340a.$$

(Beware however that this labelling of the curves in Magma of conductor 170 is not the same as that in [LMFDB].) Letting $V_0$ be the induction of $\psi$ to $G_F$ and $V$ its Asai induction to $\mathbb{Q}$, the representation $V$ is self-dual and decomposes as a direct sum

$$\mathrm{Asai}(V_0) = \chi_4 \oplus \chi_{68} \oplus \eta_{17} \oplus \overline{\eta}_{17}.$$

Here $\chi_4$ and $\chi_{68}$ are quadratic characters of conductors 4 and 68, and $\eta_{17}$ a quartic character of conductor 17. Using Magma we compute that the order of vanishing of $L(E, \mathrm{Asai}(V_0))(s)$ at $s = 1$ is (very likely to be) one when $E$ is the curve $20a$, $170c$ or $170d$, and even otherwise.

Because the diagonal restriction of the Hilbert modular theta series $\theta_\psi$ is non-zero, we need to modify our theta derivative using a Hecke operator which kills the $p$-depletion of this non-zero form. In practice we must use $U_2(U_2 - I)$ or $U_{17} + I$. (The modification is done on Fourier expansions themselves, and is very time-consuming, thus limiting the choice to the simplest possibilities.)

Let $f_{20a}$, $f_{170c}$ and $f_{170d}$ denote the weight two eigenforms attached to the three remaining curves. We observe $U_2(U_2 - I)$ kills $f_{20a}$ and $f_{170c}$, and $U_{17} + I$ kills $f_{170c}$ and $f_{170d}$. In particular, we cannot extract any non-zero values for the curve labelled $170c$ using these modifications. We shall thus focus on the curves with labels $20a$ and $170d$.

We consider first

$$E_{20a} : y^2 = x^3 + x^2 + 4x + 4,$$

which has non-split multiplicative reduction at $p = 5$. The rank one component of Mordell-Weil appearing in the Asai representation is generated by

$$P^- := (-53/4, 91\sqrt{-17}/8).$$

The frobenius element $\sigma_p$ acts by $-1$ on this point, and we find that

$$\lambda_\theta = 8 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{5^8}.$$

Working with the Hida rather than theta derivation, we can get slightly higher precision and find

$$\lambda_{\text{Hida}} = 337558 \cdot 5 = 8 \cdot \log_{\text{Tate}}(P^-) \pmod{5^{10}}.$$

Next consider

$$E_{170d} : y^2 + xy + y = x^3 - 3x + 6,$$

which has split multiplicative reduction at $p$. Here the rank one component of Mordell-Weil appearing in $V$ is generated by

$$P^+ := (-5, -10\sqrt{-1} + 2),$$

and $\sigma_p$ acts by $+1$ on this point. We find

$$\lambda_\theta = 8 \cdot \log_{\text{Tate}}(P^+) \pmod{5^9}.$$

Working with the Hida rather than theta derivation, we can again get higher precision and find

$$\lambda_{\text{Hida}} = 818912872 \cdot 5^2 = 8 \cdot \log_{\text{Tate}}(P^+) \pmod{5^{16}}.$$

Observe our precisions for $\lambda_\theta$ are much lower than in previous examples. This is because we had to further act upon the Fourier expansion of the theta derivative by small powers of $U_p$, to “improve overconvergence”.

*Remark* 5.13. The modular form $f_{20a}$ is not a full eigenform in $S_2(340)$, and there is not a canonical choice of “test vector” in the old space $\langle f_{20a}(q), f_{20a}(q^{17})\rangle \subset S_2(340)$ it generates to take the coefficient along. Given our modification by $U_{17} + I$ it was natural and convenient to choose the unique $\tilde{f}$ in this space with $(U_{17} + I)(\tilde{f}) = f_{20a}$.

5.4. **A non-Galois quartic extension.** Our final example is beyond both the reach of Heegner and Stark-Heegner points, and even outside that of CM extensions.

*Example* 5.14. Let $F = \mathbb{Q}(\sqrt{301})$ and

$$\alpha := \frac{-\sqrt{301} - 17}{2},$$

which has norm $-3$. We have $3 \cdot O_F = PP'$ where $P = (\alpha)$ and $P' = (\alpha')$ with $\alpha' := (\sqrt{301} - 17)/2$. Define $K := F(\sqrt{\alpha})$ and

$$\beta := \frac{5\sqrt{301} - 87}{4}\sqrt{\alpha} + \frac{\sqrt{301} - 18}{2} \in K, \quad \text{Norm}_{K/F}(\beta) = \alpha'.$$

The field $K$ is a non-Galois quartic extension of signature $(2, 1)$, with Galois closure a degree 8 extension of $\mathbb{Q}$. There is a Hecke character $\psi$ of order 2 of $K$ factoring through the extension $K(\sqrt{\beta})$ such that the diagonal restriction of the associated series $\theta_\psi$ (constructed using the trivial ideal $O_F$) lies in $S_2(3)$; in particular, it has *trivial* character. (Finding non-Galois quartic extensions and Hecke characters where the diagonal restriction has quadratic character is much easier. The point of this example is that the character is trivial.) Very conveniently $S_2(3)$ is zero, and so this diagonal restriction must vanish. Thus we do not need to modify our theta derivative by any Hecke operators.

Letting $V_0$ be the induction of $\psi$ to $G_F$, the Asai representation itself is self-dual and decomposes as a direct sum

$$\mathrm{Asai}(V_0) = 1 \oplus \chi_3 \oplus W_{903}$$

where $\chi_3$ is a quadratic character of conductor 3 and $W_{903}$ an irreducible 2-dimensional representation of conductor 903.

We consider small primes $p$ which are inert in $F$ and for which we have curves in levels $p$ or $3p$. First, we take $p := 13$ and consider the curve $E_{39a}$ labelled $39a$ by Cremona. The ranks of the components of the Asai representation in the Mordell-Weil group are $(0, 1, 2)$. Since we are in a rank 3 setting, Conjecture 1.2 predicts the component along the attached weight two form $f_{39a}$ of the theta derivative should be zero. And we observe this numerically (to precision $13^5$).

Now let $p := 17$. Here we may consider two curves, labelled $17a$ and $51a$. The ranks in the Asai representation of the Mordell-Weil group are respectively $(0, 0, 1)$ and $(0, 1, 0)$. The $\sigma_p$-eigenvalues on the three components of $\mathrm{Asai}(V_0)$ are $+1, -1$ and $\pm i$. The curve labelled $17a$ has split multiplicative reduction at $p$, but in any case $+1$ (or $-1$) does not occur as an $\sigma_p$-eigenvalue on $W_{903}$. So our conjecture predicts the component of the theta derivative along $f_{17a}$ should be zero, as we observe numerically (to precision $17^5$).

More interestingly, the other curve

$$E_{51a} : y^2 + y = x^3 + x^2 + x - 1$$

has non-split reduction at $p$, and here $-1$ does occur as an eigenvalue on the rank 1 component. The relevant generator is

$$P^- = ((\sqrt{-3} - 1)/2, (\sqrt{-3} - 1)/2) \in E_{51a}(\mathbb{Q}(\sqrt{-3})),$$

and we find

$$\lambda_\theta = 734096 \cdot 17 = 4 \cdot \log_{\mathrm{Tate}}(P^-) \pmod{17^6}.$$

5.5. **Beyond Hecke characters.** Let $g$ be an exotic modular form of weight 1 and quadratic character $\chi$. Given a real quadratic field $F$, one may restrict the Galois representation $\rho_g$ of $\mathbb{Q}$ to $F$, and then take the Asai induction of that to get a four dimensional representation $\mathrm{Asai}(\rho_g|_{G_F})$. Since $\chi$ is quadratic we find this Asai representation is self-dual. This is the most appealing setting in which to do further experimental work, and some partial progress has been made in that direction.

The smallest examples to consider are the exotic forms in levels 283 and 331, which have projective image $S_4$. (See for example the webpage accompanying [BL], or [LMFDB].) For each of these, using code for computing Hilbert class fields in Magma we were able to find the degree 48 fields through which each of the *linear* representations for these forms factors. Then given $F = \mathbb{Q}(\sqrt{D_F})$ with $D_F = 1 \bmod 4$ (and $D_F$ inert in $\mathbb{Q}(\sqrt{-283})$ or $\mathbb{Q}(\sqrt{-331})$, respectively), we could find both the diagonal restriction of the theta family and of its derivative. The (so far) insurmountable difficulty we then encounter is that the diagonal restriction of the theta family lies in $S_2(283)$ (or $S_2(331)$ respectively), which is a space of dimension 23 (respectively 27). And it seems difficult in practice to modify the family by a Hecke operator on Fourier expansions to kill a $p$-depletion of this form, a necessary step to move from the largely mysterious world of "$p$-adic mock modular forms" to that of (nearly) overconvergent ones. Note that forms in latter infinite dimensional spaces are far more amenable to computation, using the methods of [AL1, AL2]. It is evident that a better theoretical and computational understanding of $p$-adic mock modular forms is required in order to push the experimental work further.

**Acknowledgements.** The first author was supported by an NSERC Discovery grant, and the third author was supported by Grant MTM2015-63829-P. The third author is supported by Icrea through an Icrea Academia Grant and Proyecto de generación de conocimiento PID2022-137605NB- I00. The experimental work was carried out by the second author using the Magma Computational Algebra System.

## References

[ABSV] Fabrizio Andreatta, Massimo Bertolini, Marco Seveso, and Rodolfo Venerucci, *Endoscopy for* $\mathbf{GSp}(4)$ *and rational points of elliptic curves*, in preparation

[BF] Ivan Blanco-Chacón and Michele Fornea, *Twisted triple product p-adic L-functions and Hirzebruch-Zagier cycles*, Journal of the Institute of Mathematics of Jussieu, **19** (6) (2020) 1947–1992.

[BD1] Massimo Bertolini and Henri Darmon. *Iwasawa's Main Conjecture for elliptic curves over anticyclotomic* $\mathbb{Z}_p$*-extensions*, Annals of Math. **162** (2005) 1–64.

[BD2] Massimo Bertolini and Henri Darmon, *Hida families and rational points on elliptic curves*, Inventiones Math. **168** (2007), no.2, 371-431.

[BD3] Massimo Bertolini and Henri Darmon, *The rationality of Stark-Heegner points over genus fields of real quadratic fields*, Annals of Math. **170** (2009), 343-369.

[BL] Kevin Buzzard and Alan Lauder, *A computation of modular forms of weight one and small level*, Annales Mathématiques du Québec **41**, Issue 2, (2017), 213-219. Accompanying webpage: https://people.maths.ox.ac.uk/lauder/weight1/

[CC] S. Chen, Y. Cheng, *On Deligne's conjecture for certain automorphic L-functions for* $\mathrm{GL}(3)\times\mathrm{GL}(2)$ *and* $\mathrm{GL}(4)$, Documenta Mathematic **24** (2019), 2241-2297.

[DaH] Luca Dall'Ava and Alex Horawa. *Balanced triple product p-adic L-functions and Stark points*, submitted.

[Dar] Henri Darmon, *Integration on* $\mathcal{H}_p \times \mathcal{H}$ *and arithmetic applications*, Annals of Math. **154** (2000), 589-639.

[DLR1] Henri Darmon, Alan Lauder, and Victor Rotger, *Stark points and p-adic iterated integrals attached to modular forms of weight one.* Forum of Mathematics, Pi, (2015), Vol. 3, e8, 95 pages.

[DPV] Henri Darmon, Alice Pozzi, and Jan Vonk, *Diagonal restrictions of p-adic Eisenstein families*, Math. Annalen **379** (2021), 503–548.

[DR1] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems I: a p-adic Gross-Zagier formula*, Annales Scientifiques de l'Ecole Normale Supérieure **47**, n. 4 (2014), 779–832.

[DR2] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, Journal of the American Mathematical Society **30** Vol. 3, (2017), 601-672.

[FLPSW] Paulina Fust, Judith Ludwig, Alice Pozzi, Mafalda Santos and Hanneke Wiersema, *Real quadratic singular moduli and p-adic families of modular forms*, Women in Numbers Europe IV, Association for Women in Mathematics Series **32**, (2024), 253-291.

[GZ] Benedict Gross and Don Zagier, *Heegner points and derivatives of L-series.* Invent. Math. **84** (1986), no. 2, 225–320.

[HM] Valentin Hernandez and Santiago Molina, *Plectic points and Hida-Rankin p-adic L-functions*, Transactions of the American Mathematical Society **387**, no. 9, (2025), 6535-6583.

[Hi2] Haruzo Hida, *On the critical values of L-functions of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(2)\times\mathrm{GL}(2)$, Duke Math. J. **74** (1994), 431–529.

[Hs] Ming-Lun Hsieh, *Hida families and p-adic triple product L-functions*, American Journal of Mathematics **143**, no.2, (2021), 411-532.

[Ich] Atsushi Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J., **145 (2)** (2008), 281–307.

[Ish] I. Ishikawa, *On the construction of twisted triple product p-adic L-functions for unbalanced weights*, preprint 2025.

[Kato] Kazuya Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque **295**, 2004.

[Kazi] Ananyo Kazi, *Twisted triple product p-adic L-function for finite slope families of Hilbert modular forms*, preprint 2024.

[KL] Ananyo Kazi and David Loeffler, *p-adic Asai and twisted triple product L-functions for finite slope families*, preprint 2025.

[Ko] Victor Kolyvagin, *Finiteness of $E(\mathbb{Q})$ and $Ш(E/\mathbb{Q})$ for a subclass of Weil curves.* Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 3, 522–540, 670–671. Math. USSR-Izv. **32** (1989), no. 3, 523–541.

[AL1] Alan Lauder, Computations with classical and $p$-adic modular forms, LMS J. Comput. Math. **14** (2011), 214-231

[AL2] Alan Lauder, Efficient computation of Rankin $p$-adic L-functions, in "Computations with Modular Forms. Proceedings of a Summer School and Conference, Heidelberg, August/September 2011", Boeckle G. and Wiese G. (eds), Springer Verlag, 181-200, 2014.

[LMFDB] The LMFDB Collaboration, The L-functions and modular forms database, https://www.lmfdb.org

[LV] Matteo Longo and Stefano Vigni, *The rationality of quaternionic Darmon points over genus fields of real quadratic fields*. Int. Math. Res. Not. **13** (2014), 3632–3691.

[Mok] Chung-Pang Mok, *Heegner points and p-adic L-functions for elliptic curves over certain totally real fields*. Comment. Math. Helv., **86** (4), (2011), 867–945.

[Pra] Dipendra Prasad, *Invariant forms for representations of* $\mathrm{GL}_2$ *over a local field*, Amer. J. Math. **114** (1992), no. 6, 1317–1363.

H. D.: McGill University, Montreal, Canada
*Email address*: darmon@math.mcgill.ca

A. L.: University of Oxford, U. K.
*Email address*: lauder@maths.ox.ac.uk

V. R.: Universitat Politècnica de Catalunya, Barcelona, Spain
*Email address*: victor.rotger@upc.edu