STARK POINTS ON ELLIPTIC CURVES VIA PERRIN-RIOU'S PHILOSOPHY

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To Bernadette Perrin-Riou, on her 65th birthday

Abstract. In the early 90's, Perrin-Riou [PR] introduced an important refinement of the Mazur-Swinnerton-Dyer p-adic L-function of an elliptic curve E over $\mathbb{Q}$, taking values in its p-adic de Rham cohomology. She then formulated a p-adic analogue of the Birch and Swinnerton-Dyer conjecture for this p-adic L-function, in which the formal group logarithms of global points on E make an intriguing appearance. The present work extends Perrin-Riou's construction to the setting of a Garret-Rankin triple product (f, g, h), where f is a cusp form of weight two attached to E and g and h are classical weight one cusp forms with inverse nebentype characters, corresponding to odd two-dimensional Artin representations ϱ_g and ϱ_h respectively. The resulting p-adic Birch and Swinnerton-Dyer conjecture involves the p-adic logarithms of global points on E defined over the field cut out by $\varrho_g \otimes \varrho_h$, in the style of the regulators that arise in [DLR1], and recovers Perrin-Riou's original conjecture when g and h are Eisenstein series.

Abstract. (French). Au début des années 90, Bernadette Perrin-Riou [PR] propose un raffinement de la fonction L p-adique de Mazur-Swinnerton-Dyer associée à une courbe elliptique E sur $\mathbb{Q}$, à valeurs dans sa cohomologie de de Rham p-adique. Elle énonce ensuite une conjecture de Birch et Swinnerton-Dyer pour cette fonction L, faisant apparaître des logarithmes p-adiques de certains points de E sur des corps cyclotomiques. On se propose d'étendre la construction de Perrin-Riou à la fonction L p-adique de Garret-Rankin associée à la convolution de trois formes modulaires (f, g, h), où f est la forme cuspidale de poids deux attachée à E et g et h sont des formes modulaires de poids un, correspondant à des représentations d'Artin ϱ_g et ϱ_h impaires de dimension deux. Le conjecture de Birch et Swinnerton-Dyer qui se dégage de ce contexte fait intervenir des logarithmes p-adiques de points algébriques sur E définis sur le corps de nombre découpé par $\varrho_g \otimes \varrho_h$, dans le style des régulateurs de [DLR1]. On récupère la conjecture originale de Perrin-Riou quand g et h sont des séries d'Eisenstein de poids un.

Introduction

One expects to associate p-adic L-functions to quite general p-adic families $\mathbb{V}_p$ of Galois representations. Such families, which typically arise as continuous $\Lambda[G_{\mathbb{Q}}]$-modules over a suitable "Iwasawa algebra" Λ in one or more variables, include the "cyclotomic" collection $\{V_p(k)\}_{k\in\mathbb{Z}_p}$ interpolating the Tate twists of a fixed (motivic) p-adic Galois representation V_p, which provides the backdrop for classical Iwasawa theory and whose associated p-adic L-function is directly analogous to the complex L-function attached to V_p. When V_p is the two-dimensional Galois representation attached to a classical eigenform, this p-adic L-function was first constructed and studied by Mazur and Swinnerton-Dyer [MSD]. Other p-adic L-functions, such as those of Katz attached to families of algebraic Hecke characters of an imaginary quadratic field and those of Mazur-Kitagawa and Greenberg-Stevens attached to Hida families of elliptic modular forms, are less prone to admit direct complex avatars. The study of their leading terms reveals a rich array of phenomena going well beyond a routine transcription to the p-adic setting of known conjectures about special values of complex L-functions.

The "Perrin-Riou philosophy" alluded to in the title asserts that the p-adic L-function attached to a $\Lambda[G_{\mathbb{Q}}]$-module $\mathbb{V}_p$ should arise from a global class

$$\kappa(\mathbb{V}_p) \in \mathrm{Ext}^1_{\Lambda[G_{\mathbb{Q}}]}(\Lambda, \mathbb{V}_p) = H^1(\mathbb{Q}, \mathbb{V}_p)$$

by restricting it to the decomposition group at p and taking its image under a "Λ-adic regulator map"

$$\mathbb{EXP}^* : H^1(\mathbb{Q}_p, \mathbb{V}_p) \longrightarrow \mathbb{D}.$$

This map interpolates the Bloch-Kato dual exponential map at a suitable (typically dense) set of potentially cristalline specialisations of $\mathbb{V}_p$, and its target $\mathbb{D}$ is, accordingly, a module (over a ring $\mathscr{H}(\Lambda) \supset \Lambda$ consisting of power series whose coefficients satisfy certain growth conditions) interpolating the Dieudonné modules of the same specialisations. One expects to recover more standard p-adic L-functions attached to $\mathbb{V}_p$ by projecting $\mathscr{L}_p(\mathbb{V}_p) := \mathbb{EXP}^*(\kappa(\mathbb{V}_p))$ to various natural quotients of $\mathbb{D}$ which are typically locally free of rank one over $\mathscr{H}(\Lambda)$. But the "full" p-adic L-function $\mathscr{L}_p(\mathbb{V}_p)$ allows more satisfying and far-reaching statements about special values.

In the case where $\mathbb{V}_p(E) = \{V_p(E)(k)\}_{k\in\mathbb{Z}_p}$ is the cyclotomic family attached to the p-adic representation $V_p(E) = H^1_{\mathrm{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(1))$ of an elliptic curve E, the idoneous global class $\kappa(\mathbb{V}_p(E))$ was constructed by Kato from p-adic families of *Beilinson elements*: distinguished classes in the second K-groups of modular curves arising from pairs of modular units. The p-adic L-function

$$\mathscr{L}_p(E) := \mathbb{EXP}^*(\kappa(\mathbb{V}_p(E))) \in H^1_{\mathrm{dR}}(E/\mathbb{Q}_p) \otimes \mathscr{H}(\Lambda), \qquad \mathscr{H}(\Lambda) \supset \Lambda \simeq \mathbb{Z}_p[[T]],$$

can be viewed (thanks to Kato's deep reciprocity law) as a refinement of the Mazur-Swinnerton-Dyer p-adic L-function attached to E, which one recovers, when E is ordinary, by projecting $\mathscr{L}_p(E)$ to the unit root subspace of $H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)$ for the action of the Frobenius endomorphism.

In [PR], Perrin-Riou proves two key assertions about the leading term of $\mathscr{L}_p(E)$ at the trivial character $\mathbf{1}$, relating this leading term to the specialisation, denoted $\kappa(V_p(E))$, of the Λ-adic class $\kappa(\mathbb{V}_p(E))$ at the same character. These assertions involve the successive images of $\kappa(V_p(E))$ under the Bloch Kato *dual exponential* and *logarithm* maps:

$$\exp_p^* : \frac{H^1(\mathbb{Q}_p, V_p(E))}{H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_p(E))} \longrightarrow \mathrm{Fil}^1 H^1_{\mathrm{dR}}(E/\mathbb{Q}_p), \qquad \log_p : H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_p(E)) \longrightarrow \frac{H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)}{\mathrm{Fil}^1 H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)}.$$

Perrin-Riou's first main theorem [PR, Prop. 2.1.4] asserts that

$$\frac{(1-p^{-1}\varphi^{-1})}{(1-\varphi)} \mathscr{L}_p(E)(\mathbf{1}) = \exp_p^*(\kappa(V_p(E))), \tag{1}$$

where φ denotes the cristalline frobenius endomorphism acting on $H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)$. It also follows from Kato's reciprocity law that

$$\frac{(1-p^{-1}\varphi^{-1})}{(1-\varphi)} \mathscr{L}_p(E)(\mathbf{1}) = \frac{L(E,1)}{\Omega_E} \cdot \omega, \tag{2}$$

where Ω_E is a complex period attached to the choice of an invariant differential

$$\omega \in \Omega^1(E/\mathbb{Q}) = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(E/\mathbb{Q}).$$

In particular, the global class $\kappa(V_p(E))$ belongs to the Selmer group $H^1_{\mathrm{fin}}(\mathbb{Q}, V_p(E))$ if and only if $L(E,1) = 0$. (Cf. [PR, §3.3.1].)

When $L(E,1) = 0$, it thus becomes natural to examine the *first derivative*

$$\mathscr{L}'_p(E)(\mathbf{1}) \in H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)$$

of $\mathscr{L}_p(E)$ at the trivial character. Perrin-Riou's second main theorem [PR, Prop. 2.2.2] asserts that

$$\frac{(1-p^{-1}\varphi^{-1})}{(1-\varphi)}\mathscr{L}_p'(E)(\mathbf{1}) = \log_p(\kappa(V_p(E))) \qquad (\mathrm{mod}\ \mathrm{Fil}^1 H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)). \tag{3}$$

The counterpart of (2), conjectured in [PR, §3.3.5], asserts that

$$\frac{(1-p^{-1}\varphi^{-1})}{(1-\varphi)}\mathscr{L}_p'(E)(\mathbf{1}) = \log^2(P), \tag{4}$$

where P is a global point in $E(\mathbb{Q})\otimes\mathbb{Q}$ whose Néron-Tate canonical height differs from $L'(E,1)$ by an elementary non-zero factor. The point P is thus expected to be non-trivial precisely when $E(\mathbb{Q})$ has (algebraic or analytic) rank one. When E has good reduction at p, this conjecture was proved in [BDV] by realising the conjectural P as a Heegner point. It then follows from the Gross-Zagier formula that the global class $\kappa(V_p(E))$ is trivial at p if and only if $L'(E,1)=0$. The proof of (4) relies crucially on a comparison between Kato's cohomology class $\kappa(V_p(E))$ and the *generalised Kato classes* whose study was initiated in [BDR1], [BDR2], [DR1], and [DR2]. These generalised Kato classes are related to the Garret-Rankin triple product p-adic L-function of Hida and Harris-Tilouine in exactly the same way that the Kato class $\kappa(\mathbb{V}_p(E))$ is related to the Mazur-Swinnerton-Dyer p-adic L-function of E.

The main goal of the present work is to formulate a (largely conjectural) analogue of (4) in which $\mathscr{L}_p(E)$ is replaced by a Perrin-Riou-style refinement of the Garret-Rankin p-adic L-function $\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ attached to a triple of Hida families specialising to a triple (f,g,h) of classical elliptic modular newforms in weights $(2,1,1)$, and the expression $\log^2(P)$ involving the Heegner point P is replaced by an expression involving the *Stark points* of [DLR1] defined over the number field cut out by the tensor product $\varrho_g\otimes\varrho_h$ of the Artin representations attached to the eigenforms g and h.

The resulting Conjecture 3.2 below, which is the main contribution of this work, draws its inspiration from (at least) three sources:

(1) The seminal article [PR] where conjecture (4) is formulated, suggesting that the "enhanced" p-adic L-functions of Perrin-Riou can be parlayed into a p-adic analytic construction of global points on elliptic curves, analogous to an earlier construction of Rubin [Ru].
(2) The proof [BDV] of Perrin-Riou's conjecture relying on an alternate construction of Perrin-Riou's $\mathscr{L}_p(E)$ in which the machinery of overconvergent modular symbols is replaced by the theory of p-adic modular forms and their interpretation in terms of de Rham cohomology. This shift in point of view (which amounts roughly speaking to a passage from Betti to de Rham cohomology, thus obviating the theory of modular symbols and their overconvergent analogues) turns out to be crucial both for the proofs of [BDV] and for Conjecture 3.2 below, which applies in many settings where modular symbols are unavailable.
(3) The "elliptic Stark conjectures" of [DLR1], in which a "Stark point regulator" very similar to (but not quite the same as) the one of Conjecture 3.2 is related to the *value* of a Garrett-Hida p-adic L-function at the triple (f,g,h), which lies *outside* the range of classical interpolation defining the p-adic L-function. In contrast, Conjecture 3.2 involves the derivative at the same triple of a *different* Garret-Rankin p-adic L-function, whose range of classical interpolation *contains* (f,g,h). In that sense, the main conjecture of [DLR1] is more closely analogous to [Ru, Cor. 10.3] and to the p-adic Gross-Zagier formula of [BDP], while Conjecture 3.2 below bears a closer relationship to Perrin-Riou's conjecture, as formulated in [PR]. The bridge between the latter and Rubin's theorem (which is hinted at in the introduction of [PR]) is provided by the (generalised) Kato class itself, whose logarithms (evaluated along different lines of the

relevant Dieudonné module) encode all at once the three different p-adic L-functions attached to $(\mathbf{f},\mathbf{g},\mathbf{h})$, as described in [DR1]. When compared with [DLR1], Conjecture 3.2 has the ancillary role of clarifying the relationship between Perrin-Riou's conjecture and Rubin's theorem.

1. The Garret-Rankin p-adic L-function, d'après Perrin-Riou

Let $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \simeq \mathbb{Z}_p[\mathbb{Z}/(p-1)\mathbb{Z}][[T]]$ be the usual Iwasawa algebra. For any finite flat extension $\tilde\Lambda$ of Λ, let $M_{\tilde\Lambda}(N,\chi)$ denote the module of Λ-adic modular forms of tame level N and tame character χ, with coefficients in $\tilde\Lambda$, following the notations that were employed, for instance, in the introduction of [DR2]. Let

$$\mathbf{g}\in M_{\Lambda_g}(N_g,\chi_g),\qquad \mathbf{h}\in M_{\Lambda_h}(N_h,\chi_h),$$

be two Λ-adic eigenforms (with coefficients in certain finite flat extensions Λ_g and Λ_h of Λ respectively) whose tame characters satisfy the important self-duality condition

$$\chi_g\chi_h = 1.$$

Let

$$\mathcal{W} := \mathrm{Spf}(\Lambda) = \hom_{\mathrm{cts}}(1+p\mathbb{Z}_p,\mathbb{C}_p^\times)$$

be the usual weight space, and let

$$\mathcal{W}_g := \mathrm{Spf}(\Lambda_g),\qquad \mathcal{W}_h := \mathrm{Spf}(\Lambda_h)$$

be the rigid analytic spaces parametrising the Hida families $\mathbf{g}$ and $\mathbf{h}$ respectively. The inclusion $\Lambda \hookrightarrow \Lambda_g$ gives rise to a structure map from $\mathcal{W}_g$ to $\mathcal{W}$, called the *weight map* and denoted $x\mapsto w(x)$. A point $x\in\mathcal{W}_g$ is said to be classical if $w(x)$ (viewed as a continuous character of $1+p\mathbb{Z}_p$) is of the form $t\mapsto t^k\eta(t)$, where $k\geq 0$ and η is a finite order character, factoring through $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We set conventions so that the specialisation (denoted g_x) of $\mathbf{g}$ at a point x of classical weight $w(x)=(k,\eta)$ is an eigenform of weight k and nebentypus character $\chi_g\eta\omega^{1-k}$, where ω is the mod p cyclotomic character.

Let $S_k^{(p)}(N)$ be the space of p-adic overconvergent modular forms of weight two, trivial nebentype character, and tame level N. After setting $N=\mathrm{lcm}(N_f,N_g,N_h)$ and $\Lambda_{gh} := \Lambda_g\otimes_\Lambda\Lambda_h$, we choose a pair $(\breve{\mathbf{g}},\breve{\mathbf{h}})$ of Λ-adic test vectors in the Hecke eigenspaces of $M_{\Lambda_{gh}}(N)$ attached to the Hecke eigenvalues for $\mathbf{g}$, and $\mathbf{h}$ respectively. The fiber product

$$\mathcal{W}_{gh} := \mathrm{Spf}(\Lambda_{gh}) = \mathcal{W}_g\times_{\mathcal{W}}\mathcal{W}_h$$

naturally parametrises a family Ξ_{gh} of p-adic modular forms *of constant weight two*, whose specialisation at a pair (x,y) of common weight $w(x)=w(y)=(k,\eta)$ is given by

$$\Xi_{gh}(x,y) = (d^{1-k}\breve{g}_x\otimes\omega^{k-1}\eta^{-1})\times\breve{h}_y,$$

where d is the Atkin-Serre d operator on p-adic modular forms which raises the weight by 2 and acts as $\sum_n a_nq^n\mapsto\sum na_nq^n$ on q-expansions. When (k,η) is a classical weight, the specialisations $\breve{g}_x$ and $\breve{h}_y$ are both of weight k and of character $\chi_g\eta\omega^{1-k}$ and $\chi_h\eta\omega^{1-k}$ respectively. It follows that $\Xi_{gh}(x,y)$ belongs to the space $S_2^{(p)}(N)$.

A theorem of Coleman (cf., the discussion in [DR1, §2.1]) produces a frobenius-equivariant exact sequence

$$0\longrightarrow H^1_{\mathrm{dR}}(X_0(N))\longrightarrow\frac{S_2^{(p)}(N)}{dS_0^{(p)}(N)}\xrightarrow{\ \partial\ }\mathbb{C}_p[\mathrm{SS}],\tag{5}$$

where ∂ is a residue map along the supersingular residue discs of $X_0(N)$, and $\mathbb{C}_p[\mathrm{SS}]$ consists of the formal $\mathbb{C}_p$ linear combinations of supersingular points on $X_0(N)$ in characteristic p. The frobenius endomorphism acts on the target of ∂ with eigenvalues of complex absolute value p, and the exact sequence (5) therefore admits a canonical frobenius-equivariant splitting.

Therefore $\Xi_{gh}(x,y)$ gives rise to a class $\mathscr{L}_{gh}(x,y) \in H^1_{\mathrm{dR}}(X_0(N))$, which interpolates to a class

$$\mathscr{L}_{gh} \in H^1_{\mathrm{dR}}(X_0(N)) \otimes \Lambda_{gh}.$$

Suppose that f is a (not necessarily new) eigenform of weight two on $\Gamma_0(N)$. Assume for simplicity that f has rational Hecke eigenvalues, and hence corresponds to an elliptic curve E of level dividing N. Fix a non-constant morphism

$$\pi_f : X_0(N) \longrightarrow E,$$

and let

$$\mathscr{L}_{fgh} := \pi_{f*}(\mathscr{L}_{gh}) \in H^1_{\mathrm{dR}}(E) \otimes \Lambda_{gh}. \tag{6}$$

This Λ_{gh}-adic family of de Rham cohomology classes is called the *Perrin-Riou p-adic L-function* attached to the modular parametrisation π_f and to the pair $(\breve{\mathbf{g}}, \breve{\mathbf{h}})$ of Λ-adic test vectors. When (x_0, y_0) is the classical point of $\mathcal{W}_{gh}$ of weight one attached to the original weight one eigenforms $\breve{g}$ and $\breve{h}$, the specialisation $\mathscr{L}_{fgh}(x_0,y_0)$ is related to the central critical value of $L(f \otimes g \otimes h, 1)$, in a way that will be made more precise in Theorem 3.1 below.

When $L(f \otimes g \otimes h, 1) = 0$, it is then natural to consider the first derivative of $\mathscr{L}_{fgh}$ at (x_0, y_0) with respect to the "weight of g and h" variable. More precisely, assume that the natural projection $\mathcal{W}_{gh} \longrightarrow \mathcal{W}$ is étale at (x_0, y_0). This is known to hold whenever g and h are not theta series of a real quadratic field in which the prime p is split, thanks to a result of Bellaiche and Dimitrov [BD]. Under this unrestrictive hypothesis, one can choose a local parameter on $\mathcal{W}_{gh}$ near (x_0, y_0) which corresponds to the weight k, and consider the derivative with respect to this variable:

$$\mathscr{L}'_{fgh}(x_0, y_0) := \frac{d}{dk} \mathscr{L}_{fgh}(x_0, y_0)_{k=1} \in H^1_{\mathrm{dR}}(E).$$

The nature of this class, more precisely, its image in a suitable explicit quotient of $H^1_{\mathrm{dR}}(E)$, is the main object of this note.

2. The Perrin-Riou regulator

As in the previous section, let f be a weight two eigenform on $\Gamma_0(N_f)$, and let g and h be eigenforms of weight one, levels N_g and N_h, and with inverse nebentype characters. Under this last assumption, the tensor product $V_{gh} := V_g \otimes V_h$ of the Artin representations associated to g and h has trivial determinant and real traces. Let H_g, H_g, and H_{gh} denote the extensions of $\mathbb{Q}$ cut out by V_g, V_h, and V_{gh} respectively.

A *p-adic deformation datum* for g is the datum of

(1) a $G_{\mathbb{Q}}$-stable line in V_g, if $g = E_1(\chi_1, \chi_2)$ is a weight one Eisenstein series. Such a datum amounts to the choice of a character in $\{\chi_1, \chi_2\}$, and there are always two distinct such choices, since χ_1 and χ_2 have opposite parity and are therefore distinct.

(2) a $G_{\mathbb{Q}_p}$ stable line in V_g, if g is cuspidal. If g is regular at p, i.e., the characteristic polynomial $x^2 - a_p(g)x + \chi_g(p) = (x - \alpha_g)(x - \beta_g)$ has distinct roots, then there are exactly two distinct such p-deformation data, corresponding to a choice of α_g or β_g.

The terminology is justified by the fact that the choice of a p-adic deformation datum for g determines a Hida family specialising to a suitable p-stabilisation of g in weight one. When $g = E_1(\chi_1, \chi_2)$ is an Eisenstein series, the Hida families attached to the deformation data χ_1 and χ_2 are just

$$g_{\chi_1} = E_k(\chi_1, \chi_2), \qquad g_{\chi_2} = E_k(\chi_2, \chi_1),$$

which satisfy

$$a_p(g_{\chi_1}) = \chi_1(p), \qquad a_p(g_{\chi_2}) = \chi_2(p).$$

If g is cuspidal but *regular* at p, i.e., if $\alpha_g \neq \beta_g$, then the Hida families attached to the deformation data α_g and β_g, denoted $\mathbf{g}_\alpha$ and $\mathbf{g}_\beta$ respectively, have weight one specialisations equal to g_α and g_β, where

$$a_p(g_\alpha) = \alpha_g, \qquad a_p(g_\beta) = \beta_g.$$

Under the regularity assumption, a theorem of Bellaiche and Dimitrov guarantees that the Hida families $\mathbf{g}_\alpha$ and $\mathbf{g}_\beta$ are uniquely determined by these conditions.

On the Galois side, the choice of a pair of p-adic deformation data α_g and α_h for g and h determines a decomposition of the Galois representations V_g and V_h into a direct sum of one-dimensional spaces

$$V_g = V_g^\alpha \oplus V_g^\beta, \qquad V_h = V_h^\alpha \oplus V_h^\beta,$$

leading to a decomposition

$$V_{gh} \quad = \quad (V_g^\alpha \otimes V_h^\alpha) \oplus \cdots \oplus (V_g^\beta \otimes V_h^\beta) \quad =: \quad V_{gh}^{\alpha\alpha} \oplus \cdots \oplus V_{gh}^{\beta\beta}$$

of the tensor product V_{gh} into $G_\mathbb{Q}$ (resp. $G_{\mathbb{Q}_p}$)-stable one-dimensional subspaces when g is Eisenstein (resp. cuspidal). This choice of deformation data also determines a canonical two-dimensional subspace of V_{gh} by the rule

$$V_{gh}^{\alpha_g,\alpha_h} := V_{gh}^{\alpha\alpha} \oplus V_{gh}^{\beta\beta}.$$

Note that the p-stabilised eigenforms g_α and h_α play symmetrical roles in the definition of this subspace, which differs from the two-dimensional subspaces denoted $V_{gh}^{g_\alpha}$ and $V_{gh}^{g_\beta}$ in [DLR1].

Recall that all of the Artin representations above are viewed as $\mathbb{Q}_p$ vector spaces but come equipped with natural $G_\mathbb{Q}$-stable L-rational structures, denoted ${}_LV_g$, ${}_LV_h$, ${}_LV_{gh}$, ${}_LV_{gh}^{\alpha_g,\alpha_h}$, etc., where $L = L_{gh}$ is a finite extension of $\mathbb{Q}$ large enough to contain the eigenvalues of all the elements in the images of ϱ_g and ϱ_h. If $\dim_L \hom_{G_\mathbb{Q}}({}_LV_{gh}, E(H)) = 2$, let (P,Q) denote an L-basis for this vector space, and let (v_1, v_2) be an L-basis for ${}_LV_{gh}^{\alpha_g,\alpha_h}$. The matrix

$$R_{g_\alpha,h_\alpha}(E,\varrho_{gh}) := \begin{pmatrix} \log_{E,p}(P(v_1)) & \log_{E,p}(P(v_2)) \\ \log_{E,p}(Q(v_1)) & \log_{E,p}(Q(v_2)) \end{pmatrix}$$

depends on the choice of basis (P,Q) (resp. (v_1,v_2)) only up to multiplication on the left (resp. right) by an element of $\mathbf{GL}_2(L)$, and hence the p-adic number $\det R_{g_\alpha,h_\alpha}(E,\varrho_{gh})$ is well-defined up to $L^\times$.

Definition 2.1. The matrix $R_{g_\alpha,h_\alpha}(E,\varrho_{gh})$ is called the *Perrin-Riou matrix*, and its determinant is called the Perrin-Riou regulator, associated to the triple (f, g_α, h_α).

We will now give an explicit description of the Perrin-Riou regulator in various arithmetically interesting scenarios.

2.1. **The case where g and h are Eisenstein series.** This corresponds exactly to the case that was studied in depth by Kato and Perrin-Riou. By possibly twisting g and h, we can assume without loss of generality that

$$g = E_1(\chi_1,\chi_2), \qquad h = E_1(1,\chi), \qquad \text{where } \chi_1\chi_2\chi = 1,$$

and χ is an odd character, so that in particular $\chi \neq 1$ and $\chi_1 \neq \chi_2$. It follows that both g and h each admit two distinct deformation data, corresponding to the choice of ordering between the pairs of distinct characters corresponding to $E_1(\chi_1,\chi_2)$ and $E_1(1,\chi)$. The Artin representations attached to g and h are given by

$$V_g = \chi_1 \oplus \chi_2, \qquad V_h = 1 \oplus \chi, \qquad V_{gh} = \chi_1 \oplus \bar\chi_1 \oplus \chi_2 \oplus \bar\chi_2.$$

In particular, the subspaces of V_{gh} attached to the various choices of deformation data for g and h are equal to

$$V_{gh}^{\chi_1,1} = V_{gh}^{\chi_2,\chi} = \chi_1 \oplus \bar\chi_1, \qquad V_{gh}^{\chi_1,\chi} = V_{gh}^{\chi_2,1} = \chi_2 \oplus \bar\chi_2.$$

Since
$$L(E, V_{gh}, s) = L(E, \chi_1, s) \cdot L(E, \bar{\chi}_1, s) \cdot L(E, \chi_2, s) \cdot L(E, \bar{\chi}_2, s),$$
the L-function of $L(E, V_{gh}, s)$ admits a double zero if and only if, after eventually interchanging χ_1 and χ_2, one has
$$\mathrm{ord}_{s=1} L(E, \chi_1, s) = \mathrm{ord}_{s=1} L(E, \bar{\chi}_1, s) = 1, \qquad L(E, \chi_2, 1) \neq 0, \quad L(E, \bar{\chi}_2, 1) \neq 0.$$
In that case, the Birch and Swinnerton Dyer conjecture predicts that the χ_1 and $\bar{\chi}_1$ isotypic components of the Mordell-Weil group of E are of rank one, and spanned by points
$$P_{\chi_1} \in (E(H) \otimes L)^{\chi_1}, \qquad P_{\bar{\chi}_1} \in (E(H) \otimes L)^{\bar{\chi}_1}.$$
When $\chi_1 = \bar{\chi}_1$, i.e., when χ_1 is quadratic, the existence of $P_{\chi_1} = P_{\bar{\chi}_1}$ is known and follows from the Gross-Zagier formula, while the existence of P_{χ_1} is considerably more mysterious when $\chi_1 \neq \bar{\chi}_1$. In any case, the Perrin-Riou regulators in this scenario are given by:
$$\begin{aligned} R_{g_{\chi_1}, h_1}(E, \varrho_{gh}) = R_{g_{\chi_2}, h_\chi}(E, \varrho_{gh}) &= \log_{E,p}(P_{\chi_1}) \log_{E,p}(P_{\bar{\chi}_1}), \\ R_{g_{\chi_2}, h_1}(E, \varrho_{gh}) = R_{g_{\chi_1}, h_\chi}(E, \varrho_{gh}) &= 0. \end{aligned}$$
In this setting, Conjecture 3.2 corresponds to the original conjecture of Perrin-Riou [PR]. Note that when χ_1 is quadratic, the Perrin-Riou regulator is the square of the formal group logarithm of a global point on E defined over the quadratic field cut out by this character.

2.2. **The case where g is cuspidal and h is Eisenstein.** We can assume, without loss of generality, that $h = E(1, \chi^{-1})$, where χ is the nebentypus character of g. One then has
$$V_{gh} = V_g \oplus V_g^*.$$
After setting
$$\alpha_h = 1, \qquad \beta_h = \chi^{-1}(p) = (\alpha_g \beta_g)^{-1},$$
one finds
$$\alpha_g \alpha_h = \alpha_g, \quad \beta_g \beta_h = \alpha_g^{-1}, \qquad \alpha_g \beta_h = \beta_g^{-1}, \quad \beta_g \alpha_h = \beta_g.$$
From this one readily obtains
$$V_{gh}^{\alpha_g, 1} = V_g^{\alpha_g} (V_g^*)^{\bar{\alpha}_g}, \qquad V_{gh}^{\beta_g, 1} = V_g^{\beta_g} (V_g^*)^{\bar{\beta}_g}.$$
Since
$$L(E, V_{gh}, s) = L(E, V_g, s) \cdot L(E, V_g^*, s),$$
the L-function of $L(E, V_{gh}, s)$ admits a double zero if and only if
$$\mathrm{ord}_{s=1} L(E, V_g, s) = \mathrm{ord}_{s=1} L(E, V_g^*, s) = 1.$$
This is harmonious with the fact that the Perrin-Riou regulator always vanishes if $r(E, V_g) > 1$ or $r(E, V_g^*) > 1$. Otherwise, the V_g and V_g^* isotypic components of the Mordell-Weil group of E are of dimension one, spanned by elements
$$P_g \in \hom(V_g, E(H) \otimes L)^{G_{\mathbb{Q}}}, \qquad P_g^* \in \hom(V_g^*, E(H) \otimes L)^{\bar{\chi}_1 G_{\mathbb{Q}}}.$$
The Perrin-Riou regulators is then given by
$$\begin{aligned} R_{g_\alpha, h_1}(E, \varrho_{gh}) = R_{g_\beta, h_{\bar{\chi}}}(E, \varrho_{gh}) &= \log_{E,p}(P_g(v_\alpha)) \log_{E,p}(P_g^*(v_{\bar{\alpha}})), \\ R_{g_\beta, h_1}(E, \varrho_{gh}) = R_{g_\alpha, h_{\bar{\chi}}}(E, \varrho_{gh}) &= \log_{E,p}(P_g(v_\beta)) \log_{E,p}(P_g^*(v_{\bar{\beta}})). \end{aligned}$$

When g is the theta series attached to a *ring class character* of an imaginary quadratic field, the global points arising in the Perrin-Riou regulator can be constructed from Heegner points on modular or Shimura curves. In all other cases, no geometric construction of P_g and P_g^* seems readily available.

2.3. **The adjoint case.** This heading alludes to the setting where $h = g^*$ is the dual of g. One can then order the frobenius eigenvalues (α_h, β_h) in such a way that

$$\alpha_h = \alpha_g^{-1}, \qquad \beta_h = \beta_g^{-1}.$$

The Artin representation attached to g and h is then equal to

$$V_{gh} = 1 \oplus \mathrm{Ad}(g),$$

where $\mathrm{Ad}(g)$ denotes the three-dimensional space of trace zero endomorphisms of V_g endowed with the usual action of $G_{\mathbb{Q}}$. The subspaces of V_{gh} attached to the various choices of deformation data for g and h are equal to

$$V_{gh}^{\alpha_g,\alpha_h} = V_{gh}^{\beta_g,\beta_h} = 1 \oplus \mathrm{Ad}(g)^{\varphi=1}, \qquad V_{gh}^{\alpha_g,\beta_h} = V_{gh}^{\beta_g,\alpha_h} = \mathrm{Ad}(g)^{\varphi=\alpha_g/\beta_g} \oplus \mathrm{Ad}(g)^{\varphi=\beta_g/\alpha_g}.$$

The decomposition of V_{gh} implies that

$$L(E, V_{gh}, s) = L(E, s) \cdot L(E, \mathrm{Ad}(g), s).$$

In particular, the L-function of $L(E, V_{gh}, s)$ admits a double zero at the center if and only if

$$(\mathrm{ord}_{s=1} L(E, s), \mathrm{ord}_{s=1} L(E, \mathrm{Ad}(g), s)) = (2, 0), (1, 1), \text{ or } (0, 2).$$

We discuss these cases in turn.

Case 1: $\mathrm{ord}_{s=1} L(E, s) = 2$ and $L(E, \mathrm{Ad}(g), 1) \neq 0$.
The Birch and Swinnerton Dyer conjecture predicts that $E(\mathbb{Q})$ has rank two and that the $\mathrm{Ad}(g)$-isotypic components of the Mordell-Weil group of E is trivial. In this case one always has

$$R_{g_\alpha,h_\alpha}(E, \varrho_{gh}) = R_{g_\beta,h_\beta}(E, \varrho_{gh}) = R_{g_\alpha,h_\beta}(E, \varrho_{gh}) = R_{g_\beta,h_\alpha}(E, \varrho_{gh}) = 0.$$

Case 2: $\mathrm{ord}_{s=1} L(E, s) = \mathrm{ord}_{s=1} L(E, \mathrm{Ad}(g), s) = 1$.
Both $E(\mathbb{Q}) \otimes L$ and $\hom(\mathrm{Ad}(g), E(H) \otimes L)^{G_{\mathbb{Q}}}$ are then expected to be of dimension one over L. Let P and P_g denote generators of these L-vector spaces. One then has, up to scalars in $L^\times$,

$$\begin{aligned} R_{g_\alpha,h_\alpha}(E, \varrho_{gh}) = R_{g_\beta,h_\beta}(E, \varrho_{gh}) &= \log_{E,p}(P) \log_{E,p}(P_g(v_1)), \\ R_{g_\alpha,h_\beta}(E, \varrho_{gh}) = R_{g_\beta,h_\alpha}(E, \varrho_{gh}) &= 0. \end{aligned}$$

Case 3: $L(E, 1) \neq 0$ and $\mathrm{ord}_{s=1} L(E, \mathrm{Ad}(g), s) = 2$.
It is then expected that $E(\mathbb{Q})$ has rank zero while $\hom(\mathrm{Ad}(g), E(H) \otimes L)^{G_{\mathbb{Q}}}$ is a two-dimensional vector space; let (P_g, Q_g) be a basis for the latter, and let $v_{\alpha/\beta}$ and $v_{\beta/\alpha}$ be vectors of $\mathrm{Ad}(g)$ on which the frobenius element acts with the eigenvalue α_g/β_g and β_g/α_g respectively. Up to scalars in $L^\times$, one finds in this case

$$R_{g_\alpha,h_\alpha}(E, \varrho_{gh}) = R_{g_\beta,h_\beta}(E, \varrho_{gh}) = 0. \tag{7}$$

The regulators $R_{g_\alpha,h_\beta}(E, \varrho_{gh})$ and $R_{g_\beta,h_\alpha}(E, \varrho_{gh})$ are more interesting, and are both equal to

$$\log_{E,p}(P_g(v_{\alpha/\beta})) \log_{E,p}(Q_g(v_{\beta/\alpha})) - \log_{E,p}(P_g(v_{\beta/\alpha})) \log_{E,p}(Q_g(v_{\alpha/\beta})).$$

This represents the simplest instance where the Perrin-Riou regulator is "not factorable", i.e., is not simply a product of logarithms of global points on E.

2.4. **The dihedral case.** Let K be a real or imaginary quadratic field and suppose that g and h are the Hecke theta series attached to finite order characters ψ_g and ψ_h of G_K. When K is real, it is therefore assumed that ψ_g and ψ_h are of *mixed signature*, so that the induced Artin representations are odd and the associated theta series are holomorphic. Let

$$\psi_1 := \psi_g \psi_h, \qquad \psi_2 := \psi_g \psi_h',$$

where ψ_h' denotes the character of K obtained by viewing ψ_h as an idéle class character and pre-composing it with the Galois involution on K. The running assumption that the nebentype characters of g and h are inverses of each other implies that both ψ_1 and ψ_2 are ring class characters, i.e.,

$$\psi_1' = \psi_1^{-1}, \qquad \psi_2' = \psi_2^{-1}.$$

The Artin representation V_{gh} then decomposes as

$$V_{gh} = V_1 \oplus V_2, \qquad V_1 := \mathrm{Ind}_K^{\mathbb{Q}}(\psi_1), \quad V_2 := \mathrm{Ind}_K^{\mathbb{Q}}(\psi_2).$$

The discussion can now be broken up into two cases:

Case 1. The prime p is inert in K. There is a root of unity ξ for which (after eventually re-ordering (α_g, β_g) and (α_h, β_h) appropriately),

$$\alpha_g = \xi, \qquad \beta_g = -\xi, \qquad \alpha_h = \xi^{-1}, \qquad \beta_h = -\xi^{-1}.$$

The subspaces of V_{gh} attached to the various choices of deformation data for g and h are equal to

$$V_{gh}^{\alpha_g,\alpha_h} = V_{gh}^{\beta_g,\beta_h} = V_1^+ \oplus V_2^+,$$
$$V_{gh}^{\alpha_g,\beta_h} = V_{gh}^{\beta_g,\alpha_h} = V_1^- \oplus V_2^-,$$

where the superscripts of $+$ and $-$ denote the 1 and -1 eigenspaces respectively for the action of the frobenius element at p. One finds that the Perrin-Riou regulators are non-zero if and only if $r(E, V_1) = r(E, V_2) = 1$, and one then has, after letting P_1 and P_2 be L-vector space generators of $\hom(V_1, E(H) \otimes L)^{G_{\mathbb{Q}}}$ and $\hom(V_2, E(H) \otimes L)^{G_{\mathbb{Q}}}$ respectively, and letting (v_1^+, v_1^-) be a frobenius eigenbasis for V_1, and likewise (v_2^+, v_2^-) an eigenbasis for V_2:

$$\begin{aligned} R_{g_\alpha,h_\alpha}(E, \varrho_{gh}) = R_{g_\beta,h_\beta}(E, \varrho_{gh}) &= \log_{E,p}(P_1(v_1^+)) \log_{E,p}(P_2(v_2^+)), \\ R_{g_\alpha,h_\beta}(E, \varrho_{gh}) = R_{g_\beta,h_\alpha}(E, \varrho_{gh}) &= \log_{E,p}(P_1(v_1^-)) \log_{E,p}(P_2(v_2^-)). \end{aligned}$$

Case 2. The prime p is split in K. After eventually re-ordering (α_g, β_g) and (α_h, β_h), we may assume without loss of generality that the frobenius element at p acts on V_1 with eigenvalues

$$\alpha_1 = \alpha_g \alpha_h, \qquad \beta_1 = \beta_g \beta_h,$$

and on V_2 with eigenvalues

$$\alpha_2 = \alpha_g \beta_h, \qquad \beta_2 = \beta_g \alpha_h.$$

The subspaces of V_{gh} attached to the various choices of deformation data for g and h are equal to

$$V_{gh}^{\alpha_g,\alpha_h} = V_{gh}^{\beta_g,\beta_h} = V_1,$$
$$V_{gh}^{\alpha_g,\beta_h} = V_{gh}^{\beta_g,\alpha_h} = V_2.$$

One then finds that the first Perrin-Riou regulator attached to the deformation data (α_g, α_h) is non-zero if and only if $r(E, V_1) = 2$ and $r(E, V_2) = 0$. After letting (P_1, Q_1) be a basis for the vector space $\hom(V_1, E(H) \otimes L)^{G_{\mathbb{Q}}}$, and letting (v, w) be a basis for V_1, one then has

$$R_{g_\alpha,h_\alpha}(E, \varrho_{gh}) = R_{g_\beta,h_\beta}(E, \varrho_{gh}) = \log_{E,p}(P_1(v)) \log_{E,p}(Q_1(w)) - \log_{E,p}(P_1(w)) \log_{E,p}(Q_1(v)).$$

The discussion is exactly the same for the Perrin-Riou regulator attached to the deformation data (α_g, β_h), but with the roles of V_1 and V_2 interchanged.

3. The main conjecture

Let $\mathscr{L}_p(f,\breve{g},\breve{h}) \in H^1_{\mathrm{dR}}(E)\otimes\Lambda_{gh}$ be the class denoted $\mathscr{L}_{fgh}$ in (6) with $g=\breve{g}$ and $h=\breve{h}$. We begin by showing that the value $\mathscr{L}_p(f,\breve{g},\breve{h})$ belongs to a specific translate of the Hodge filtration in $H^1_{\mathrm{dR}}(X_0(N))$, and explaining how its non-vanishing is directly related to that of the central critical value $L(f\otimes g\otimes h,1)=L(E,V_{gh},1)$. Let φ denote the cristalline frobenius acting on $H^1_{\mathrm{dR}}(X_0(N))[f]$, and set

$$\mathcal{E}(g,h;x) = (1-\alpha_g\alpha_h\cdot x)\times(1-\alpha_g\beta_h\cdot x)\times(1-\beta_g\alpha_h\cdot x)\times(1-\beta_g\beta_h\cdot x).$$

The operator $\mathcal{E}(g,h;\varphi)$ acts invertibly on $H^1_{\mathrm{dR}}(X_0(N))$, since the eigenvalues of φ have complex absolute value $\sqrt{p}$ while the zeroes of $\mathcal{E}(g,h,x)$ are roots of unity.

Theorem 3.1. *Let $\mathscr{L}_p(f,\breve{g},\breve{h}) \in H^1_{\mathrm{dR}}(X_0(N))[f]$ be the class attached to the triple $(f,\breve{g},\breve{h})$. Then*

$$\frac{(1-\varphi^2)}{\mathcal{E}(g,h;\varphi)}\mathscr{L}_p(f,\breve{g},\breve{h}) \quad \textit{belongs to} \quad \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_0(N))[f] = \Omega^1(X_0(N))[f]. \tag{8}$$

It vanishes for all choices $(\breve{g},\breve{h})$ of test vectors if and only if $L(E,V_{gh},1)=0$.

Proof. The class $\mathscr{L}_p(f,\breve{g},\breve{h})$ is the image of the class in $H^1_{\mathrm{dR}}(X_0(N))$ represented by the weight two overconvergent modular form $\breve{g}^{[p]}\breve{h}$. The case $(k,\ell,m)=(2,1,1)$ of [DR1, Cor. 4.13] shows that, after applying the ordinary projection,

$$e_{f^*}(\breve{g}^{[p]}\times\breve{h}) = \frac{\mathcal{E}(g,h,\varphi)}{(1-\varphi^2)}e_{f^*}(\breve{g}\times\breve{h}).$$

The very same identity holds on the slope one subspace of $H^1_{\mathrm{dR}}(X_0(N))[f]$, hence it is true on all of $H^1_{\mathrm{dR}}(X_0(N))[f]$: this follows from the same calculation as was used to deduce Cor. 4.13 of loc.cit. The assertion (8) now follows from the fact that $\breve{g}\breve{h}$ is a holomorphic modular form of weight two on $X_0(N)$, hence represents a regular differential, whose class in de Rham cohomology belongs to $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_0(N))[f]$. The second assertion follows from the main result of [HK] relating the non-vanishing of the central critical value $L(f\otimes g\otimes h,1)$ to that of an invariant trilinear form on the tensor product of the automorphic representations associated to f, g and h. □

Recall that L_{gh} is the field generated by the fourier coefficients of g and h, viewed as a subfield of $\bar{\mathbb{Q}}_p$ after fixing a p-adic embedding.

Conjecture 3.2. *Assume that $\mathscr{L}_p(f,\breve{g},\breve{h})=0$ for all pairs $(\breve{g},\breve{h})$, i.e., that $L(E,V_{gh},1)=0$. If $r(E,V_{gh})>2$, then the first derivative $\mathscr{L}_p'(f,\breve{\mathbf{g}},\breve{\mathbf{h}})$ vanishes as well. Otherwise, for all $\omega_{\breve{f}}$ in $\Omega^1(X_0(N)/\mathbb{Q})$,*

$$\left\langle \frac{(1-\varphi^2)}{\mathcal{E}(g,h;\varphi)}\mathscr{L}_p'(f,\breve{\mathbf{g}},\breve{\mathbf{h}}),\ \omega_{\breve{f}}\right\rangle \textit{ belongs to } \mathrm{Reg}_{g_\alpha,h_\alpha}(E)L_{gh},$$

where $\mathrm{Reg}_{g_\alpha,h_\alpha}(E)$ is the Perrin-Riou regulator of Definition 2.1 and $\langle\ ,\ \rangle$ denotes the Poincaré pairing on $H^1_{\mathrm{dR}}(X_0(N))$.

So far, Conjecture 3.2 remains open in all but a very few instances. The case where g and h are Eisenstein series corresponds to the conjecture of [PR], and was proved in [BDV] when g and h are associated to quadratic characters.

When g and h are theta series attached to the same imaginary quadratic field, the global points that enter into the Perrin-Riou regulator can be expressed in terms of Heegner points, and Conjecture 3.2 might be amenable to an attack via the techniques of [DLR1] .

When g and h are theta series of the same real quadratic field, the points that enter into the Perrin-Riou regulator can be expressed in terms of *Stark-Heegner points* defined over ring class fields of real quadratic fields. It would thus be of great interest to relate the quantity $\mathscr{L}_p'(f, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of Conjecture 3.2 to Stark-Heegner points. This raises a significant challenge, given that E has good rather than multiplicative reduction at the prime p of Conjecture 3.2.

All other scenarios lie tantalizingly beyond the reach of the limited repertoire of techniques currently at our disposal for constructing global points on elliptic curves, and for these cases we must largely content ourselves with numerical evidence, some of which is summarised in the next section.

4. Numerical evidence

This section presents numerical evidence which illustrates and supports Conjecture 3.2. As in [DLR1] all the computations decribed in this section were done using the Magma computer algebra system, and the authors are deeply grateful to those who develop and support it.

The experiments here were more difficult than those in [DLR1]. This is in part because here we consider a derivative of a p-adic L-series rather than the value taken by one. One must appreciate also though that the experimental range is limited to rather small *tame* levels, but in [DLR1] we were free to take p dividing the conductor of the elliptic curve, and it turned out that many of the most interesting examples found were of this type [DLR1, Examples 3.14, 3.15, 5.3, 5.4 (curves 26*b*, 52*b*), 5.5, 5.6 (curves 629*a*, 629*d*), 7.1, 7.2]. (Note there are two typographical errors in the table in [DLR1, Appendix A], the levels for Examples 3.14 and 3.15 being 69 and 161 and not as stated 57 and 35, respectively.) In the current paper we insist that the elliptic curve has good reduction at p, and so we could not revisit this rich vein of examples (it would be interesting to relax this condition though).

As a result we are not able to present such an impressive panoply of examples as in [DLR1]. Note though that the experimental work in this paper has played a rather different role. The elliptic Stark conjecture of [DLR1] was arrived at after several years of experimental work and theoretical calculation, and the detailed experimental investigation of many different cases was key to its eventual formulation. By contrast, the conjecture in the present paper was derived largely by pure thought, guided by a little experimentation, taking as a starting point the elliptic Stark conjecture and its strengthening in [DR3] and also the conjecture of Perrin-Riou (proved in [BDV]). So here the experiments played a supporting rather than leading part.

We first recall some notations. We define

$$\begin{aligned} \mathcal{E}_1(f_\alpha) &= (1-\alpha_f{}^{-2}) \\ \mathcal{E}(f_\alpha, g, h) &= (1-\alpha_f^{-1}\alpha_g\alpha_h)(1-\alpha_f^{-1}\alpha_g\beta_h)(1-\alpha_f^{-1}\beta_g\alpha_h)(1-\alpha_f^{-1}\beta_g\beta_h). \end{aligned}$$

Here $x^2 - a_p(f)x + p = (x-\alpha_f)(x-\beta_f)$ with α_f the unit root. Likewise

$$x^2 - a_p(g)x + \chi_g(p) = (x-\alpha_g)(x-\beta_g),\ x^2 - a_p(h)x + \chi_h(p) = (x-\alpha_h)(x-\beta_h)$$

where $\chi_h = \chi_g^{-1}$. We define $\mathcal{E}_1(f_\beta)$ and $\mathcal{E}(f_\beta, g, h)$ in the same way but replacing α_f by β_f. The precise incarnation of the elliptic Stark conjecture in this setting asserts that

$$\frac{\mathcal{E}_1(f_\alpha)}{\mathcal{E}(f_\alpha, g, h)} \cdot \beta_f \cdot \ell_\alpha + \frac{\mathcal{E}_1(f_\beta)}{\mathcal{E}(f_\beta, g, h)} \cdot \alpha_f \cdot \ell_\beta = \frac{R_{\alpha\alpha,\beta\beta}(E, \varrho_{gh})}{[\omega_E, \phi(\omega_E)]}.$$

Here ℓ_α and ℓ_β are defined via the equation

$$\ell_\alpha \cdot f_\alpha + \ell_\beta \cdot f_\beta = \frac{\partial}{\partial k}\left(e_f\left(d^{1-k}(\mathbf{g}_k)\cdot \mathbf{h}_k^{[p]}\right)\right)\Big|_{k=1}$$

where $\mathbf{g}$ is the Hida family through g_α and $\mathbf{h}$ that through h_α. Note these Hida families are unique in the examples considered (except for the case of an Eisenstein series $E_1(1,\chi)$ with $\chi(p) = 1$ when there are two Eisenstein families and also a CM-family meeting in weight one).

The values ℓ_α and ℓ_β were found numerically using an extension of the methods in [La]. The computation of ℓ_β was greatly facilitated by a beautiful observation of David Loeffler on computing higher slope projections of overconvergent modular forms. See the final sentence of [LSZ, Section 6.3].

The period $[\omega_E, \phi(\omega_E)]$ can be computed by using Kedlaya's algorithm [K]. Namely, relative to a suitable Weierstrass model for E, the regular differential ω_E is equal to dx/y and Kedlaya's algorithm allows the calculation of

$$\phi(dx/y) \equiv a dx/y + b x dx/y$$

in de Rham cohomology, and from this one obtains $[\omega_E, \phi(\omega_E)] = b$, since $[dx/y, dx/y] = 0$ and $[dx/y, xdx/y] = 1$.

In all of the numerical examples we considered, the Perrin-Riou regulator is "factorisable" (in the language [DLR1, Introduction]) and

$$R_{\alpha\alpha,\beta\beta}(E, \varrho_{gh}) = \log_{E,p}(P) \cdot \log_{E,p}(Q)$$

for points P and Q in the $\alpha_g\alpha_h$ and $\beta_g\beta_h$ eigenspace, respectively, for Frobenius on $E(H) \otimes V_{gh}$.

4.1. **CM forms and points over class fields of imaginary quadratic fields.** The first examples involve CM forms, cuspidal and Eisenstein, and both primes which are split and primes which are inert in the associated imaginary quadratic field. Note that even in this setting the case of inert primes is still well beyond what might be reached in theory with current techniques. (For CM forms and an inert prime even the attached Hida families are less well understood, but see [DLR3] for some relevant results in this direction.)

We start with CM Eisenstein series, examples with both split and inert primes, including an inert prime for which the elliptic curve is supersingular rather than ordinary.

Example 4.1. Let χ be the (odd) quadratic character of conductor 43. Take $g = h = E_1(1, \chi)$ and let $\mathbf{g} = \mathbf{h} = \mathbf{E}(1, \chi)$ be the Eisenstein family with characters 1 and χ. Note that under the running convention on Eisenstein series the family $\mathbf{E}(1, \chi)$ specialises in weight one to the stabilisation $E_1(1, \chi)_{\chi(p)}$ and so $\alpha = \chi(p)$ and $\beta = 1$.

We have $V_{gh} = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}(\chi) \oplus \mathbb{Q}(\chi)$. With E the curve $43a$ one finds $r_{\mathrm{an}}(E) = 1$ and $r_{\mathrm{an}}(E, \chi) = 0$, with $E(\mathbb{Q})$ generated by $P = (0, -1)$. Let f be the weight two newform attached to E.

First take $p = 5$ which is inert in K and so $\chi(p) = -1$. We find that

$$\begin{aligned} \ell_\alpha &= 30543808205613888178751887205032 1 \bmod 5^{50} \\ \ell_\beta &= 17159887710805062902739977049820 59 \bmod 5^{45}. \end{aligned}$$

To 47-digits of 5-adic precision we have

$$\frac{\mathcal{E}_1(f_\alpha)}{\mathcal{E}(f_\alpha, g, h)} \cdot \beta_f \cdot \ell_\alpha + \frac{\mathcal{E}_1(f_\beta)}{\mathcal{E}(f_\beta, g, h)} \cdot \alpha_f \cdot \ell_\beta = 2 \cdot p \cdot \frac{\log_{E,p}(P)^2}{[\omega_E, \phi(\omega_E)]}. \tag{9}$$

Next take $p = 11$, which is split in K and so $\chi(p) = 1$. Then we find

$$\begin{aligned} \ell_\alpha &= 3684866901965786406005203 1 \bmod 11^{25} \\ \ell_\beta &= 2137235593546798625453411 \bmod 11^{19}, \end{aligned}$$

and in fact (9) holds again to 21-digits of 11-adic precision.

Finally consider $p = 7$, which is inert in K. Here the curve E is supersingular at p and so we do not have distinguished slope 0 and 1 stabilisations of f, but rather two stabilisations of slope $\frac{1}{2}$ which we may arbitrarily label f_α and f_β. Observe that the lefthand side of the formula above is in actual fact symmetric in α and β. We compute

$$\begin{aligned} \ell_\alpha &= (660006332767319361052 93\sqrt{-7} + 652578133435989411694779) \cdot 7 \bmod 7^{28} \\ \ell_\beta &= (-660006332767319361052 93\sqrt{-7} + 652578133435989411694779) \cdot 7 \bmod 7^{28}. \end{aligned}$$

To 28 digits of 7-adic precision we find now

$$\frac{\mathcal{E}_1(f_\alpha)}{\mathcal{E}(f_\alpha,g,h)}\cdot\beta_f\cdot\ell_\alpha+\frac{\mathcal{E}_1(f_\beta)}{\mathcal{E}(f_\beta,g,h)}\cdot\alpha_f\cdot\ell_\beta=\frac{1}{2}\cdot\frac{\log_{E,p}(P)^2}{[\omega_E,\phi(\omega_E)]}.$$

The next example involves a cuspidal form with projective image S_3 and split prime.

Example 4.2. Let χ be the (odd) quadratic character of conductor 83. We take $g\in S_1(83,\chi)$ and $h=E_1(1,\chi)$. Let $K=\mathbb{Q}(\sqrt{-83})$ and H be the Hilbert class field of K, and write $\mathrm{Gal}\,(H/K)=\langle\sigma\rangle$. Then g is the theta series θ_{ψ_g} where ψ_g is a cubic character of $\mathrm{Gal}\,(H/K)$. The representation ρ_g is equal to the induced representation V_{ψ_g}, and $V_{gh}=V_{\psi_g}\oplus V_{\psi_g}$. Here V_{ψ_g} is an L-vector space with $L=\mathbb{Q}(\zeta_3)$.

Let E be the elliptic curve labelled $83a$ and f the attached newform. The representation V_{ψ_g} occurs in the Mordell-Weil group of E with multiplicity 1. Precisely, the V_{ψ_g} component of the Mordell-Weil group has as an L-basis P and $\sigma(P)$, where,

$$P=(t,-t^2-2)\in E(H)\text{ with }t^3-t^2+t-2=0$$

is the Heegner point in $E(H)$.

Let $p=7$, which splits in K. We have $\alpha_g=\zeta_3$, $\beta_g=\zeta_3^2$, $\alpha_h=\chi(p)=1$ and $\beta_h=1$. Define

$$P_{\psi_g}^{\zeta_3}=P+\zeta_3^2\cdot\sigma(P)+\zeta_3\cdot\sigma^2(P),\ P_{\psi_g}^{\zeta_3^2}=P+\zeta_3\cdot\sigma(P)+\zeta_3^2\cdot\sigma^2(P).$$

We find

$$\begin{aligned}\ell_\alpha &= 866900775985779195132568 47\cdot 7^2 \bmod 7^{35}\\ \ell_\beta &= 111304939462498464367895297 \bmod 7^{29}\end{aligned}$$

and

$$\frac{\mathcal{E}_1(f_\alpha)}{\mathcal{E}(f_\alpha,g,h)}\cdot\beta_f\cdot\ell_\alpha+\frac{\mathcal{E}_1(f_\beta)}{\mathcal{E}(f_\beta,g,h)}\cdot\alpha_f\cdot\ell_\beta=-p\cdot\frac{\log_{E,p}(P_{\psi_g}^{\zeta_3})\log_{E,p}(P_{\psi_g}^{\zeta_3^2})}{16[\omega_E,\phi(\omega_E)]}$$

to 31-digits of 7-adic precision, as predicted.

Let us now take f and g as before with $p=7$, but instead $h=g$. We now have

$$V_{gh}=V_{\psi_g}\oplus\mathbb{Q}(\chi)\oplus\mathbb{Q}$$

occurring in the Mordell-Weil group of E with ranks $1,0,1$. Here one may choose stabilisations in two essentially different ways, so that

$$(\alpha_g\cdot\alpha_h,\beta_g\cdot\beta_h)=(\zeta_3^2,\zeta_3)\text{ or }(1,1).$$

However, V_{ψ_g} does not have 1 as a frobenius eigenvalue, and likewise the trivial representation does not have either ζ_3 or ζ_3^2. So under both choices of stabilisations the regulator vanishes. Experimentally one observes that in both cases each of the coefficients ℓ_α and ℓ_β vanishes numerically, which is consistent with Conjecture 3.2.

4.2. **Points over real quadratic fields.** Weight one forms whose associated representations are induced from characters of a real quadratic field but no imaginary quadratic field are rather rare in small level. The smallest level is 145, where one finds two (up to Galois conjuguate) distinct forms, which have projective image D_8. In [DLR1, Example 4.1] we consider one of these, and the elliptic curve $17a$ with prime $p=17$; here the ranks of the appropriate parts of the Mordell-Weil group are favourable to test Conjecture 3.2, and all computations can be done in tame level 145. In the current setting though we require a prime of good reduction, and so this example is not appropriate. Unfortunately there are no suitable examples with D_8 forms within computational reach. The next example instead has D_4 projective image; here the representation is induced from a character of a real quadratic field, but also two imaginary quadratic fields (so it would have been equally at home in the preceding section c.f. [DLR1, Example 4.3]).

Example 4.3. Let $\chi_{39} = \chi_3 \cdot \chi_{13}$ be the (odd) quadratic character of conductor 39. The space $S_1(39, \chi_{39})$ is one dimensional and spanned by the eigenform

$$g = q - q^3 - q^4 + q^9 + q^{12} - q^{13} + \cdots .$$

The representation ϱ_g is induced from characters of $\mathbb{Q}(\sqrt{13})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-39})$. Letting $h = g$ we find that

$$V_{gh} = \mathbb{Q} \oplus \mathbb{Q}(\chi_{13}) \oplus \mathbb{Q}(\chi_3) \oplus \mathbb{Q}(\chi_{39}).$$

The curve

$$E : y^2 + xy = x^3 + x^2 - 4x - 5,$$

labelled $39a$ in Cremona's tables, has

ranks $0, 1, 1, 0$ over the fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{13})$,$\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-39})$, respectively.

Let $p = 5$. Note this prime is *inert* in the real quadratic field $\mathbb{Q}(\sqrt{13})$ (the case of split primes is exceptional, see [DLR2] for a discussion of that setting). We have Hecke polynomial $x^2 - a_5(g)x + \chi(5) = x^2 + 1$, and we take $\alpha_g = i$ and $\beta_g = -i$. Since $h = g$ we may choose stabilisations in two essentially different manners, so that either

$$(\alpha_g \cdot \alpha_h, \beta_g \cdot \beta_h) = (-1, -1) \text{ or } (+1, +1).$$

Observe 5 is inert in $\mathbb{Q}(\sqrt{-3})$ and split in $\mathbb{Q}(\sqrt{-39})$. So with the latter "$(+1, +1)$ choice" of stabilisations an inspection of the Mordell-Weil ranks shows the regulator is zero. A numerical computation of the Perrin-Riou derivative in that setting shows also the values ℓ_α and ℓ_β vanish.

The more interesting "$(-1, -1)$ choice" of stabilisation yields the following. We take points

$$P_{13} = (11, (-21\sqrt{13} - 11)/2) \text{ and } P_3 = (-7/3, (-13\sqrt{-3} + 21)/18)$$

which lie in $E(\mathbb{Q}(\sqrt{13}))$ and $E(\mathbb{Q}(\sqrt{-3}))$, in the (-1)-eigenspace for frobenius in each case. Numerically we find

$$\begin{aligned} \ell_\alpha &= -1771831980707645206022405768^2 \bmod 5^{40} \\ \ell_\beta &= -819594150830239792388179918 \bmod 5^{35} \end{aligned}$$

and

$$\frac{\mathcal{E}_1(f_\alpha)}{\mathcal{E}(f_\alpha, g, h)} \cdot \beta_f \cdot \ell_\alpha + \frac{\mathcal{E}_1(f_\beta)}{\mathcal{E}(f_\beta, g, h)} \cdot \alpha_f \cdot \ell_\beta = \frac{i}{8} \cdot p \cdot \frac{\log_{E,p}(P_{13}) \log_{E,p}(P_3)}{[\omega_E, \phi(\omega_E)]}$$

to 35-digits of 5-adic precision, as predicted.

4.3. **Points over cyclotomic fields.** The next example involves (non-CM) Eisenstein series and points over cyclotomic fields. Note again that the two cyclotomic field examples in [DLR1, Examples 7.1 and 7.2] were not suitable because the elliptic curves in those examples had multiplicative reduction at the prime p. Here the authors were rather fortunate to find another example in which the curve does have good reduction. (In this example we include some additional details which we suppressed in earlier ones, to avoid boring the reader.)

Example 4.4. Let χ be the (odd) quadratic character of conductor 11, and ε an (even) cubic character of conductor 9. Let

$$f = q + q^2 - q^4 + 4q^5 - 2q^7 - 3q^8 + 4q^{10} + q^{11} + \cdots$$

be the weight 2 newform attached to the elliptic curve

$$E : y^2 + xy = x^3 - x^2 - 15x + 8$$

labelled $99c$ in Cremona's tables. Taking $p = 7$ we see that f is ordinary at p and the Hecke polynomial $x^2 - a_p(f)x + p = (x - \alpha_f)(x - \beta_f)$ has distinct roots $\alpha_f, \beta_f \in \mathbb{Q}_7$, with α_f of valuation zero. We denote by

$$f_{\alpha_f} = f(q) - \beta_f f(q^p), \quad f_{\beta_f} = f(q) - \alpha_f f(q^p)$$

the ordinary and slope 1 stabilisations, respectively, and define ℓ_α and ℓ_β by

$$\ell_\alpha \cdot f_{\alpha_f} + \ell_\beta \cdot f_{\beta_f} = \frac{\partial}{\partial k}\left(e_f\left(d^{1-k}\left(E_k(1,\chi\cdot\varepsilon)\right)\cdot E_k(\overline{\varepsilon},\overline{\chi})^{[p]}\right)\right)\Big|_{k=1}.$$

Let $\varepsilon(2) = \zeta_6 - 1$ and embed $\mathbb{Q}(\varepsilon)$ into $\mathbb{Q}_7$ by sending ζ_6 to the 6th root of unit congruent to 3 modulo 7. Then with this embedding one computes

$$\begin{aligned} \ell_\alpha &= -950947249173860558304772145 05\cdot 7^2 \bmod 7^{35} \\ \ell_\beta &= 2065089524429283082254754 6879 \bmod 7^{31}. \end{aligned}$$

With $g = E_1(1,\chi\cdot\varepsilon)$ and $h = E_1(\overline{\varepsilon},\overline{\chi})$ one has

$$V_{gh} = \overline{\varepsilon} \oplus \overline{\chi} \oplus \chi \oplus \varepsilon$$

and one finds that

$$r_{\mathrm{an}}(E,\chi) = 0, \quad r_{\mathrm{an}}(E,\varepsilon) = r_{\mathrm{an}}(E,\overline{\varepsilon}) = 1$$

and likewise for algebraic ranks. The cyclic cubic field attached to the character ε is

$$K = \mathbb{Q}(a), \quad a^3 - 3a - 1 = 0$$

and the Mordell-Weil group of E over K is generated as a $\mathrm{Gal}\,(K/\mathbb{Q})$-module by

$$P = (3a^2 - 6a - 2, 12a^2 - 15a - 8).$$

Define

$$P_\varepsilon = P + \zeta_3\cdot\sigma(P) + \zeta_3^2\cdot\sigma^2(P), \quad P_{\overline{\varepsilon}} = P + \zeta_3^2\cdot\sigma(P) + \zeta_3\cdot\sigma^2(P)$$

where $\zeta_3 = \zeta_6 - 1$ and $\mathrm{Gal}\,(K/\mathbb{Q}) = \langle\sigma\rangle$. Testing Conjecture 3.2 also requires the numerical calculation of

$$[\omega_E, \phi(\omega_E)] = -4444791943294574370736083602370 27\cdot 7 \bmod 7^{40}.$$

Then with the notation for Euler factors defined earlier in the paper one finds that

$$\frac{\mathcal{E}_1(f_\alpha)}{\mathcal{E}(f_\alpha,g,h)}\cdot\beta_f\cdot\ell_\alpha + \frac{\mathcal{E}_1(f_\beta)}{\mathcal{E}(f_\beta,g,h)}\cdot\alpha_f\cdot\ell_\beta = \frac{1}{12}\cdot(-\zeta_6+2)\cdot p\cdot\frac{\log_{E,7}(P_\varepsilon)\log_{E,7}(P_{\overline{\varepsilon}})}{[\omega_E,\phi(\omega_E)]}$$

to 32 digits of 7-adic precision.

Note here that $\chi(7) = -1$ and $\varepsilon(7) = \zeta_6 - 1$, and so in the setting of [DLR1] the 7-adic iterated integral attached to the triple (f,g,h) would in fact have vanished.

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