ELLIPTIC STARK CONJECTURES AT IRREGULAR WEIGHT ONE CM FORMS

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To Henri Darmon on his 60th birthday,
with gratitude, love and admiration.

Abstract. In a paper [DLR3] by Henri Darmon and the authors a p-adic elliptic Stark conjecture is proposed for triples of eigenforms (f, g, h) of weight $(2, 1, 1)$ at primes p for which g is irregular. In the case where g is the θ-series of a character of an imaginary quadratic field, we give an explicit conjectural expression for a mysterious p-adic period which occurs in the formulation of that conjecture. We verify this numerically by completing two examples left unfinished in that paper.

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1. Review of the elliptic Stark conjecture at irregular primes

This note is an addendum to [DLR3], offered as a gift from the last two authors of that paper to the first. We identify explicitly a mysterious p-adic number which appears in Examples 3.5 and 3.6 of that paper—a puzzle which frustrated the authors of that paper for seven years and was eventually left unsolved.

Crucial for our purposes have been the recent developments on Katz's p-adic L-function at characters that are trivial at p by various authors, including Betina-Dimitrov [BeDi], Hsieh-Chida [ChHs], Dimitrov-Maksoud [DiMa], Buyukboduk-Sakamoto [BuSa], and the ongoing work [KuRo] of Kumar and the second author of this article.

We briefly recall the notation and main conjecture of [DLR3]. Let $g = \sum_{n\geq 1} a_n(g)q^n$ be a normalized eigenform of weight 1, level N_g and nebentypus χ. Let $p \nmid N_g$ be an irregular prime, that is to say, one for which the p-th Hecke polynomial $T^2 - a_p(g)T + \chi(p) = (T-\alpha)^2$ has a double root. Let L denote the finite extension of $\mathbb{Q}$ generated by the Fourier coefficients of g and α, and let $g_\alpha \in S_1(N_g p, \chi)_L$ denote the p-stablization of g on which U_p acts with eigenvalue α.

Let $\mathbb{T}$ denote the Hecke L-algebra acting on $S_1(N_g p, \chi)_L$ and write $I_{g_\alpha} \subset \mathbb{T}$ for the annihilator ideal of g_α. The g_α-eigenspace is

$$S_1(N_g p, \chi)_L[g_\alpha] := \ker(I_{g_\alpha} | S_1(N_g p, \chi)_L) = \langle g_\alpha, g(q^p)\rangle.$$

Fix embeddings $L \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$. Let $S_1^{\mathrm{ord}}(N_g, \chi)_{L_p}$ denote the L_p-vector space of overconvergent ordinary p-adic modular forms.

The generalised eigenspace

$$S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]] := \cup_{n\geq 1} \ker(I_{g_\alpha}^n | S_1^{\mathrm{ord}}(N_g,\chi)_{L_p})$$

associated to g_α is expected to be simply $S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]] = \ker(I_{g_\alpha}^2 | S_1^{\mathrm{ord}}(N_g,\chi)_{L_p})$, as conjectured in [DLR2, §4], [DLR3, §3.1], and we shall assume throughout this is the case. This L_p-vector space decomposes naturally as a direct sum

$$S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]] = S_1(N_g p,\chi)_{L_p}[g_\alpha] \oplus S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]]_0,$$

where

$$S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]]_0 = \{\phi \in S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]] : a_1(\phi) = a_p(\phi) = 0\}.$$

The latter space is expected to be 2-dimensional over L_p (cf. [DLR2, §4], [DLR3, §3.1]) and the work of Betina and Dimitrov [BeDi] show this is indeed the case when g is a CM form, conditionally on the non-vanishing of certain $\mathcal{L}$-invariants.

Let $\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathbf{GL}(V_g) = \mathbf{GL}_2(L)$ denote the odd two-dimensional Artin representation associated to g by Deligne-Serre and

$$W_g = \mathrm{Ad}(\varrho_g) := \mathrm{End}^0(V_g)$$

denote the *adjoint representation* associated to g, equipped with the conjugation action of $G_{\mathbb{Q}}$ on the space of trace zero endomorphisms of V_g. Let H denote the number field cut out by W_g. As shown e.g. in [DLR1, Proposition 1.5], the L-vector space

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(W_g, \mathcal{O}_H^\times \otimes L) \tag{1}$$

is one-dimensional, and we let u_g denote a basis, which is a well-defined element up to scalars in $L^\times$. For every prime $\ell \nmid N_g$ we have

$$\dim(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_{\mathbb{Q}}} = \begin{cases} 2 & \text{if } g \text{ is regular at } \ell \\ 4 & \text{if } g \text{ is irregular at } \ell. \end{cases} \tag{2}$$

When if g is regular at ℓ, there is thus a well-defined element

$$u_g(\ell) \in (\mathcal{O}_H[1/\ell]^\times \otimes W_g)^{G_{\mathbb{Q}}} \tag{3}$$

up to scaling and multiples of u_g. According to [DLR2, Th. 5.3] there exists a canonical isomorphism of two-dimensional L_p-vector spaces

$$\Phi : \frac{W_g \otimes_L L_p}{L_p \cdot \log_p(u_g)} \longrightarrow S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]]_0 \tag{4}$$

satisfying $a_\ell(\Phi(w)) = 0$ for all primes $\ell \nmid N_g p$ at which g is irregular. Hence $\Phi(w)$ is supported at regular primes, and loc. cit. provides a recipe for the ℓ-th Fourier coefficient of $\Phi(w)$ in terms of w and the p-adic logarithms of u_g and $u_g(\ell)$.

In addition to g, consider an eigenform $f \in S_2(N_f)$ of weight 2, trivial nebentypus and rational Fourier coefficients, and an eigenform $h \in S_1(N_h,\bar\chi)_L$ of weight 1 and nebentypus $\bar\chi$, the inverse of that of g. In [DLR3] we associated to the triple (f,g,h) two invariants of an analytic and motivic nature, respectively. Namely:

- an overconvergent generalized eigenform notated $\Phi_{f,g_\alpha,h} \in S_1^{\mathrm{ord}}(N_g,\chi)[[g_\alpha]]_0$ and
- an explicit regulator $R_p(f,g,h) \in W_g \otimes_L L_p$ involving p-adic logarithms of units in the field H_{gh} cut out by the Artin representation $\varrho_{gh} = \varrho_g \otimes \varrho_h$ and of points on the Mordell-Weil group $E(H_{gh})$ of the elliptic curve E associated to f.

The main conjecture of the paper was that when $\mathrm{ord}_{s=1} L(E,\varrho_{gh},s) = 2$ we have up to a factor in $L^\times$ that

$$\Phi_{f,g_\alpha,h} \stackrel{?}{=} \frac{\Phi(R_p(f,g,h))}{\mathcal{L}_{g_\alpha}} \tag{5}$$

for some p-adic period $\mathcal{L}_{g_\alpha} \in L_p$ that should depend only upon g. See [DLR3, §3.2] for a more precise statement, and in particular Conjecture 3.2 therein.

This conjecture was tested numerically in [DLR3, Example 3.9] in the case ϱ_g has D_4 projective image, as an explicit expression for $\mathcal{L}_{g_\alpha}$ is presented in that special setting [DLR3, Conjecture 3.7]. It was tested numerically in the setting of S_3 projective image in [DLR3, Examples 3.5 and 3.6]. In that case no explicit expression for $\mathcal{L}_{g_\alpha}$ was known, and in the experiments the unknown quantity $\mathcal{L}_{g_\alpha}$ was cancelled by fixing g and considering different choices of f and h.

The purpose of this note is to give an *explicit* expression for the period $\mathcal{L}_{g_\alpha}$ in all cases in which g is an arbitrary CM form, prove that it is consistent with the numerical experiments obtained in the D_4 examples, and verify (5) numerically by completing the two S_3 examples left unfinished in that paper.

2. The period $\mathcal{L}_{g_\alpha}$ in the CM case

2.1. **A description of the period.** Let K be an imaginary quadratic field of discriminant $-D$ and let $\mathcal{O}_K$ denote its ring of integers. Write $\epsilon_K : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ for the quadratic character associated to $K/\mathbb{Q}$.

Let $\psi : G_K \to L^\times$ be a finite order character of conductor $\mathfrak{f} \subseteq \mathcal{O}_K$ with values in a finite extension $L/\mathbb{Q}$. Set $N = D\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{f})$ and let

$$g = \theta_\psi = \sum_\beta \psi(\beta) q^{\mathbf{N}_{K/\mathbb{Q}}(\beta)} \in M_1(N, \epsilon_K\chi)_L \qquad (6)$$

denote the theta series associated to ψ, where the sum ranges over all ideals β of $\mathcal{O}_K$. The eigenform g has weight 1, level N and nebentype character $\epsilon_K\chi$ where $\chi : (\mathbb{Z}/\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{f})\mathbb{Z})^\times \longrightarrow L^\times$ is the transfer of ψ to $G_\mathbb{Q}$. Moreover g has CM by K, which amounts to saying that $g \otimes \epsilon_K = g$, and the Artin representation associated to g is $\rho_g = \mathrm{Ind}_\mathbb{Q}^K(\psi)$, the induced representation of ψ from G_K to $G_\mathbb{Q}$. In particular, for a prime $\ell = \mathfrak{L}\bar{\mathfrak{L}}$ that splits in K we have $a_\ell(g) = \psi(\mathfrak{L}) + \psi(\bar{\mathfrak{L}})$, while at primes ℓ that remain inert in K we have $a_\ell(g) = 0$.

Let $c(z) = \bar{z}$ denote complex conjugation, define the conjugate of ψ by the rule $\psi^c(\sigma) = \psi(c^{-1}\sigma c)$ and set

$$\psi_\heartsuit = \psi/\psi^c. \qquad (7)$$

We assume throughout $\psi_\heartsuit$ is non-trivial, ensuring g is cuspidal. Let H be the (non-trivial) extension of K cut out by $\psi_\heartsuit$ and $C = \mathrm{Gal}(H/K)$ denote its Galois group. By construction $\psi_\heartsuit$ is anticyclotomic in the sense that $\psi_\heartsuit^c = \bar{\psi}_\heartsuit$.

The $\psi_\heartsuit$-isotypical component

$$\mathcal{O}_H^\times[\psi_\heartsuit] = \{x \in \mathcal{O}_H^\times \otimes L : x^\sigma = \psi_\heartsuit(\sigma)x, \ \forall \sigma \in C\}$$

of the group of global units is 1-dimensional over L. Fix a generator of this space, which may be then expressed as

$$u_{\psi_\heartsuit} = \sum_{\sigma \in C} \psi_\heartsuit(\sigma)\sigma^{-1}(u) \in \mathcal{O}_H^\times \otimes L$$

for some choice of global unit $u \in \mathcal{O}_H^\times$. Any other choice yields a multiple of $u_{\psi_\heartsuit}$ by a scalar in L. Set also

$$\bar{u}_{\psi_\heartsuit} = \sum_{\sigma \in C} \psi_\heartsuit(\sigma) c\sigma^{-1}(u),$$

which as one readily checks is a basis of $\mathcal{O}_H^\times[\bar{\psi}_\heartsuit]$. Since $u_{\bar{\psi}_\heartsuit}$ is also a non-trivial element of the 1-dimensional L-vector space $\mathcal{O}_H^\times[\bar{\psi}_\heartsuit]$, it follows that

$$u_{\bar{\psi}_\heartsuit} = \bar{u}_{\psi_\heartsuit} \pmod{L^\times},$$

but some of the formulas below are sensible to the scalar difference between these two units, so we keep the notational distinction. In any case the generator u_g of $\mathrm{Hom}_G(W_g, \mathcal{O}_H^\times \otimes L)$ chosen in (1) satisfies

$$u_g(W_g) = \langle u_{\psi_\heartsuit}, u_{\bar{\psi}_\heartsuit} \rangle_L = \langle u_{\psi_\heartsuit}, \bar{u}_{\psi_\heartsuit} \rangle_L.$$

Let p be a prime that splits in K and fix an embedding $K \subset H \hookrightarrow \bar{\mathbb{Q}}_p$. The latter determines a prime ideal $\wp$ in K above p such that $p\mathcal{O}_K = \wp\bar{\wp}$. Assume $\psi(\wp) = \psi(\bar{\wp})$, so that $g = \theta_\psi$ is irregular at p, that is to say g admits a unique p-stabilization g_α on which U_p acts with eigenvalue $\alpha = \psi(\wp)$. It follows that $\psi(\wp) = 1$ and thus $H \subset \mathbb{Q}_p$. In light of (2), we also have $\dim(\mathcal{O}_H[1/p]^\times \otimes W_g)^{G_\mathbb{Q}} = 4$ and hence a priori there is no natural definition of a canonical p-unit $u_g(p)$ up to scaling and multiples of u_g as in (3), because $\dim(\mathcal{O}_H[1/p]^\times \otimes W_g)^{G_\mathbb{Q}}/(\mathcal{O}_H^\times \otimes W_g)^{G_\mathbb{Q}} = 3 > 1$. Nevertheless, we can circumvent this by noticing that there is a decomposition of $\mathrm{Gal}\,(H/\mathbb{Q})$-modules $W_g \simeq L\epsilon_K \oplus V_{\psi_\heartsuit}$ as the direct sum of the 1-dimensional representation afforded by the quadratic character ϵ_K and the 2-dimensional irreducible representation induced from the character $\psi_\heartsuit$.

We may consider the group of $\wp$-units $\mathcal{O}_H[1/\wp]^\times$ as a C-module. Then the C-module $V_{\psi_\heartsuit}$ decomposes as $V_{\psi_\heartsuit} \simeq L\psi_\heartsuit \oplus L{\psi_\heartsuit}^{-1}$ and

$$\dim(\mathcal{O}_H[1/\wp]^\times[\psi_\heartsuit]) = 2, \quad \dim(\mathcal{O}_H[1/\wp]^\times[\psi_\heartsuit])/\langle u_{\psi_\heartsuit} \rangle = 1.$$

As a consequence we obtain that there does exist a canonical $\wp$-unit $u_g(\wp)$ up to multiples of $u_{\psi_\heartsuit}$, which may be constructed as follows. Let $\wp_H$ denote the prime ideal of H above $\wp$ determined by the choice of embedding $H \hookrightarrow \mathbb{Q}_p$ made above. Then, every prime of H above $\wp$ is $\sigma(\wp_H)$ for some $\sigma \in C$. Choose a $y_0 \in \mathcal{O}_H[1/\wp_H]^\times$ with valuation 1 at $\wp_H$ and define

$$u_{\psi_\heartsuit}(\wp) = \sum_{\sigma \in C} \psi_\heartsuit(\sigma)\sigma^{-1}(y_0),$$

$$\bar{u}_{\psi_\heartsuit}(\wp) = \sum_{\sigma \in C} \psi_\heartsuit(\sigma)c\sigma^{-1}(y_0).$$

Finally, let $u_K(p) \in \mathcal{O}_K[1/\wp]^\times$ be any $\wp$-unit of valuation 1 at $\wp$. Note that the analogous $\bar{\wp}$-unit $u_K(\bar{\wp})$ satisfies $\log_p u_K(\bar{\wp}) = -\log_p u_K(p)$; we caution the reader that in some situations the choice of one or another may be harmless if the final output is unaffected up to scalars in $L^\times$, but here the sign difference is relevant.

Define

$$\mathcal{L}_{K,\psi_\heartsuit} = \frac{\log_p(u_{\psi_\heartsuit})\log_p \bar{u}_{\psi_\heartsuit}(\wp) - \log_p(\bar{u}_{\psi_\heartsuit})\log_p u_{\psi_\heartsuit}(\wp)}{\log_p(\bar{u}_{\psi_\heartsuit})} + 2\log_p(u_K(p)) \tag{8}$$

and

$$\mathcal{L}_{K,\bar{\psi}_\heartsuit} = \frac{\log_p(u_{\bar{\psi}_\heartsuit})\log_p \bar{u}_{\bar{\psi}_\heartsuit}(\wp) - \log_p(\bar{u}_{\bar{\psi}_\heartsuit})\log_p u_{\bar{\psi}_\heartsuit}(\wp)}{\log_p(\bar{u}_{\bar{\psi}_\heartsuit})} + 2\log_p(u_K(p)).$$

Building on the earlier works of Betina-Dimitrov [BeDi], Hsieh-Chida [ChHs], Dimitrov-Maksoud [DiMa], Buyukboduk-Sakamoto [BuSa], and the ongoing work of Kumar-Rotger [KuRo] as part of the Ph.D thesis of the former at Duke University, we propose the following:

Conjecture 2.1. *Equation* (5) *holds true with*

$$\mathcal{L}_{g_\alpha} = \log_p(\bar{u}_{\psi_\heartsuit})\log_p(\bar{u}_{\bar{\psi}_\heartsuit})(\mathcal{L}_{K,\psi_\heartsuit} + \mathcal{L}_{K,\bar{\psi}_\heartsuit}). \tag{9}$$

The next two subsections are devoted to summarizing the numerical evidence we have gathered in support of the above conjecture.

2.2. **Quartic CM characters.** Assume in this subsection that ψ is an anticyclotomic character of order 4, that is to say, $\psi^c = \bar{\psi}$ and $\psi_\heartsuit = \bar{\psi}_\heartsuit = \psi^2$ is a quadratic character.

Letting H_g and H denote as usual the extension of K cut out by $\mathrm{Ind}_{\mathbb{Q}}^K(\psi)$ and $\psi_\heartsuit$ respectively, one readily notes that $H_g \supset H$, both extensions are Galois over $\mathbb{Q}$ and

$$\mathrm{Gal}\,(H_g/\mathbb{Q}) = D_8, \quad \mathrm{Gal}\,(H/\mathbb{Q}) = D_4 = C_2 \times C_2.$$

There are thus quadratic fields F and K', real and imaginary respectively, such that $H = FK = FK'$. Label the elements in $\mathrm{Gal}\,(H/\mathbb{Q})$ as $\{1, c, \tau, c\tau\}$ where we set K (resp. K') to be the fixed field of τ (resp. of $c\tau$).

Note that $u_{\psi_\heartsuit} = u_{\bar{\psi}_\heartsuit} = u_F$ is the fundamental unit of F. This implies

$$\mathcal{L}_{K,\psi_\heartsuit} = \mathcal{L}_{K,\bar{\psi}_\heartsuit} = \log_p(\bar{u}_{\psi_\heartsuit}(\wp)) - \log_p(u_{\psi_\heartsuit}(\wp)) + 2\log_p(u_K(p)).$$

Since $\psi_\heartsuit(\wp) = 1$, the prime p splits completely in H, say as $p = \mathcal{P}\bar{\mathcal{P}}\mathcal{P}^\tau\bar{\mathcal{P}}^\tau$. Let $y_0 \in \mathcal{O}_H[1/\mathcal{P}]^\times$ be a $\mathcal{P}$-unit with $\mathrm{ord}_{\mathcal{P}}(y_0) = 1$. Then

$$\log_p u_{\psi_\heartsuit}(\wp) = \log_p y_0 - \log_p y_0^\tau,$$
$$\log_p \bar{u}_{\psi_\heartsuit}(\wp) = \log_p \bar{y}_0 - \log_p \bar{y}_0^\tau.$$

Hence

$$\begin{aligned}\mathcal{L}_{K,\psi_\heartsuit} = \mathcal{L}_{K,\bar{\psi}_\heartsuit} &= \log_p \bar{y}_0 - \log_p \bar{y}_0^\tau - \log_p y_0 + \log_p y_0^\tau + 2\log_p y_0 + 2\log_p y_0^\tau \\ &= 2\log_p y_0 + 2\log_p \bar{y}_0 + 4\log_p y_0^\tau;\end{aligned}$$

the latter equality holds because $y_0\bar{y}_0 y_0^\tau \bar{y}_0{}^\tau = \pm p$ and hence the logarithm of this product vanishes.

The p-unit in K (whose valuation is 1 at $\wp$, and 0 at $\bar{\wp}$) is

$$\log_p u_K(p) = \log_p y_0 + \log_p y_0^\tau$$

while the p-unit in K' (whose valuation is 1 at $\wp' = \mathcal{P}\mathcal{P}^{c\tau}$, and 0 at $\bar{\wp}'$) is

$$\log_p u_{K'}(p) = \log_p y_0 + \log_p \bar{y}_0^\tau = -(\log_p \bar{y}_0 + \log_p y_0^\tau).$$

We conclude that

$$\mathcal{L}_{K,\psi_\heartsuit} + \mathcal{L}_{K,\bar{\psi}_\heartsuit} = 4(\log_p u_K(p) - \log_p u_{K'}(p)),$$

and hence

$$\mathcal{L}_{g_\alpha} = \log_p(u_F)^2(\log_p u_K(p) - \log_p u_{K'}(p)) \pmod{L^\times}$$

which is [DLR3, Conjecture 3.7].

In other words, the numerical computations accomplished in [DLR3, §3.4] provide numerical evidence in support of our Conjecture 2.1.

2.3. **Cubic CM characters: unfinished business.** Assume in this paragraph that $g \in S_1(D, \chi_K)$ is a theta series attached to a cubic anticyclotomic class character ψ of K. Then $\psi_\heartsuit = \bar{\psi}$ and the cyclic extension $H = H_g$ cut out by ψ (or $\psi_\heartsuit$) is Galois over $\mathbb{Q}$ with Galois group $\mathrm{Gal}\,(H/\mathbb{Q}) = D_6 = S_3$. The roots of the ℓ-th Hecke polynomial $x^2 - a_\ell(g)x + \chi_K(\ell)$ are

$$\begin{cases} \alpha_{g,\ell} = 1, \quad \beta_{g,\ell} = -1 & \text{if } \ell \text{ is inert in } K, \\ \alpha_{g,\ell} = \psi(\mathfrak{L}), \quad \beta_{g,\ell} = \psi(\bar{\mathfrak{L}}) & \text{if } \ell = \mathfrak{L}\bar{\mathfrak{L}} \text{ splits in } K. \end{cases}$$

Recall the notations and conventions adopted in the general set-up fixed in §2.1, so that g is irregular at $p = \wp\bar{\wp}$ with $\psi(\wp) = \psi_\heartsuit(\wp) = 1$ and $x^2 - a_p(g)x + 1 = (x-1)^2$.

We now are able to complete the calculations in [DLR3, Examples 3.5 and 3.6]. These were eventually left unfinished after a long struggle through the failure of the authors to identify explicitly the numerical constant $\mathcal{L}_{g_1}$ which occurs on the denominator.

Example 2.2. We continue [DLR3, Example 3.5], first briefly recalling the setting. Let χ be the quadratic character of conductor 83. The space $S_1(83,\chi)$ is one dimensional and spanned by the S_3-form

$$g = q - q^3 + q^4 - q^7 - q^{11} - q^{12} + q^{16} - q^{17} + \cdots .$$

Let

$$h = 3/2 + q + 2q^3 + q^4 + 2q^7 + 3q^9 + 2q^{11} + 2q^{12} + q^{16} + \cdots$$

be the Eisenstein series in $M_1(83,\chi)$. We take $p = 23$, which is split in $K = \mathbb{Q}(\sqrt{-83})$, and note $a_{23}(g) = 2$. Thus we are in the irregular case, and there is a unique p-stabilisation g_1. Let H be the Hilbert class field of K, and $L := \mathbb{Q}(w) = \mathbb{Q}(\sqrt{-3})$ with w a primitive cube root of unity. Note that g is the weight one form attached to a cubic character ψ of $\mathrm{Gal}\,(H/K)$.

We consider two curves of conductor dividing $23 \cdot 83 = 1909$ namely

$$\begin{array}{l} E_{83a} : y^2 + xy + y = x^3 + x^2 + x \\ E_{1909a} : y^2 + y = x^3 - 4x + 2 \end{array}$$

labelled $83a$ and $1909a$ in Cremona's table, with associated newforms f_{83a} and f_{1909a}.

On the analytic side, for $f = f_{83a}$ and f_{1909a}, we have the projections

$$\begin{array}{lllll} \Phi_{f,g_1,g} & = & e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times g) & = & \alpha g_1 + \tilde{\beta}\tilde{g}_1^\flat + \tilde{\gamma}\tilde{g}_2^\flat + \delta g(q^p) \\ \Phi_{f,g_1,h} & = & e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times h) & = & \alpha' g_1 + \tilde{\beta}'\tilde{g}_1^\flat + \tilde{\gamma}'\tilde{g}_2^\flat + \delta' g(q^p). \end{array}$$

Here $\tilde{g}_i^\flat$ denotes the canonical flat form $g_i^\flat$ from [DLR3, Equation (18)], but scaled to have leading coefficient 1. This is computationally more convenient, and in [DLR3, Example 3.5] only a ratio which cancels leading terms was considered. Thus $\tilde{g}_1^\flat = q^2 + \cdots$ and $\tilde{g}_2^\flat = q^3 + \cdots$.

The values of the coefficients $\alpha, \tilde{\beta}, \tilde{\gamma}, \delta$ and $\alpha', \tilde{\beta}', \tilde{\gamma}', \delta'$ were computed modulo 23^{15} and are displayed in tables in [DLR3, Example 3.5]. Since here we shall *not* be considering a ratio which cancels leading terms, we need to work with the canonical flat forms themselves; in particular, the form $g_2^\flat = \log_p(u(3)_K) \cdot \tilde{g}_2^\flat$, where $u(3)_K$ is the ratio of the roots of $x^2 - 5x + 27$ in $\mathbb{Q}_p$.

We shall focus our attention first on $\Phi_{f,g_1,h}$ for $f = f_{1909a}$. The reason is simply that the constant in $L^\times$ which arises here happens to be appealingly simple. From this calculation taken alongside the calculations of ratios in $\mathbb{Q}^\times$ from [DLR3, Example 3.5], one can immediately deduce similar results in the remaining cases (as shown below).

Dividing the value of $\tilde{\gamma}'$ for $f = f_{1909a}$ by $\log_p(u(3)_K)$, the coefficient of $g_2^\flat$ in $\Phi_{f_{1909a},g_1,h}$ is

$$\gamma'_{1909a} := -523887791842652201543/23 \bmod 23^{15}.$$

Ten years after making the original calculations one is then ecstatic to finally discover that

$$\gamma'_{1909a} = \frac{11}{23 \cdot \sqrt{-3}} \cdot \frac{\log_p(u_\psi)(\log_{E',p}(Q'_\psi))^2 - \log_p(u_{\bar{\psi}})(\log_{E',p}(Q'_{\bar{\psi}}))^2}{\mathcal{L}_{g_1}}$$

to 15 digits of 23-adic precision. The points Q'_ψ, $Q'_{\bar{\psi}} \in E_{1909a}(H) \otimes L$ are given in [DLR3, Example 3.5]. The factor $\mathcal{L}_{g_1}$ is as defined in Section 2.1. Namely, recalling that $\psi_\heartsuit = \bar{\psi}$ we have

$$\mathcal{L}_{g_1} := \log_p(u_{\bar{\psi}}) \log_p(u_\psi)(\mathcal{L}_{K,\psi_\heartsuit} + \mathcal{L}_{K,\bar{\psi}_\heartsuit})$$

where

$$\mathcal{L}_{K,\psi_\heartsuit} := \frac{\log_p(u_{\bar{\psi}}) \log_p(\bar{u}_{\bar{\psi}}(\wp)) - \log_p(\bar{u}_{\bar{\psi}}) \log_p(u_{\bar{\psi}}(\wp))}{\log_p(\bar{u}_{\bar{\psi}})} + 2\log_p u_K(p)$$

and $\mathcal{L}_{K,\bar{\psi}_\heartsuit}$ is likewise but with $\overline{\psi}$ replaced by ψ.

The units $u_\psi, u_{\bar{\psi}}, \bar{u}_\psi$ and $\bar{u}_{\bar{\psi}}$ are constructed by choosing the global unit $u \in \mathcal{O}_H^\times$ to be a root of $x^3 - 2x^2 - 2x - 1$ in H, and following the description given in Section 2.1. (Note here in fact $\log_p(u_\psi) = w \log_p(\bar{u}_{\bar{\psi}})$ and $\log_p(u_{\bar{\psi}}) = w^2 \log_p(\bar{u}_\psi)$.) Having fixed an embedding of H into $\mathbb{Q}_p$, the $\wp_H$-units $u_\psi(\wp), u_{\bar\psi}(\wp), \bar u_\psi(\wp), \bar u_{\bar\psi}(\wp)$ are constructed by taking the root $y_0 \in \mathcal{O}[1/\wp_H]^\times$ of $x^6 - x^5 + 7x^4 + 4x^3 + 9x^2 + 21x + 23$ in $\mathbb{Q}_p$ of valuation 1, and again following the description from Section 2.1. (This construction is quite delicate, e.g. it is essential that one starts with the root of valuation 1.) The $\wp$-unit $u_\wp$ is the fundamental $\wp$-unit in K of valuation 1; namely, the root of $x^2 + 3x + 23$ of valuation 1.

Note that $\gamma_{1909a} = 0$, as expected because a certain component of the Mordell-Weil group occurs with rank 4. Looking finally at the curve $83a$ we then find

$$\gamma_{83a} = \frac{2^3 \cdot 7^2}{23 \cdot 11 \cdot \sqrt{-3}} \cdot \frac{\log_p(u_\psi)\log_{E,p}(P)\log_{E,p}(Q_{\bar\psi}) - \log_p(u_{\bar\psi})\log_{E,p}(P)\log_{E,p}(Q_\psi)}{\mathcal{L}_{g_1}}$$

and

$$\gamma'_{83a} = \frac{2^2 \cdot 7^2}{23 \cdot 11 \cdot \sqrt{-3}} \cdot \frac{\log_p(u_\psi)(\log_{E,p}(Q_\psi))^2 - \log_p(u_{\bar\psi})(\log_{E,p}(Q_{\bar\psi}))^2}{\mathcal{L}_{g_1}}$$

to 15 digits of 23-adic precision. Here $P = (0, -1) \in E_{83a}(\mathbb{Q})$, and the points $Q_\psi, Q_{\bar\psi} \in E_{83a}(H) \otimes L$ are given explicitly in [DLR3, Example 3.5].

Example 2.3. Now we consider [DLR3, Example 3.6]. We focus on the curve $1003a$, noting again that the unknown factor $\mathcal{L}_{g_1}$ is the same for all of the curves in this example, and it is enough to identify it numerically for a single curve given the ratio calculations in that example.

Briefly recalling the setting, we work in the one-dimensional space $S_1(59, \chi)$ spanned by the S_3-form g where χ is the quadratic character, and let $h \in M_1(59, \chi)$ be the Eisenstein series. We take $p = 17$ which is split in $K = \mathbb{Q}(\sqrt{-59})$, and note that $a_{23}(g) = 2$ so there is a unique p-stabilisation g_1. Again H denotes the Hilbert class field of K and $L = \mathbb{Q}(\sqrt{-3})$.

We considered four curves of conductor $1003 = 59 \cdot 17$, and here shall focus on

$$E_a : y^2 + y = x^3 - x^2 + x + 1$$

labelled $1003a$ by Cremona, with associated newform f_a (following our notation in [DLR3, Example 3.6]).

On the analytic side, for $f = f_a$ we have the projection

$$\Phi_{f,g_1,g} \quad = \quad e_{g_1} \cdot e_{\mathrm{ord}}(d^{-1}(f) \times g) \quad = \quad \alpha g_1 + \tilde\beta \tilde g_1^\flat + \tilde\gamma \tilde g_2^\flat + \delta g(q^p).$$

To higher precision for $1003a$ the value of $\tilde\gamma$ in the first line of the first table in [DLR3, Example 3.6] is

$$\tilde\gamma_{1003a} = -11390590277790628272532699684971124503246076639671118555/17 \bmod 17^{40}.$$

With the canonical scaling we have $\gamma = \tilde\gamma / \log_p(u_K(3))$ and we find

$$\gamma_{1003a} = 4514951116461789873892600876556969294803578236569923O3/17^2 \bmod 17^{40}.$$

Here $u_K(3)$ is the ratio of roots of $x^2 + 7x + 27$ in $\mathbb{Q}_p$.

Our calculations now reveal that

$$\gamma_{1003a} = \sqrt{-3} \cdot \frac{3^3}{2^2 \cdot 17} \cdot \frac{\log_p(u_\psi)\log_{E,p}(P)\log_{E,p}(Q_{\overline\psi}) - \log_p(u_{\overline\psi})\log_{E,p}(P)\log_{E,p}(Q_\psi)}{\mathcal{L}_{g_1}}$$

to 40 digits of 17-adic precision. Here $P = (1, -2) \in E_a(\mathbb{Q})$ and $Q_\psi, Q_{\bar\psi} \in E_a(H) \otimes L$ are described in [DLR3, Example 3.6] (though not explicitly written down as the underlying $Q \in E_a(H)$ has quite large height).

In this case the units are constructed from a global unit u which is a root of $x^3 + 2x + 1$ in H. After fixing the embedding of H into $\mathbb{Q}_p$, the $\wp_H$-units are found by starting with the root y_0 of $x^6 + 2x^5 + 2x^3 + 18x^2 + 28x + 17$ in H of valuation 1. The $\wp$-unit $u_K(p)$ is the fundamental $\wp$-unit of valuation 1 in K; namely, the root of $x^2 + 3x + 17$ of valuation 1.

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