Massimo Bertolini — p-adic Beilinson's formulas for Rankin p-adic L-functions and applications.

Goal: i) p-adic analogs of Beilinson's formula for $L(f\otimes g, 2)$, where $f\in S_2(N,\chi_f)$ and $g\in M_2(N,\chi_g)$ eigenforms.

ii) applications to Euler systems

p-adic Rankin L-series:

Let $p\nmid N$, f and g ordinary at p, with Hida families $\mathbf{f}=(f_k)$ and $\mathbf{g}=(g_\ell)$, where k and ℓ are in U_f and U_g resp ($\subseteq$ weight space). We make a p-adic interpolation of $L(f_k\otimes g_\ell, j)$ with $j\in\mathbb{Z}$ critical. If $\ell<k$, j is critical $\Leftrightarrow$ $j\in[\ell, k-1]$.

In this situation, we have a special value formula due to Shimura: ($t:=k-1-j$; $m:=k-\ell-2t$)

$j\in\left[\frac{\ell+k-1}{2}, k-1\right]$

$$\delta_m := \frac{1}{2\pi i}\left(\frac{d}{dz}+\frac{im}{2y}\right): M_m^{nh}(N,\varphi)\longrightarrow M_{m+2}^{nh}(N,\varphi)$$

$$L(f_k\otimes g_\ell, j) = \begin{pmatrix}\text{elementary}\\ \text{constant}\neq 0\end{pmatrix}\times \frac{\left(f_k^*, \delta_m^t E_{m,\chi}\times g_\ell\right)_{k,N}}{(f_k^*, f_k^*)_{k,N}} \in\overline{\mathbb{Q}}$$

Peterson

where $E_{m,\chi} = \frac{1}{2}L(\chi, 1-m) + \sum_{n\geq 1}\sigma_{m-1,\chi}(n)q^n$

$\chi = (\chi_f\chi_g)^{-1}$

$\Longrightarrow$ $L_p(\mathbf{f},\mathbf{g})(k,\ell,j)$ $\quad (k,\ell,j)\in U_f\times U_g\times\mathbb{Z}_p$

$\Longrightarrow$ $L_p(\mathbf{f},\mathbf{g})(2,2,2)$ in terms of p-adic integration. Can be interpreted as p-adic regulator

p-adic regulator: $\boxed{\text{Case 1:}}$ $g\in S_2(N,\chi_g)$, K p-adic field of defn, $X=X_1(N)_{/K}$

$$\mathrm{reg}_p: CH^2(X\times X, 1)\xrightarrow{\mathrm{reg}_{et}} H^1_f(K, H^2_{et}(\overline{X\times X}, \mathbb{Q}_p)(2))$$

rep. by elements in $\bigoplus_{C\subset X\times X} K(C)^\times$

$\downarrow$ log + P.D

$$\mathrm{Fil}^1 H^2_{dR}(X\times X)^\vee$$

$\boxed{\text{Case 2:}}$ $g = E_{2,\chi}$

$$\mathrm{reg}_p: CH^2(X,2)\xrightarrow{\mathrm{reg}_{et}} H^1(K, H^1_{et}(\overline{X},\mathbb{Q}_p)(2))\xrightarrow{\log+\text{P.D}} H^1_{dR}(X)^\vee$$

$\shortparallel$

$K_2(X)\otimes\mathbb{Q}$

It now follows from Besser's theory that:

Theorem 1 (Case 1) (Darmon, Rotger, B.)

diagonal in $X\times X$ $\quad\in H^1_{dR}(X)^{f,ur}$

$$L_p(f,g)(2,2,2) = (\text{Euler factor})\times \mathrm{reg}_p(\Delta, u_\chi)(\omega_g\otimes\eta_f^{ur})$$

Theorem 2 (Case 2) (Kato-Brunault, Bannai-Kings + Niklas, Darmon-B.)

$$L_p(f, E_\chi)(2,2,2) = L_p(f,2)\cdot L_p(f,\chi,1) = (\text{Euler factor})\times \mathrm{reg}_p\{u_\chi, u_{\chi^{-1}}\}(\eta_f^{ur})$$

For applications to Euler systems, need a 3rd kind of p-adic Beilinson formula:

$f = \underline{\theta} = (\theta_k)$ family of theta series attached to im. quadr. field $F = \mathbb{Q}(\sqrt{-D})$

$g = (g_\ell)$ e.g. $g = g_2 \longleftrightarrow A = A_g$ elliptic curve over $\mathbb{Q}$

Note: p splits in F

Assume F satisfies the Heegner hypothesis relative to g

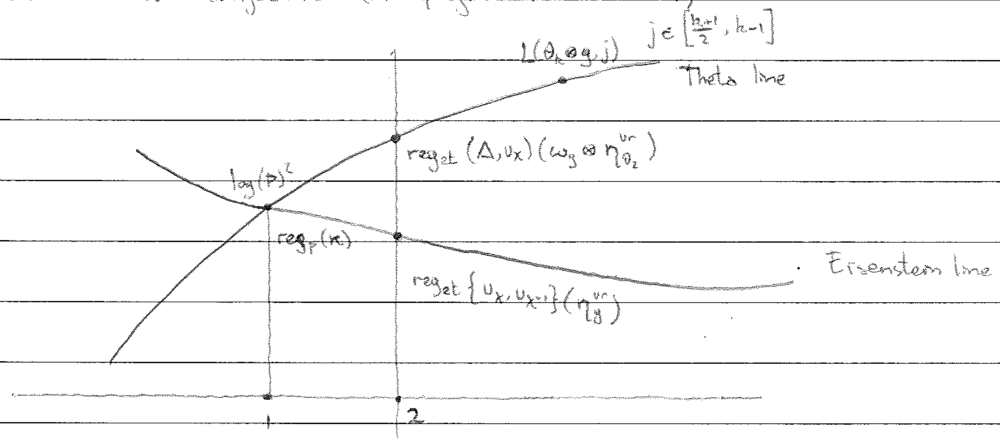
$\Longrightarrow \exists P \in A(F)$ Heegner point & $L(A/F, s)$ vanishes to odd order.

Theorem 3 (Darmon-Prasanna-B.)

$$L_p(\underline{\theta}, g)(1,2,1) = (\text{Euler factor} \neq 0) \times \log(P)^2$$

Applications to Kato's Euler system:

Perrin-Riou's conjecture (in progress with Darmon)



κ = Kato's class $\in H^1(\mathbb{Q}, V_g(1))$

By Kato's reciprocity: $\kappa \in H^1_f(\mathbb{Q}, V_g(1))$

$\mathrm{reg}_p\, \kappa \in H^1_f(\mathbb{Q}_p, V_g(1))$

Perrin-Riou's conj: $\mathrm{res}_p\, \kappa = \log^2$ of a rational pt