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Integral Eisenstein cocycles on  $GL_n(\mathbb{Z})$ 

&amp; p-adic zeta functions of tot. real fields.

(Joint with Dasgupta)

$$\mathbb{Z}^{n-1}(GL_n(\mathbb{Z}), M_{\mathbb{Q}}) \quad \begin{matrix} F \\ \mathbb{Z} \\ \mathbb{Q} \end{matrix} \quad GL \approx \mathbb{Z}^{n-1}$$

 $\equiv$  Kubota Leopoldt

$$v \in \mathbb{Q}/\mathbb{Z} \quad B_v(v) = \frac{1}{2\pi i} \sum_{m \neq 0} \frac{\exp(2\pi i m v)}{m} = \begin{cases} \sqrt{1 - \frac{1}{v^2}}, & v \notin \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

distr relation

$$\sum_{Nw=v} B_1(w) = B_1(v)$$

$w \in \mathbb{Q}/\mathbb{Z}$

 $\mathbb{Z}_p$ 

$\ell$ -smooth  
 $\ell$  prime  $\geq 3$

$$\begin{aligned} B_1^\ell(v) &= B_1(\ell v) - \ell B_1(v) \\ &= \ell v - (\ell v)^{-\frac{1}{2}} - \ell(v^{-\frac{1}{2}} - v^{-\frac{1}{2}}) \\ &\in \mathbb{Z} \end{aligned}$$

measure :  $\mu_{n-2}^\ell(a + p^N \mathbb{Z}_p) := B_1^\ell\left(\frac{a}{p^N}\right) \rightsquigarrow K-L$  p-adic  
zeta function

$$\underline{n=2} \quad (\text{F real quad}) \quad v \in \mathbb{Q}/\mathbb{Z}^2, g_v(z) \quad \text{Siegel unit}$$

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$$(\gamma, v) \mapsto \frac{1}{2\pi i} \int_{\mathcal{C}}^{\gamma \cdot \mathcal{C}} d\log g_v(z)$$

weight = Eisenstein series

dist inv

1 cocycle integrality?  $\ell$  smooth:  $v \in \overline{T_0(\ell)}$  +

change  $d\log g_v(z)$  for  $a \log g_{\ell v_1 v_2}(\ell) - \ell d\log g_v(z)$

Coates-Schmidt 79' p-adic zeta-real quad

Darmont-Darsgupta, Chenevier (Stark units)

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}}^{\mathcal{C}'(c)} d\log g_v(z) &= \frac{1}{2\pi i} \int_{\mathcal{C}}^{\mathcal{C}'(c)} \sum_{m, n} \exp(2\pi i(mv_1 + nv_2)) \frac{dz}{(mz + n)^2} \\ &= \frac{1}{2\pi i} \sum_{m, n} e\left(\frac{mv_1}{n} \frac{v_2}{n}\right) \int_{\mathcal{C}} \frac{dz}{(mz + n)^2} \\ &\quad \underbrace{\left[ \frac{-1}{m\left(m\frac{a}{c} + n\right)} \right]_{\mathcal{C}}}^{a/c} \end{aligned}$$

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periods

$$= \frac{1}{2\pi i} \sum_{m,n} e\left(\frac{m}{n}\nu_1 - \nu_2\right) (-c)$$

$$\overline{\langle \frac{m}{n}|_0 \rangle} < \langle \frac{m}{n}|_c \rangle$$

$$= \Psi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, v\right)$$

generalized (SCRECH 93')

n arbitrary

$$A = (A_1, \dots, A_n) \in \mathcal{M}_n(\mathbb{Z})^n$$

$$\sigma_i = A_i \text{ (i) } \text{ first column}$$

$$\sigma = (\sigma_1, \dots, \sigma_n) \in \mathcal{M}_n(\mathbb{Z})$$

$$\Psi(A_1, \dots, A_n, v) := \frac{\det \sigma}{(2\pi i)^n} \sum_{\gamma \in \mathbb{Z}^n} e(\frac{\gamma}{v}) \quad / \quad Q$$

$$(m_1\sigma_1, \dots, m_n\sigma_n)$$

$$Q \in \mathbb{R}^n$$

$$\& \begin{pmatrix} x=0 \\ x \in Q^n \end{pmatrix} \Rightarrow x=0 \quad \sum_{|Q|m < t} \quad \lim_{t \rightarrow \infty}$$

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Chy var  $\tilde{m} = \sigma^t m \in \sigma^t \mathbb{Z}^n$

$$= \frac{1}{(2\pi i)^n} \sum_{h \in \mathbb{Z}^n / \sigma \mathbb{Z}^n} \sum_{m \in \mathbb{Z}^n} \frac{e(m + \sigma^{-1}(h+v))}{m_1 \cdots m_n} |_{Q \cdot (\sigma^{-1})^t}$$

$$= \sum_{h \in \mathbb{Z}^n / \sigma \mathbb{Z}^n} B_1(\sigma^{-1}(h+v), Q \cdot \sigma^{-1}) \text{ still elementary.}$$

$\ell$ -smooth:

Then

$$A \mapsto \varphi(A, \circ, \circ) \in \mathbb{Z}^{n-1}(\mathcal{GL}_n(\mathbb{Z}), M_Q)$$

$$M_Q = \left\{ f: \mathbb{Q}^n / \mathbb{Z}^n \rightarrow Q \right\}$$

$$(v, Q) \mapsto f(v, Q)$$

+ dust rel

$\ell$ -smooth:

$$A_i \in T_0(\ell) = \left\{ A \in \mathcal{GL}_n(\mathbb{Z}), A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\ell} \right\}$$

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$$\pi_\ell = \begin{pmatrix} \ell & \\ 0 & \ddots \\ \vdots & \ddots \\ 0 & \end{pmatrix}$$

$$\forall r \in \mathbb{Q}^h$$

def  $\Psi_\ell(A, v, Q) = \psi(\pi_\ell A \pi_\ell^{-1}, \pi_\ell v, \pi_\ell Q) \in \mathbb{Z}\left[\frac{1}{\ell}\right]$

Th  
(C.D.)  $\Rightarrow \Psi_\ell \in \mathbb{Z}^{n-1} \left( \pi_0(\ell), M^n \right)$   
 $\mathbb{Z}\left[\frac{1}{\ell}\right]$

*if  $\ell > n+1$ , is in  $\mathbb{Z}$*

Consequences pairing

$F_n$   $f \in G_F, a \bmod f$   
 $Q$   $a^{-1}f = \sum_{j=1}^n \mathbb{Z} w_j$  a basis,  $I = \sum_{j=1}^n v_j w_j \mapsto v \in \mathbb{Z}^n$

$\sum_{j=1}^n \varepsilon_j \equiv 1 \bmod f$   
 $A_1, \dots, A_{n-1} \in GL_n(\mathbb{Z}) \rightsquigarrow \begin{matrix} \text{homog} \\ + \text{sym} \end{matrix} \quad A \in \mathbb{Z}\left[\pi_0(\ell)^n\right]$

$$Q_A = \text{any colc}(w_1, \dots, w_n) \in \mathbb{R}^n$$

$$S_f(x, 1) = \sum_{\substack{b \sim a \\ \text{mod } F}} \frac{1}{N^{\frac{1}{2}}}$$

Thm (Schach)

$$S_f(x_0) = \psi(\pi_a, v_a, Q_a) \in \mathbb{Q}$$

smoothed  $c \in \mathcal{O}_F$ , Norm  $c = \ell$

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Thm

$$S_{f,c}(\alpha, s) = \bar{S}_f(\alpha c, s) - e^{1-s} S_f(\alpha, s)$$

Thm (C.D.)

$$\bar{S}_{f,c}(\alpha, 0) = \psi_c(\pi, v, \alpha) \in \mathbb{Z}\left[\frac{1}{e}\right]$$

In fact  $\psi(A, P, v, \alpha) \in \mathbb{Z}[x_1, \dots, x_n]$   
 $\psi_c$

$$\text{Thm } \psi_c(\pi, P, v_a, \alpha) = \bar{S}_{f,c}(\alpha, -k) \in \mathbb{Z}\left[\frac{1}{e}\right]$$

pf

$$\mu_c(a + p^n \mathbb{Z}_p^n) = \psi_c(A, v_a, \alpha)$$

$a \in [0, \dots, p-1]^n \rightarrow$  get a measure on  $v + \mathbb{Z}_p^n \subset Q_p^n$   
& co-cycle.

Prop

$$\int_{v + \mathbb{Z}_p^n} P(x) d\mu_c(A, v) = \psi_c(A, P, v)$$

$\beta \mid p$  all divide  $f$   $S_{f,c,p}(\alpha, s) = \int_{v + \mathbb{Z}_p^n} N_{F_p/\mathbb{Q}_p}(x|w)^{-s} d\mu(\pi, v)$   
 $s \in w$

order of vanishing at  $s=0$        $\not\equiv 0 \pmod{f}$   
 $w$  Techm.

$$L_{c,p}(q_w, s) := \sum_{\alpha} \not\equiv \alpha_c \zeta_{f,c,p}(\alpha, s)$$

$$p_1, \dots, p_r \mid p \quad \text{s.t. } \chi(p_i) = 1$$

Conjecture (Gross)       $\text{ord}_{s=0} L_{c,p}(q_w, s) = 2$

Thm: \_\_\_\_\_  $\geq n$ .  
 (Wiles)

(C.D. Sperre) Compute  $L_{c,p}^{(k)}(q_w, 0)$   $\boxed{k < 2}$

amounts to:

$$\underbrace{\langle \psi_k, \mathfrak{A}_\alpha \otimes \mathbb{D}_0 - (\log_N(x|w))^k \rangle}_{\text{zero in homology}} \quad (\text{Sperre Thm})$$

Final Gross Stark (refined) conjecture

$p \equiv 1 \pmod{f}$

$$\overline{\zeta}_{f,c,p}(\alpha, s) = \int_{\mathbb{Z}_p^n / p \mathbb{Z}_p^n} \langle N_{F_p/\mathbb{Q}_p}(x|w) \rangle^s d_\mu(x, v) \quad s \in w$$

$$\overline{\zeta}_{f,c,p}(\alpha, 0) = 0$$

$$\overline{\zeta}_{f,c,p}(\alpha, 0) = \int \log_p(N_{F_p/\mathbb{Q}_p}(x|w)) d_\mu(x, v)$$

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Gross cony : predicts

$$= \log_p N_{LB/Q_p}(u^{\sigma_a}) \quad u \in \begin{matrix} \mathbb{H}_f \\ \mathbb{L} \\ \mathbb{F} \\ \mathbb{I}_n \\ \mathbb{Q} \end{matrix}$$

$$\log_p(u^{\sigma_a}) \stackrel{?}{=} \int_{\mathbb{L}_p - p\mathbb{Z}_p} \log_p(x|_W) d\mu(t, v)$$

Showed computation on screen.