Günther Cornelissen "Recovering curves from L-series"

game:

Mathematician 1 $3, 11, 21, 107, 288, 719, 2271, 6563$

asks M_2: is this sequence $\#X(\mathbb{F}_{q^n})$ for $n = 1, \ldots$

& find X. curve$/\mathbb{F}_q$

Issues:

- Infinitely many #'s needed?
- know q, genus g?
- is X unique? (no, we'll see about this).

$$\zeta_X(s) = \sum_{\substack{D \geq 0 \\ \text{effective} \\ \text{divisors}}} \frac{1}{q^{\deg D \cdot s}} = \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n}\right)$$

$$= \frac{\text{poly of degree } 2g}{(1-T)(1-qT)} \qquad T = q^{-s}$$

ζ_X doesn't determine X.

The above sequence occurs for $X_{\pm}: y^2 = x^5 \pm x^3 + x^2 - x - 1$

over $k = \mathbb{F}_3$

(E. Howe): $\zeta_{X_+} = \zeta_{X_-} = \dfrac{1 - T + T^2 - 3T^3 + 9T^4}{(1-T)(1-3T)}$

- $X_+ \not\cong_{\bar{k}} X_-$
- Jacobians are abs. simple.

Thm Tate 1966

X, Y two curves of k (finite field) then

$\zeta_X = \zeta_Y \iff \mathrm{Jac}(X) \sim \mathrm{Jac}(Y)$

point counting — isogenous

Thm X, Y two curves over k (finite field) then

$X \cong^F_k Y \iff \exists\ G^{ab}_{k(X)} \xrightarrow{\varphi} G^{ab}_{k(Y)}$ group iso

$\boxed{\begin{array}{l} F = \text{up to Frobenius twist} \\ \exists \sigma \in \mathrm{Aut}(k) : X \simeq Y^\sigma \end{array}}$ $L(X, \chi) = L(Y, \chi \circ \varphi^{-1})\ \forall \chi \in G^{ab}_{k(X)}$

Rmk

- $X \cong^F_k Y$ is the same as $k(X) \simeq k(Y)$ field iso, not k algebra iso.
- need this "F" e.g. E ell. curve over $\mathbb{F}_{q^2}$ with $j(E) \in \mathbb{F}_{q^2} - \mathbb{F}_q$ then $E^{(q)} \not\cong_{\bar{k}} E$

- L series are "weighted point counting"

$$L(X,\chi) = \exp\Big(\sum_{n\geq 1} \frac{T^n}{n} \sum_{d|n} \sum_{\zeta\in\mu_\infty} \zeta^{n/d} \sum_{m|d} \mu(\tfrac{d}{m}) \underbrace{N(X,m,\zeta,\chi)}\Big)$$

$\# \{ x \in X(\mathbb{F}_{q^n})$ place $[x]$ unramified for χ and $\chi(\mathrm{Frob}[x]) = \zeta \}$

- $\exists$ thm in # fields (C + Marcolli 2010).

<u>Improvement</u> (B. de Smit):

$\forall$ # field K $\exists \chi$ of order 3 s.t $L_K(\chi) \neq L_L(\chi')$ for any other L & $\chi' \in \check{G}_L^{ab}$

- <u>Use less:</u> a) no group iso, just a bijection?

(don't know how to do any of this.)

b) only zetas of Abelian covers?

$$\{\zeta_{X'}\}_{X'/X \text{ ab}} = \{\zeta_{Y'}\}_{Y'/Y \text{ ab}}$$

c) only finitely many χ's

- Example

$X_\pm$ both have 4 places of degree 2: P, Q, R, S

⇝ 12 choices of $D = 2P + Q + R$, remaining place S

consider Abelian covers of degree 3 corresponding to ray class groups for D with S totally split

→ for every choice of S on X_- $\exists$ such a cover with no k-rational points

→ for exactly two choices of S on X all such covers have k-rational points

- Analog in differential geometry

$$\zeta_{R.M\ X} = \sum_{\substack{\lambda \in \text{spectrum} \\ \text{of } \Delta_X}} \frac{1}{\lambda^s}$$

$\zeta_X \cong \zeta_Y \not\Rightarrow X$ isometric Y

an analog of L-series does the truck in this case.

§ Method "dynamical class field theory"

dynamical system (S_X, I_X)

§ Method

dynamical system (S_X, I_X)

S_X : top space

I_X : semi group of eff. divisors

$$S_X := S^{ab}_{k(X)} \times_{\hat{O}_X^*} \hat{O}_X$$

integral adeles

$(\gamma, \rho) \sim \gamma\,(\mathrm{rec}_X(u)^{-1}, u\rho)$ for any $u \in \hat{O}_X^*$

Now $\underline{n} \in I_X$ acts by

$$\underline{n} * [(\gamma, \rho)] := [(\gamma \cdot \mathrm{rec}_X(s(n))^{-1}, s(n)\rho)]$$

$$\begin{tikzcd} \mathbb{A}_X^{*} \supset \arrow[r] & I_X^{+} \arrow[l, bend right, "s"'] \end{tikzcd}$$

$$(x_p) \longmapsto \prod p^{\mathrm{ord}_p(x_p)}$$

dynamical system defined by S_X and this action.

an isomorphism of $(S_X, I_X) \simeq (S_Y, I_Y)$ is a

homeo $\Phi : S_X \xrightarrow{\sim} S_Y$

a semigroup iso $\varphi : I_X \xrightarrow{\sim} I_Y$

s.t. $\Phi(\underline{n} * x) = \varphi(n) * \Phi(x) \quad \forall x \in S_X \; \forall n \in I_X$

Main thm:

$$X \underset{k}{\overset{\sim}{=}} Y \iff (S_X, I_X) \simeq (S_Y, I_Y)$$

degree preserving i.e. $\deg(\varphi(\underline{n})) = \deg(\underline{n})$

Rmk – For $\mathbb{Q}$ this dynamical system is the Bost–Connes system.

– For # field due to Ha–Paugam.

"Proofs" Main thm $\Rightarrow$ thm

given condition (rhs of thm) $\rightsquigarrow$ rep th. $\forall$

$$\begin{array}{ccc} k(X)^{ab} & & k(Y)^{ab} \\ | \; N & & | \; \varphi(N) \\ K & & L \\ | & & | \\ k(X) & & k(Y) \end{array}$$

$\downarrow$

$I_X \xrightarrow[\sim]{\varphi} I_Y$

if p is unramified in K then $\varphi(p)$ is unramified in L & $\psi(\mathrm{Frob}_p) = \mathrm{Frob}_{\varphi(p)}$

so get a dynamical system isomorphism

Main lemma for proof of main thm:

⇐ Uchida' proof of anabelian theorem for global function fields:

$X \cong_k^{\Gamma} Y \Leftrightarrow$ $\begin{cases} \cdot \ \mathrm{Places}(X) \xrightarrow{\varphi} \mathrm{Places}(Y) \text{ bijection} \\ \circ \text{ group iso } G_{k(X)}^{ab} \xrightarrow{\Phi} G_{k(Y)}^{ab} \\ \circ \ \forall v \in \mathrm{Places}(X) \text{ iso of mult.} \end{cases}$

groups.

$$\widehat{k(X)}_v^* \xrightarrow{\Phi_v} \widehat{k(Y)}^*_{\varphi(x)}$$

$$\text{s.t. } \mathrm{ord}_{\varphi(v)} \Phi_v(x) = \mathrm{ord}_v(x)$$

$$\begin{array}{ccc} \mathbb{A}_X^* & \xrightarrow{rec_X} & k(X) \\ \downarrow \prod \Phi_v & & \downarrow \Phi \\ \mathbb{A}_Y^* & \xrightarrow[rec_X]{} & k(Y) \end{array}$$

$k(X)^* \simeq k(Y)^*$