Rational points on curves

p-adic and computational aspects

p-adic iterated integrals and rational points on curves

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(Joint with Alan Lauder and Victor Rotger)

Also based on work with Bertolini and Kartik Prasanna

Rational points on elliptic curves

Let E be an elliptic curve over a number field K.

BSD0: If $L(E/K, 1) \neq 0$, then $E(K)$ is finite.

BSD1: If $\mathrm{ord}_{s=1} L(E/K, s) = 1$, then $\mathrm{rank}(E(K)) = 1$.

BSDr: If $\mathrm{ord}_{s=1} L(E/K, s) = r$, then $\mathrm{rank}(E(K)) = r$.

We will have nothing to say (hélas) about BSDr when $r > 1$.

(For this see John Voight's lecture on Monday.)

Equivariant BSD

Let E be an elliptic curve over $\mathbb{Q}$;

Let $\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \mathsf{GL}_n(\mathbb{C})$ be an Artin representation.

Equivariant BSD conjecture $(\mathrm{BSD}_{\rho,r})$.

$$\mathrm{ord}_{s=1} L(E, \rho, s) = r \quad \stackrel{?}{\Rightarrow} \quad \dim_{\mathbb{C}} \hom_{G_{\mathbb{Q}}}(V_\rho, E(K) \otimes C) = r.$$

Question: if $L(E, \rho, s)$ has a simple zero, produce the point in the ρ-isotypic part of Mordell-Weil predicted by the BSD conjecture.

Heegner points

$K =$ imaginary quadratic field, χ a *ring class character* of K,

$$\rho = \mathrm{Ind}_K^{\mathbb{Q}} \chi.$$

Theorem. (Gross-Zagier, Kolyvagin, Bertolini-D, Zhang, Longo-Rotger-Vigni, Nekovar,...) *$BSD_{\rho,0}$ and $BSD_{\rho,1}$ are true.*

Key ingredient in the proof: the collection of Heegner points on E defined over various ring class fields of K.

More general geometric constructions

If V is a modular variety equipped with a (preferably infinite) supply $\{\Delta\}$ of interesting algebraic cycles of codimension j, and $\Pi : V \cdots \to E$ is a correspondence, inducing

$$\Pi : \mathrm{CH}^j(V)_0 \longrightarrow E,$$

we may study the points $\Pi(\Delta)$.

Bertolini-Prasanna-D: Generalised Heegner cycles in $V = W_r \times A^s$; (cf. Kartik Prasanna's lecture on Monday);

Zhang, Rotger-Sols-D: Diagonal cycles, or more interesting "exceptional cycles", on $V = W_r \times W_s \times W_t$; (cf. Victor Rotger's lecture on Thursday).

Stark-Heegner points

$K =$ real quadratic field, χ a *ring class character* of K,

$$\rho = \mathrm{Ind}_K^{\mathbb{Q}} \chi.$$

Stark-Heegner points: Local points in $E(\mathbb{C}_p)$ defined (*conjecturally*) over the field H cut out by χ. They can be computed in practice, (cf. the lecture of Xavier Guitart on Monday reporting on his recent work with Marc Masdeu).

Conjecture. $L(E, \rho, s)$ has a simple zero at $s = 1$ if and only if $\hom(V_\rho, E(H) \otimes \mathbb{C})$ is generated by Stark-Heegner points.

Stark-Heegner points

The *completely conjectural* nature of Stark-Heegner points prevents a proof of $\mathrm{BSD}_{\rho,0}$ and $\mathrm{BSD}_{\rho,1}$ along the lines of the proof of Kolyvagin-Goss-Zagier when ρ is induced from a character of a real quadratic field.

Goal: Describe a more indirect approach whose goal is to

1. Prove $\mathrm{BSD}_{\rho,0}$.
2. Construct the *global cohomology classes* $\kappa_{E,\rho} \in H^1(\mathbb{Q}, V_p(E) \otimes V_\rho)$ which *ought to arise* from Stark-Heegner points via the connecting homomorphism of Kummer theory.

p-adic deformations of geometric constructions

A Λ-adic Galois representation is a finite free module $\underline{V}$ over Λ equipped with a continuous action of $G_{\mathbb{Q}}$.

Specialisations: $\xi \in \mathcal{W} := \mathrm{hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) = \mathrm{hom}_{cts}(\Lambda, \mathbb{C}_p)$,

$$\xi : \underline{V} \longrightarrow V_\xi := \underline{V} \otimes_{\Lambda,\xi} \mathbb{Q}_{p,\xi}.$$

Suppose there is a dense set of points $\Omega_{\mathrm{geom}} \subset \mathcal{W}$ and, for each $\xi \in \Omega_{\mathrm{geom}}$, a class

$$\kappa_\xi \in H^1_{\mathrm{fin}}(\mathbb{Q}, V_\xi).$$

Definition

The collection $\{\kappa_\xi\}_{\xi \in \Omega_{\mathrm{geom}}}$ *interpolates p-adically* if there exists $\underline{\kappa} \in H^1(\mathbb{Q}, \underline{V})$ such that

$$\xi(\underline{\kappa}) = \kappa_\xi, \qquad \text{for all } \xi \in \Omega_{\mathrm{geom}}.$$

p-adic limits of geometric constructions

Suppose that $V_p(E) = H^1_{\text{et}}(\overline{E}, \mathbb{Q}_p)(1)$ arises as the specialisation

$$\xi_E : \underline{V} \longrightarrow V_p(E)$$

for some ξ_E *not necessarily belonging to* Ω_{geom}. One may then consider the class

$$\kappa_E := \xi_E(\underline{\kappa}) \in H^1(\mathbb{Q}, V_p(E))$$

and attempt to relate it to $L(E, 1)$ and to the arithmetic of E.

The class κ_E is a p-adic limit of geometric classes, but need not itself admit a geometric construction.

Basic examples

Coates-Wiles: $\underline{V}$ is induced from a family of Hecke characters of a quadratic imaginary field,

$$\Omega_{\mathsf{geom}} = \{\text{finite order Hecke characters}\},$$

the κ_ξ arise from the images of *elliptic units* under the Kummer map, and ξ_E corresponds to a Hecke character of infinity type $(1, 0)$ attached to a CM elliptic curve E.

Kato. $\underline{V} = V_p(E)(1) \otimes \Lambda_{\mathsf{cyc}}$,

$$\Omega_{\mathsf{geom}} = \{ \text{ finite order } \chi : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times \},$$

the $\kappa_\chi \in H^1(\mathbb{Q}, V_p(E)(1)(\chi))$ arise from the images of Beilinson elements in $K_2(X_1(Np^s))$ ($s =$cond(χ)), and $\xi_E = -1$.

The Perrin-Riou philosophy

Perrin-Riou. p-adic families of global cohomology classes are a powerful tool for studying p-adic L-functions.

I will illustrate this philosophy in the following contexts:

1. Classes arising from Beilinson-Kato elements, and the Mazur-Swinnerton-Dyer p-adic L-function (as described in Massimo Bertolini's lecture);
2. Classes arising from diagonal cycles and the Harris-Tilouine triple product p-adic L-function (as discussed in Victor Rotger's lecture).

Modular units

Manin-Drinfeld: the group $\mathcal{O}^\times_{Y_1(N)_{/\mathbb{C}}}/\mathbb{C}^\times$ has "maximal possible rank", namely $\#(X_1(N) - Y_1(N)) - 1$.

The logarithmic derivative gives a surjective map

$$\mathrm{dlog} : \mathcal{O}^\times_{Y_1(N)_{/\mathbb{Q}(\mu_N)}} \otimes \mathbb{Q} \longrightarrow \mathrm{Eis}_2(\Gamma_1(N), \mathbb{Q})$$

to the space of weight two Eisenstein series.

Let $u_\chi \in \mathcal{O}^\times_{Y_1(N)} \otimes \mathbb{Q}_\chi$ be the modular unit characterised by

$$\mathrm{dlog}\, u_\chi = G_{2,\chi},$$

$$G_{2,\chi} = 2^{-1} L(\chi, -1) + \sum_{n=1}^{\infty} \sigma_\chi(n) q^n, \quad \sigma_\chi(n) = \sum_{d|n} \chi(d) d.$$

Beilinson elements

Given χ of conductor Np^s,

$$\begin{array}{rcl} \alpha_\chi := \delta(u_\chi) & \in & H^1_{\text{et}}(X_1(Np^s), \mathbb{Z}_p(1)), \\ \beta_\chi := \delta(w_\zeta u_\chi) & \in & H^1_{\text{et}}(X_1(Np^s)_{\mathbb{Q}(\mu_{Np^s})}, \mathbb{Z}_p(1)) \\ \tilde{\kappa}_\chi := \alpha_\chi \cup \beta_\chi & \in & H^2_{\text{et}}(X_1(Np^s)_{\mathbb{Q}(\mu_{Np^s})}, \mathbb{Z}_p(2)), \\ \kappa_\chi := \text{ its image} & \text{in} & H^1(\mathbb{Q}(\mu_{Np^s}), H^1_{\text{et}}(X_1(Np^s)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))). \end{array}$$

The latter descends to a class

$$\kappa_\chi \in H^1(\mathbb{Q}, H^1_{\text{et}}(X_1(Np^s)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))(\chi^{-1})).$$

Let $X_1(N) \longrightarrow E$ be a modular elliptic curve, and

$$\kappa_E(G_{2,\chi}, G_{2,\chi}) \in H^1(\mathbb{Q}, V_p(E)(1)(\chi^{-1}))$$

be the natural image.

Kato's Λ-adic class

Key Remark: The Eisenstein series $G_{2,\chi_0\chi}$ (with $\mathfrak{f}_\chi = p^s$) are the weight two specialisations of a *Hida family* $\underline{G}_{\chi_0}$.

Theorem (Kato)

There is a Λ-adic cohomology class

$$\kappa_E(\underline{G}_{\chi_0}, \underline{G}_{\chi_0}) \in H^1(\mathbb{Q}, V_p(E)(\chi_0) \otimes \Lambda_{\mathrm{cyc}}(-1)),$$

satisfying

$$\xi_{2,\chi}(\kappa_E(\underline{G}_{\chi_0}, \underline{G}_{\chi_0})) = \kappa_E(G_{2,\chi_0\chi}, G_{2,\chi_0\chi})$$

at all "weight two" specialisations $\xi_{2,\chi}$.

The Kato–Perrin-Riou class

We can now specialise the Λ-adic cohomology class $\kappa_E(\underline{G}_{\chi_0}, \underline{G}_{\chi_0})$ to Eisenstein series *of weight one*.

$$\kappa_E(G_{1,\chi_0}, G_{1,\chi_0}) := \nu_1(\kappa_E(\underline{G}_{\chi_0}, \underline{G}_{\chi_0})).$$

Theorem (Kato)

The class $\kappa_E(G_{1,\chi_0}, G_{1,\chi_0})$ is cristalline if and only if $L(E,1)L(E,\chi_0^{-1},1) = 0$.

Corollary

$BSD_{\chi,0}$ is true for E.

Hida families

To prove $\mathrm{BSD}_{\rho,0}$ for larger classes of ρ, we will

1. replace the Beilinson elements $\kappa_E(G_{2,\chi}, G_{2,\chi}) \in H^1(\mathbb{Q}, V_p(E)(1)(\chi^{-1}))$ by *geometric elements* $\kappa_E(g,h) \in H^1(\mathbb{Q}, V_p(E) \otimes V_g \otimes V_h(k-1))$ attached to a pair of cusp forms g and h of the *same weight* $k \geq 2$.
2. Interpolate these classes in *Hida families* $\rightarrow \kappa_E(\underline{g}, \underline{h})$.
3. Consider the weight one specialisations

$$\kappa_E(g_1, h_1) \in H^1(\mathbb{Q}, V_p(E) \otimes V_{\rho_{g_1}} \otimes V_{\rho_{h_1}}).$$

Of special interest is the case where ρ_{g_1} and ρ_{g_2} are *Artin representations*.

Gross-Kudla-Schoen diagonal classes

étale Abel-Jacobi map:

$$\begin{aligned} \mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^2(X_1(N)^3)_0 &\longrightarrow H^4_{\mathrm{et}}(X_1(N)^3, \mathbb{Q}_p(2))_0 \\ &\longrightarrow H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\overline{X_1(N)}^3, \mathbb{Q}_p(2))) \\ &\longrightarrow H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\overline{X_1(N)}, \mathbb{Q}_p)^{\otimes 3}(2)) \end{aligned}$$

Gross-Kudla Schoen class:

$$\kappa_E(g, h) := \mathrm{AJ}_{\mathrm{et}}(\Delta)^{f,g,h} \in H^1(\mathbb{Q}, V_p(E) \otimes V_g \otimes V_h(1)).$$

Hida Families

Weight space: $\Omega := \hom(\Lambda, \mathbb{C}_p) \subset \hom((1 + p\mathbb{Z}_p)^\times, \mathbb{C}_p^\times)$.

The integers form a dense subset of Ω via $k \leftrightarrow (x \mapsto x^k)$.

Classical weights: $\Omega_{\text{cl}} := \mathbb{Z}^{\geq 2} \subset \Omega$.

If $\tilde{\Lambda}$ is a finite flat extension of Λ, let $\tilde{\mathcal{X}} = \hom(\tilde{\Lambda}, \mathbb{C}_p)$ and let

$$\kappa : \tilde{\mathcal{X}} \longrightarrow \Omega$$

be the natural projection to weight space.

Classical points: $\tilde{\mathcal{X}}_{\text{cl}} := \{x \in \tilde{\mathcal{X}} \text{ such that } \kappa(x) \in \Omega_{\text{cl}}\}$.

Hida families, cont'd

Definition

A *Hida family* of tame level N is a triple $(\Lambda,\Omega,\underline{g})$, where

1. Λ_g is a finite flat extension of Λ;
2. $\Omega_g \subset \mathcal{X}_g := \hom(\Lambda_g, \mathbb{C}_p)$ is a non-empty open subset (for the p-adic topology);
3. $\underline{g} = \sum_n \mathbf{a}_n q^n \in \Lambda_g[[q]]$ is a formal q-series, such that $\underline{g}(x) := \sum_n x(\mathbf{a}_n) q^n$ is the q series of the *ordinary p-stabilisation* $g_x^{(p)}$ of a normalised eigenform, denoted g_x, of weight $\kappa(x)$ on $\Gamma_1(N)$, for all $x \in \Omega_{g,\mathrm{cl}} := \Omega_g \cap \mathcal{X}_{g,\mathrm{cl}}$.

Λ-adic Galois representations

If $\underline{g}$ and $\underline{h}$ are Hida families, there are associated Λ-adic Galois representations $\underline{V}_g$ and $\underline{V}_h$ of rank two over Λ_g and Λ_h respectively (cf. Adrian Iovita's lecture on Thursday).

A p-adic family of global classes

Theorem (Rotger-D)

Let $\underline{g}$ and $\underline{h}$ be two Hida families. There is a $\Lambda_g \otimes_\Lambda \Lambda_h$-adic cohomology class

$$\kappa_E(\underline{g}, \underline{h}) \in H^1(\mathbb{Q}, V_p(E) \otimes (\underline{V}_g \otimes_\Lambda \underline{V}_h) \otimes_\Lambda \Lambda_{\mathrm{cyc}}(-1)),$$

where $\underline{V}_g, \underline{V}_h =$ Hida's Λ-adic representations attached to $\underline{g}$ and $\underline{h}$, satisfying, for all "weight two" points $(y, z) \in \Omega_g \times \Omega_h$,

$$\xi_{y,z}(\kappa_E(\underline{g}, \underline{h})) = *\kappa_E(g_y, h_z).$$

This Λ-adic class generalises Kato's class, which one recovers when $\underline{g}$ and $\underline{h}$ are Hida families of Eisenstein series.

Generalised Kato Classes

Lei, Loeffler and Zerbes are studying similar families of "twisted Beilinson-Flach elements". There is a strong parallel between the three settings:

1. **Beilinson-Kato elements**, leading to the Kato class $\kappa_E(G_{1,\chi}, G_{1,\chi}) \in H^1(\mathbb{Q}, V_p(E)(\chi^{-1}))$;
2. **Twisted diagonal cycles**, leading to classes $\kappa_E(g, h) \in H^1(\mathbb{Q}, V_p(E) \otimes V_g \otimes V_h)$ where g and h are cusp forms of weight one with $\det(V_g \otimes V_h) = 1$;
3. The **twisted Beilinson-Flach elements** in David Loeffler's lecture, leading to classes $\kappa_E(g, G_{1,\chi}) \in H^1(\mathbb{Q}, V_p(E) \otimes V_g)$, where V_g is a *not-necessarily-self-dual* representation.

All three will be called **generalised Kato classes** for E.

A reciprocity law for diagonal cycles

As in Kato's reciprocity law, one can consider the specialisations of $\kappa_E(\underline{g}, \underline{h})$ when $\underline{g}$ and $\underline{h}$ are evaluated at points of *weight one*.

Theorem (Rotger-D; still in progress)

Let $(y, z) \in \Omega_g \times \Omega_h$ be points with $\mathrm{wt}(y) = \mathrm{wt}(z) = 1$. The class $\kappa_E(g_y, h_z)$ is cristalline if and only if $L(V_p(E) \otimes g_y \otimes h_z, 1) = 0$.

Main ingredients:

1. The p-adic Gross-Zagier formula for diagonal cycles described in Rotger's lecture, and its extension to levels divisible by powers of p;

2. Perrin-Riou's theory of Bloch-Kato logarithms and dual exponential maps "in p-adic families".

BSD_ρ in analytic rank zero.

Corollary

Let E be an elliptic curve over $\mathbb{Q}$ and ρ_1, ρ_2 odd irreducible two-dimensional Galois representations. Then $BSD_{\rho_1 \otimes \rho_2, 0}$ is true for E.

Proof. Use the ramified class $\kappa_E(g, h) \in H^1(\mathbb{Q}, V_p(E) \otimes \rho_1 \otimes \rho_2)$ to bound the image of the global points in the local points.

Corollary

Let χ be a dihedral character of a real quadratic field K, and let $\rho = \mathrm{Ind}_K^{\mathbb{Q}} \chi$. Then $BSD_{\rho,0}$ is true.

Proof. Specialise to the case $\rho_1 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_1$ and $\rho_2 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_2$.

Analytic rank one, and Stark-Heegner points?

Question. Assume that

1. g and h are attached to classical modular forms, and hence to Artin representations ρ_g and ρ_h;
2. $L(E, \rho_g \otimes \rho_h, 1) = 0$, so that $\kappa_E(g, h)$ is cristalline.

Project with Lauder and Rotger: Give an explicit, computable formula for

$$\log_p(\kappa_E(g, h)) \in (\Omega^1(E/\mathbb{Q}_p) \otimes D(V_{\rho_g}) \otimes D(V_{\rho_h}))^\vee.$$

This would be useful both for theoretical and experimental purposes.

Perrin-Riou's formula for the log of the Kato class

Recall there are two Mazur-Swinnerton-Dyer p-adic L-functions: $L_{p,\alpha}(E/\mathbb{Q}, s)$ and $L_{p,\beta}(E/\mathbb{Q}, s)$

$$x^2 - a_p x + p = (x - \alpha)(x - \beta), \quad \mathrm{ord}_p(\alpha) \leq \mathrm{ord}_p(\beta).$$

$\mathrm{ord}_p(\beta) = 1$: Kato-Perrin-Riou; Pollack-Stevens; Bellaiche.

$$L_{p,\dagger}(E, s) := \left(1 - \frac{1}{\beta}\right)^2 L_{p,\alpha}(E, s) - \left(1 - \frac{1}{\alpha}\right)^2 L_{p,\beta}(E, s).$$

Perrin-Riou's formula for the Kato class

Theorem (Perrin-Riou)

If $L(E, \chi, 1) = 0$, *there exists* $\omega \in \Omega^1(E/\mathbb{Q})$ *such that*

$$L'_{p,\dagger}(E, \chi, 1) = \frac{\alpha - \beta}{[\varphi\omega, \omega]} \log_{\omega,p}(\kappa_E(G_{1,\chi}, G_{1,\chi})).$$

Conjecture (Perrin-Riou)

If $L(E, \chi, 1) = 0$, *there exists a point* $P_\chi \in (E(\mathbb{Q}^{\mathrm{ab}}) \otimes \mathbb{Q}_\chi)^\chi$ *and* $\omega \in \Omega^1(E/\mathbb{Q})$ *such that*

$$L'_{p,\dagger}(E, \chi, 1) = \frac{\alpha - \beta}{[\varphi\omega, \omega]} \log^2_{\omega,p}(P_\chi).$$

Experimental evidence

Numerical verifications have been carried out by Bernardi and Perrin-Riou, and pushed further by M. Kurihara and R. Pollack using the Pollack-Stevens theory of *overconvergent modular symbols* to compute $\log_p(\kappa_E(G_{1,\chi}, G_{1,\chi}))$ p-adically when p is a supersingular prime.

Example: The curve $X_0(17)$ is supersingular at $p = 3$.

$$X_0(17)_{193} : y^2 + xy + y = x^3 - x^2 - 25609x - 99966422$$

$$(x, y) = \left(\tfrac{915394662845247271}{25061097283236}, \tfrac{-878088421712236204458830141}{125458509476191439016}\right).$$

The logarithms of the generalised Kato classes

Main idea: The p-adic logarithms of the (generalised) Kato classes should be expressed a limits of "p-adic iterated integrals".

Alan Lauder has devised highly efficient algorithms to compute these iterated integrals numerically.

Caveat: The p-adic integrals that will be introduced in this talk are *very different* from the ones that arose in the lecture of Jennifer Balakrishnan on Monday.

We are a bit baffled by this last fact.

The cohomological interpretation of modular forms

A modular form g of weight $k = 2 + r \geq 2$ can be interpreted as an element

$$\omega_g \in H^0(X, \underline{\omega}^r \otimes \Omega^1_X), \qquad X = X_1(N).$$

$$\underline{\omega}^r \subset \mathcal{L}_r = \mathrm{sym}^r H^1_{\mathrm{dR}}(\mathcal{E}/X).$$

Gauss-Manin connection:

$$0 \longrightarrow \mathcal{L}_r \xrightarrow{\nabla} \mathcal{L}_r \otimes \Omega^1_X \longrightarrow 0.$$

Hodge filtration exact sequence

$$0 \longrightarrow H^0(X, \underline{\omega}^r \otimes \Omega^1_X) \longrightarrow H^1_{\mathrm{dR}}(X, \mathcal{L}_r, \nabla) \longrightarrow H^1(X, \underline{\omega}^{-r}) \longrightarrow 0.$$

p-adic modular forms

p a prime not dividing N;

$\mathcal{A} \subset X$ the ordinary locus;

$\mathcal{W} \supset \mathcal{A}$ a wide open neighborhood of this affinoid region.

$$\begin{aligned} H^1_{\mathrm{dR}}(X, \mathcal{L}_r, \nabla) &= \frac{H^0(\mathcal{W}, \mathcal{L}_r \otimes \Omega^1_X)}{\nabla H^0(\mathcal{W}, \mathcal{L}_r)} \\ &= \frac{H^0(\mathcal{W}, \underline{\omega}^r \otimes \Omega^1_X)}{\nabla H^1(\mathcal{W}, \mathcal{L}_r) \cap H^0(\mathcal{W}, \underline{\omega}^r \otimes \Omega^1_X)} \\ &= \frac{M_k^{\mathrm{oc}}(N)}{d^{1+r} M_{-r}^{\mathrm{oc}}(N)}. \end{aligned}$$

The d operator

Here $d = q\frac{d}{dq}$ is the d operator on p-adic modular forms.

$$d^j(\sum_n a_n q^n) = \begin{cases} \sum_n n^j a_n q^n & \text{if } j \geq 0; \\ \sum_{p \nmid n} n^j a_n q^n & \text{if } j < 0. \end{cases}$$

Fact: If $g \in M_k^{\mathrm{oc}}(N)$, then $d^{1-k} g \in M_{2-k}^{\mathrm{oc}}(N)$.

p-adic iterated integrals: Type I

Suppose $\gamma \in H^1_{\mathrm{dR}}(X/\mathbb{Q}_p)$, and $g, h \in M_k(N)$ $(k = r + 2 \geq 2)$.

Definition

The *p-adic iterated integral* of g and h along γ is

$$\int_\gamma \omega_g \cdot \omega_h := \langle \gamma, d^{1-k} g \times h \rangle_X,$$

where $\langle \ , \ \rangle_X$ is the Poincaré duality on $H^1_{\mathrm{dR}}(X/\mathbb{Q}_p) = H^1_{\mathrm{rig}}(\mathcal{W})$.

Key fact: If $\gamma \in H^1_{\mathrm{dR}}(X)^{\mathrm{ur}}$, the unit root subspace, then

$$\int_\gamma \omega_g \cdot \omega_h = \langle \gamma, e_{\mathrm{ord}}(d^{1-k} g \times h) \rangle_X,$$

where e_{ord} is Hida's *ordinary projector*.

p-adic iterated integrals: Type II

Suppose $\gamma \in H^1_{\mathrm{dR}}(X, \mathcal{L}_r, \nabla)$, and that $f \in M_2(N)$, $g \in M_k(N)$ $(k = r + 2 \geq 2)$.

Definition

The *p-adic iterated integral* of f and g along γ is

$$\int_\gamma \omega_f \cdot \omega_g := \langle \eta, d^{-1} f \times g \rangle_r,$$

where $\langle \ , \ \rangle_r$ is the Poincaré duality on $H^1_{\mathrm{dR}}(X, \mathcal{L}_r, \nabla)$.

Logarithms of Generalised Kato classes

Consider these classes in the "range of geometric interpolation": f is of weight two, attached to an elliptic curve E, and g and h are of weight $k = r + 2 \geq 2$ (and level prime to p).

Then $\kappa_E(g, h) \in H^1_f(\mathbb{Q}, V_p(E) \otimes V_g \otimes V_h(r+1))$.

Bloch-Kato logarithm:

$$\log_p(\kappa_E(g, h)) \in \mathrm{Fil}^{2r+3}(H^1_{\mathrm{dR}}(X) \otimes H^{r+1}_{\mathrm{dR}}(W_r) \otimes H^{r+1}_{\mathrm{dR}}(W_r))^\vee.$$

Theorem (Rotger, D)

1. $\log_p(\kappa_E(g, h))(\eta_f^{\mathrm{ur}} \otimes \omega_g \otimes \omega_h) = * \int_{\eta_f^{\mathrm{ur}}} \omega_g \cdot \omega_h$. *(This is an iterated integral "of Type I".)*

2. $\log_p(\kappa_E(g, h))(\omega_f \otimes \omega_g \otimes \eta_h^{\mathrm{ur}}) = * \int_{\eta_h^{\mathrm{ur}}} \omega_f \cdot \omega_g$. *(This is an iterated integral "of type II".)*

The p-adic logarithms of generalised Kato classes

Suppose now that g and h are of *weight one*, and that $L(E, \rho_g \otimes \rho_h, 1) = 0$, so that $\kappa_E(g, h)$ is cristalline.

Goals of the current project with Lauder and Rotger.

1. Express the p-adic logarithms of the generalised Kato classes $\kappa_E(g, h)$ in terms of (p-adic limits of) p-adic iterated integrals.

2. Compute these logarithms using Alan Lauder's fast algorithms for computing ordinary projections. (The computational aspects will be described in Alan's lecture.)

A final question

When g and h are *Eisenstein series of weight one*, the relation between $\log_p(\kappa_E(G_{1,\chi}, G_{1,\chi}))$ and p-adic iterated integrals suggests a strategy for proving Perrin-Riou's conjecture:

$$\log_p(\kappa_E(G_{1,\chi}, G_{1,\chi})) \stackrel{?}{=} \times \log_p^2(P_\chi),$$

for χ quadratic, as described in Bertolini's lecture of yesterday.

Question: When ρ_g and ρ_h are induced from characters of a real quadratic field K, show that

$$\log_p(\kappa_E(g, h)) = * \log_p^2(P_{g,h}),$$

where $P_{g,h}$ are *Stark-Heegner points* attached to E and K.

This would relate the (a priori purely local, and poorly understood) Stark-Heegner points to the *global classes* $\kappa_E(g, h)$.

Thank you for your attention!