Toby Gee "Patching functors and the cohomology of Shimura curves"

joint with Matthew Emerton
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- p-adic local Langlands / deformation rings / types
- Taylor Wiles method / patching functors
- results

$p > 2$

f cuspidal eigenform of level $\Gamma_1(N)$, weight 2

$f \rightsquigarrow \pi(f)$ automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$

$$\pi(f) = \underset{\substack{\circlearrowright \\ GL_2(\mathbb{A}^\infty_{\mathbb{Q}})}}{\pi^\infty(f)} \otimes \pi_\infty(f)$$

$$\pi^\infty(f) = \bigotimes_l{}' \underset{\substack{\circlearrowright \\ GL_2(\mathbb{Q}_l)}}{\pi_l(f)}$$

$l \neq p$

$$\varinjlim_N H^1_{et}(X_1(N)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_l) = \bigoplus \underset{\substack{\circlearrowleft \\ GL_2(\mathbb{A}^\infty_{\mathbb{Q}})}}{\overset{\substack{\pi(f) \\ \parallel}}{\pi}} \otimes \underset{\substack{\circlearrowleft \\ Gal(\overline{\mathbb{Q}}/\mathbb{Q})}}{\rho_\pi}$$

$$\varinjlim_n H^1_{et}(X_1(Np^n)_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_l) = \bigoplus \underset{\substack{\circlearrowleft \\ GL_2(\mathbb{Q}_p)}}{\pi_p} \otimes \rho_{\pi_p}$$

$\pi_p = \pi_p(f)$

$\rho_{\pi_p} = \rho_\pi | Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$\pi_p \leftrightarrow \rho_{\pi_p}$ = "local Langlands correspondence"

Irreducible smooth admissible $\bar{\mathbb{Q}}_\ell$ reps of $GL_2(\mathbb{Q}_p)$ $\longleftrightarrow$ 2-dim $\overset{\text{cont}}{\overline{\mathbb{Q}}_\ell}$ reps of $G_{\mathbb{Q}_p}$

easy to classify (since $\ell \neq p$)
difficult part is pro p.

classify: restrict to compact open subgroups,
e.g. $GL_2(\mathbb{Z}_p)$

Restricting to $GL_2(\mathbb{Z}_p) \rightsquigarrow$ theory of types

$$\pi \circlearrowright GL_2(\mathbb{Q}_p) \longmapsto \pi|_{GL_2(\mathbb{Z}_p)} \supseteq \sigma(\pi) = \text{"smallest" subrepresentation} \circlearrowleft GL_2(\mathbb{Z}_p)$$

e.g. $\sigma(\pi) = K \Rightarrow \pi$ is unramified principal series
$\Leftrightarrow \rho_\pi$ is unramified

[Globally: $p \nmid N$]

Inertial local Langlands correspondence (Henniart):

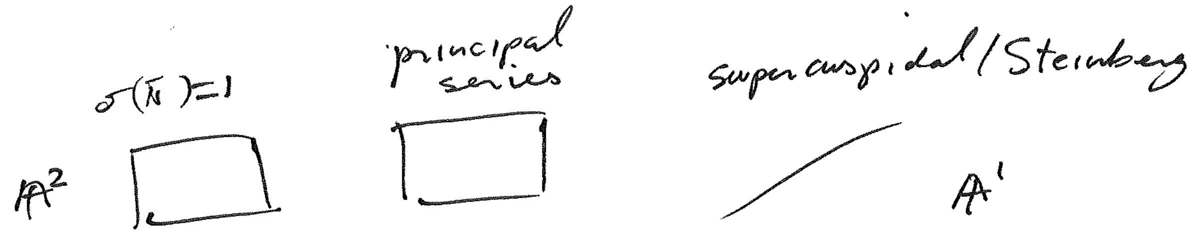
$$\sigma(\pi) \longleftrightarrow \rho_\pi|_{I_p}$$

What information do we lose in $\pi \rightsquigarrow \sigma(\pi)$?

e.g. unramified principal series: lose "characteristic polynomial of Frobenius / Hecke operator T_p"

<u>Answer</u>: Bernstein centre.

Algebraic variety with ∞ many components indexed by $\sigma(\pi)$
closed points are the π.



this is classical and well understood

p-adic Langlands: what if $\ell = p$??

Do I still have a bijection? no!

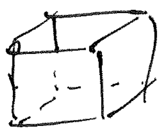
Q. (Breuil) what extra information is needed to get a bijection?

<u>Galois deformations</u>

Fix $\bar{\rho} : G_{\mathbb{Q}_p} \to GL_2(\bar{\mathbb{F}}_p)$

Then there is a "3d" space whose point parametrize the p-adic reps of $G_{\mathbb{Q}_p}$ which lift $\bar{\rho}$ (with fixed determinant)

Fix central chars



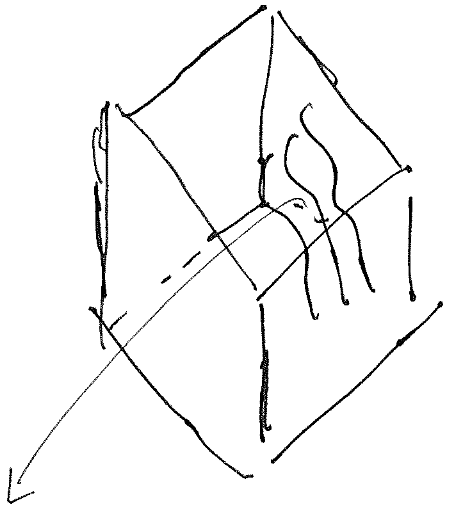
If $\rho : G_{\mathbb{Q}_p} \to GL_2(\bar{\mathbb{Q}}_p)$ comes from $H^1(\,, \mathbb{Q}_p)$, then ρ is potentially semistable with ~~Hodge~~ Hodge Tate weights 0, 1

If ρ comes from f, $\pi_p(f)$ type $\sigma_f(p) := \sigma(\pi_p(f))$, then ρ has type $\sigma_p(\rho)$ as well.

Locus of points which are pot. semistable of some fixed type is a 1D subspace.

<u>Fact</u> If $\sigma(\pi) = 1$, we can recover ρ_π from π.

For supercuspidal case things are not so nice, Steinberg case...



Global deformations

$$\uparrow$$

e.g. $\varinjlim_{n\to\infty} H^1(X(Np^n), \overline{\mathbb{Q}}_p)$

If $\bar\rho = \bar r|_{G_{\mathbb{Q}_p}}$

$\bar r : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$

Then global lifts of $\bar r$ (of fixed tame ~~kernel~~ level) also give points in my space

2D subspace

<u>Idea</u> Use the global points to study the local picture

<u>Want</u> a way to vary the tame level in a controlled fashion

<u>Taylor–Wiles method</u> (not enough time to explain)

<u>Patching functors</u>:

$$\left\{\begin{matrix}\text{cts reps of } GL_2(\mathbb{Z}_p) \text{ on} \\ \text{f.g. } \mathbb{Z}_p\text{-modules}\end{matrix}\right\} \xrightarrow{M_\infty} \left\{\begin{matrix}\text{coherent sheaves} \\ \text{on } X\end{matrix}\right\}$$

$$L \rightsquigarrow H^1(X(1), \mathcal{L})_m \qquad R_p[[\ldots]]$$

<u>Fact</u>: there is a purely local construction of such a functor using p-adic Local Langlands for $GL_2(\mathbb{Q}_p)$

$\Rightarrow$ construction of patching functor is canonical

[Not known at all for e.g. $GL_2(\mathbb{Z}_{p^2})\ldots$]

<u>Facts</u> $M_\infty(\sigma(\pi)^\circ)$ is supported on the subspace of

lattice in $\sigma(\pi)$

Hodge Tate

pot semistable HT 0,1 type $\sigma(\pi)$

& is maximal CM over this

Cohen Macaulay

e.g. if the subspace is regular $M_\infty(-)$ is locally free

Work with Shimura curves / totally real fields unramified at p.

<u>Breuil</u>: additional structure on π_p to make a bijective correspondence:

data of a $GL_2(\mathbb{Q}_p)$-stable lattice on π

$\cup$

Easier: $GL_2(\mathbb{Z}_p)$ ~~~~~~ $\sigma(\pi)$

Breuil conjectured that the lattice on $\sigma(\pi)$

induced by $H^1(X, \overline{\mathbb{Z}}_p)$ is determined by

$\rho_\pi | G_p$ by an explicit receipe if π is tamely ramified

<u>Thm</u> Breuil's conjecture is true

$$\varinjlim_n H^1(X(Np^n), \overline{\mathbb{F}}_p) \circlearrowleft Gl_2(\mathbb{Q}_p)$$

Breuil & Paskunas construct reps of $Gl_2(\mathbb{Q}_p)$ & conj that they occur in cohomology

<u>Thm</u> each indecomposable sub contains one of BP's reps