Triple product p-adic L-functions for balanced weights

Matthew Greenberg (Calgary)
and Marco Seveso (Milan)

Rational points on curves
Oxford, Sept. 24-28, 2012

Our project

$i = 1, 2, 3$

$p \nmid N_i$ tame levels

$\Omega_i \subset \mathcal{W}$ affinoid disks, $\mathcal{W}$ = weight space

We're given an N_i-new Coleman family

$$f_i = \sum_{n=1}^{\infty} a_n(f_i) q^n \in \mathcal{O}(\Omega_i)[[q]].$$

Thus, if $k_i \in \Omega_{i,\mathrm{cl}}$, then

$$f_{i,k_i} = \sum_{n=1}^{\infty} a_n(f_i, k_i) q^n \in S_{k_i+2}(\Gamma_0(N_i p))^{N_i\text{-new}}$$

is a normalized eigenform.

f_{i,k_i} is p-new for at most one $k_i \in \Omega_{i,\mathrm{cl}}$.

When f_{i,k_i} is p-old, let

$$f^{\sharp}_{i,k_i} \in S_{k_i+2}(\Gamma_0(N_i))$$

be the unique normalized newform such that

$$f^{\sharp}_{i,k_i} | T_\ell = a_\ell(f_i, k_i) f^{\sharp}_{i,k_i} \quad \text{for all } \ell \nmid N_i p.$$

Explicitly,

$$\begin{aligned} f_{i,k_i}(q) &= f^{\sharp}_{i,k_i}(q) - \frac{p^{k_i+1}}{a_p(f_i, k_i)} f^{\sharp}_{i,k_i}(q^p), \\ L_p(f^{\sharp}_{i,k_i}, s)^{-1} &= \left(1 - a_p(f_i, k_i) q^{-s}\right) \left(1 - \frac{p^{k_i+1}}{a_p(f_i, k_i)} q^{-s}\right). \end{aligned}$$

Shorthand:

$$f^\sharp_{\vec{k}} = f^\sharp_{1,k_2} \times f^\sharp_{2,k_2} \times f^\sharp_{3,k_3}, \qquad \Omega = \Omega_1 \times \Omega_2 \times \Omega_3$$

A triple (k_1, k_2, k_3) of classical weights is *balanced* if $k_1 + k_2 + k_3$ is even and

$$k_1 \leq k_2 + k_3, \quad k_2 \leq k_1 + k_3, \quad \text{and} \quad k_3 \leq k_1 + k_2.$$

Project goal: Construct a *p-adic L-function*

$$\mathcal{L}_p(f_1 \otimes f_2 \otimes f_3) \in \mathcal{O}(\Omega_1 \times \Omega_2 \times \Omega_3)$$

such that

$$\mathcal{L}_p(f_1 \otimes f_2 \otimes f_3, \vec{k}) \longleftrightarrow L\left(f^\sharp_{\vec{k}}, \frac{k_1 + k_2 + k_3}{2} + 2\right)^{\mathrm{alg}}$$

for all balanced triples $\vec{k} \in \Omega_{\mathrm{cl}}$.

This talk

Theorem: (G-Seveso) Suppose

$$N_1 = N_2 = N_3 = N$$

with N squarefree. Then there is a function $\mathcal{L}_p$ satisfying (a precise version of) the above interpolation property for balanced triples of even weights.

The assumption $N_1 = N_2 = N_3 = N$ is for ease of exposition.

We can deal with odd weights, although there are some issues beyond simply admitting forms with compatible nebentype.

Cyclotomic variable?

ε-factors

Since we assume N is squarefree,

$$\varepsilon_v(f^\sharp_{\vec{k}}, \tfrac{1}{2}) = \begin{cases} -1 & \text{if } v \mid N, \\ -1 & \text{if } v = \infty \text{ and } \vec{k} \text{ is balanced,} \\ +1 & \text{otherwise} \end{cases}$$

If $\varepsilon(f^\sharp_{\vec{k}}, \frac{1}{2}) = -1$, then the interpolation problem is trivial.

If $\varepsilon(f^\sharp_{\vec{k}}, \frac{1}{2}) = +1$ and $\vec{k}$ is unbalanced, then the corresponding $\mathcal{L}_p$ was constructed by Harris & Tilouine (*arXiv* 1996, *Math. Ann.* 2001).

If $\varepsilon(f^\sharp_{\vec{k}}, \frac{1}{2}) = +1$ and $\vec{k}$ is balanced, then $\omega(N)$ is odd.

The quaternion algebra B

Assume $\varepsilon(f^\sharp_{\vec{k}}, \frac{1}{2}) = +1$ for balanced $\vec{k} \in \Omega_{\mathrm{cl}}$.

$$B = \text{quaternion } \mathbb{Q}\text{-algebra ramification set } \{v \mid N\infty\}$$

π_{k_i}: automorpic representation of $\mathrm{GL}_2(\mathbb{A})$ with new vector $f^\sharp_{i,k_i}$

Theorem: (Prasad 1990) B is characterized by:

1. each π_{k_i} admits a Jacquet-Langlands lift $\pi^B_{k_i}$ to $B^\times(\mathbb{A})$,
2. $\mathrm{Hom}_{B_v^\times}(\pi^B_{k_1,v} \otimes \pi^B_{k_2,v} \otimes \pi^B_{k_3,v}, \mathbb{C}) \neq 0$ for all v.

Theorem: (Harris & Kudla 1991) With B as above,

$$\mathrm{Hom}_{B^\times(\mathbb{A})}(\pi^B_{k_1} \otimes \pi^B_{k_2} \otimes \pi^B_{k_3}, \mathbb{C}) \neq 0 \iff L(f^\sharp_{\vec{k}}, \tfrac{k_1+k_2+k_3}{2} + 2) \neq 0.$$

Trilinear forms and special values

Theorem: (Harris & Kudla 1991, Gross & Kudla 1992, Böcherer & Schulze-Pillot 1996, Watson 2002, Ichino 2008) With B as above, there exist

- local factors $C_v \neq 0$, $v \mid N\infty$,
- a quantity $T(f_{\vec{k}}^{\sharp}) \in \mathbb{Q}(f_{\vec{k}}^{\sharp})$

such that

$$L(f_{\vec{k}}^{\sharp}, \tfrac{k_1+k_2+k_3}{2}+2) = \langle f_{\vec{k}}^{\sharp}, f_{\vec{k}}^{\sharp} \rangle \prod_{v \mid N\infty} C_v \cdot T(f_{\vec{k}}^{\sharp}).$$

Let φ_i be a Jacquet-Langlands lift of the Coleman family f_i to $B^\times$ (more soon) and set

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3.$$

We produce an analytic function $\vec{k} \mapsto t_3(\varphi_{\vec{k}})$ on Ω such that

$$\frac{t_3(\varphi_{\vec{k}})^2}{\langle \varphi_{\vec{k}}, \varphi_{\vec{k}} | W_p \rangle} = \mathcal{E}^2_{p,\vec{k}} \cdot \left(1 - p^{k_3^*} \frac{a_p(f_{3,k_3})}{a_p(f_{1,k_1}) a_p(f_{2,k_2})}\right)^2 \cdot T(f^\sharp_{\vec{k}}),$$

for $\vec{k} \in \Omega_{\mathrm{cl}}$, where

$$k_3^* = \frac{k_1 + k_2 - k_3}{2}$$

Symmetrically, we also get functions $\vec{k} \mapsto t_1(\varphi_{\vec{k}})$ and $\vec{k} \mapsto t_2(\varphi_{\vec{k}})$.

The t_i are trilinear forms on spaces of *quaternionic Coleman families*, and are the subject of the remainder of the talk.

Quaternionic modular forms

$$G = \mathrm{GL}_2(\mathbb{Q}_p),\ K = \mathrm{GL}_2(\mathbb{Z}_p),\ K_0 = \left\{ A \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod{p} \right\} \subset K$$

$R \subset B$ maximal order, $R_0 \subset R$ suborder of level $p \nmid N$, splitting $B_p = B \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Q}_p)$ such that

$$\begin{tikzcd}
R_{0,p}^{\times} \arrow[r, "\subset"] \arrow[d, "\wr"'] & R_p^{\times} \arrow[r, "\subset"] \arrow[d, "\wr"] & B_p^{\times} \arrow[d, "\wr"] \\
K_0 \arrow[r, "\subset"'] & K \arrow[r, "\subset"'] & G
\end{tikzcd}$$

is commutative.

Semigroup: $\Sigma = G \cap \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$

Let V be a right $\mathbb{Q}_p[\Sigma]$-module.

V-valued quaternionic modular forms of level $J = K$ or K_0:

$$M(V, J) = \left\{ f : \widehat{B}^\times / B^\times \longrightarrow V : f(kx)k_p = f(x) \text{ for all } k \in J \right\}.$$

$M(V, K_0)$ admits an action of T_ℓ, $\ell \nmid Np$, and of U_p.

Jacquet-Langlands correspondence: There is a Hecke-equivariant, $\mathbb{Q}_p$-linear isomorphism

$$S_{k+2}(\Gamma_0(Np), \mathbb{Q}_p)^{N\text{-new}} \xrightarrow{\sim} M(V_k, K_0)/\mathrm{Eis},$$

where V_k is the irred. rep. of G of dimension $k+1$.

Trilinear forms

Theorem: (Clebsch & Gordan)

$$\dim_{\mathbb{Q}_p} \operatorname{Hom}_G\Big(V_{k_1} \otimes V_{k_2} \otimes V_{k_3}, \mathbb{Q}_p(k_1 + k_2 + k_3)\Big)$$
$$= \begin{cases} 1 & \text{if } \vec{k} \text{ is balanced,} \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly: $P_k = P_k^{x,y} = \mathbb{Q}_p[x,y]_k,\ V_k = \mathrm{Hom}_{\mathbb{Q}_p}(P_k, \mathbb{Q}_p)$

If $\vec{k}$ is balanced, then

$$v_{\vec{k}} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^{k_3^*} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}^{k_2^*} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^{k_1^*} \in P_{\vec{k}}(-k_1 - k_2 - k_3)^G,$$

where

$$k_i^* = \frac{-k_i + k_{i'} + k_{i''}}{2}, \quad P_{\vec{k}} = P_{k_1}^{x_1,y_1} \otimes P_{k_2}^{x_2,y_2} \otimes P_{k_3}^{x_3,y_3}.$$

When it's nonzero,

$$\mathrm{Hom}_G\left(V_{k_1} \otimes V_{k_2} \otimes V_{k_3}, \mathbb{Q}_p(k_1+k_2+k_3)\right) = \mathbb{Q}_p \cdot t_{\vec{k}},$$

where

$$t_{\vec{k}}(\mu) = \int v_{\vec{k}}\, d\mu.$$

Trilinear forms on coefficients induce trilinear forms on modular forms:

$$M(V_{k_1}) \otimes M(V_{k_2}) \otimes M(V_{k_3}) \longrightarrow M(V_{k_1} \otimes V_{k_2} \otimes V_{k_3})$$
$$\xrightarrow{t_{\vec{k}}} M(\mathbb{Q}_p(k_1+k_2+k_3)) \xrightarrow{\langle \cdot, \mathrm{Eis} \rangle} \mathbb{Q}_p$$

Abusing notation slightly,

$$t_{\vec{k}} : M(V_{k_1}) \otimes M(V_{k_2}) \otimes M(V_{k_3}) \longrightarrow \mathbb{Q}_p$$

for the composition of the above maps.

If $\psi_i \in M(V_{k_i}, K)[f^{\sharp}_{i,k_i}]$ is nonzero and $\psi = \psi_1 \otimes \psi_2 \otimes \psi_3$, then

$$T(f^{\sharp}_{\vec{k}}) = \frac{t_{\vec{k}}(\psi)^2}{\langle \psi, \psi \rangle}$$

p-adic families of trilinear forms

The p-adic L-function arises from deforming $t_{\vec{k}}$ over the weight space.

Let $\mathbb{k}_i : \mathbb{Z}_p^\times \longrightarrow \mathcal{O}_i = \mathcal{O}(\Omega_i)$ be associated to $\Omega_i \subset \mathcal{W}$.

Let $X = \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3$.

We might want (but won't quite get) a diagram

$$\begin{tikzcd} M(D_{\mathbb{k}_1}(X)) \otimes M(D_{\mathbb{k}_2}(X)) \otimes M(D_{\mathbb{k}_3}(X)) \arrow[r, "t_{\vec{\mathbb{k}}}"] \arrow[d, "\vec{\mathbb{k}} \mapsto \vec{k}"'] & \mathcal{O} \arrow[d, "\vec{\mathbb{k}} \mapsto \vec{k}"] \\ M(V_{k_1}) \otimes M(V_{k_2}) \otimes M(V_{k_3}) \arrow[r, "t_{\vec{k}}"'] & \mathbb{Q}_p \end{tikzcd}$$

coming from a trilinear form $D_{\mathbb{k}_1}(X) \otimes D_{\mathbb{k}_2}(X) \otimes D_{\mathbb{k}_3}(X) \xrightarrow{t_{\vec{\mathbb{k}}}} \mathcal{O}$.

Canonical isomorphism:

$$D_{\mathbb{k}_1}(X) \otimes D_{\mathbb{k}_2}(X) \otimes D_{\mathbb{k}_3}(X) \longrightarrow D_{\mathbb{k}_1 \oplus \mathbb{k}_2 \oplus \mathbb{k}_3}(X^3)$$

where, for $t \in \mathbb{Z}_p^\times$,

$$t^{\mathbb{k}_1 \oplus \mathbb{k}_2 \oplus \mathbb{k}_3} := t^{\mathbb{k}_1} \otimes t^{\mathbb{k}_2} \otimes t^{\mathbb{k}_3} \in \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3 = \mathcal{O}.$$

First crack at defining $t_{\vec{\mathbb{k}}}$:

$$t_{\vec{\mathbb{k}}}(\mu) = \int_{X^3} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^{\mathbb{k}_3^*} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}^{\mathbb{k}_2^*} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^{\mathbb{k}_1^*} d\mu,$$

$$\text{where} \quad \mathbb{k}_i^* = \frac{\ominus \mathbb{k}_i \oplus \mathbb{k}_{i'} \oplus \mathbb{k}_{i''}}{2}.$$

Problem: These determinants need not belong to $\mathbb{Z}_p^\times$.

Fix: If $Y = p\mathbb{Z}_p \times \mathbb{Z}_p^\times = X \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$, then

$$\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}^{\mathbb{k}_2^*} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^{\mathbb{k}_1^*}$$

makes sense on $X^2 \times Y$, while if

$$Z = \left\{ \big((x_1, y_1), (x_2, y_2)\big) \in X^2 : x_1 y_2 - x_2 y_1 \in p\mathbb{Z}_p \right\},$$

then

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^{\mathbb{k}_3^*}$$

makes sense on $X^2 - Z$.

Therefore, we may define $t^\circ_{\vec{\Bbbk}} : D_{\Bbbk_1 \oplus \Bbbk_2 \oplus \Bbbk_3}((X^2 - Z) \times Y) \to \mathcal{O}$ by

$$t^\circ_{\vec{\Bbbk}}(\mu) = \int_{(X^2 - Z) \times Y} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^{\Bbbk_3^*} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}^{\Bbbk_2^*} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^{\Bbbk_1^*} d\mu$$

Dualizing the extension-by-zero map on analytic functions, we get

$$D_{\Bbbk_1 \oplus \Bbbk_2 \oplus \Bbbk_3}(X^2 \times Y) \longrightarrow D_{\Bbbk_1 \oplus \Bbbk_2 \oplus \Bbbk_3}((X^2 - Z) \times Y),$$

so we may view $t^\circ_{\vec{\Bbbk}}$ as a map $D_{\Bbbk_1 \oplus \Bbbk_2 \oplus \Bbbk_3}(X^2 \times Y) \longrightarrow \mathcal{O}$.

We get an induced map on quaternionic modular forms:

$$t^\circ_{\vec{\Bbbk}} : M(D_{\Bbbk_1}(X)) \otimes M(D_{\Bbbk_2}(X)) \otimes M(D_{\Bbbk_3}(Y)) \longrightarrow \mathcal{O}.$$

The p-adic L-function

Let

$$\varphi_i \in M(D_{\Bbbk_i}(X))[f_i]$$

be the Jacquet-Langlands lift of the Coleman family f_i.

We (finally) define

$$\mathcal{L}_p(f_1 \otimes f_2 \otimes f_3) \in \mathcal{O}(\Omega_1 \times \Omega_2 \times \Omega_3)$$

by

$$\mathcal{L}_p(f_1 \otimes f_2 \otimes f_3) = \frac{t^{\circ}_{\vec{\Bbbk}}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 | W_p)^2}{\langle \varphi_1, \varphi_1 | W_p \rangle \langle \varphi_2, \varphi_2 | W_p \rangle \langle \varphi_3, \varphi_3 | W_p \rangle}.$$

Interpolation

New problem: $t^\circ_{\vec{\mathbb{k}}}$ no longer specializes to $t_{\vec{k}}$ – the trilinear form that evaluates special values – for $\vec{k} \in \Omega_{\mathrm{cl}}$.

However, if we let

$$t^\circ_{\vec{k}} : M(D_{k_1}(X)) \otimes M(D_{k_2}(X)) \otimes M(D_{k_3}(Y)) \longrightarrow \mathbb{Q}_p$$

be the weight-$\vec{k}$ specialization of $t^\circ_{\vec{\mathbb{k}}}$, then we have:

Key Proposition: Let $\psi_i \in M(D_{k_i}(X))$, $i = 1, 2, 3$, be such that $\psi_i | U_p = a_i \psi_i$. Then

$$t^\circ_{\vec{k}}(\psi_1 \otimes \psi_2 \otimes \psi_3 | W_p) = \left(1 - p^{k_3^*} \frac{a_3}{a_1 a_2}\right) t_{\vec{k}}(\psi_1 \otimes \psi_2 \otimes \psi_3).$$

Elements of the proof:

1. Analyze the difference between $(\psi_1 \otimes \psi_2)|U_p$ and $\psi_1|U_p \otimes \psi_2|U_p$:
$$(\psi_1 \otimes \psi_2)|U_p = a_1 a_2 \Big(\psi_1 \otimes \psi_2 - (\psi_1 \otimes \psi_2)|_{X^2-Z}\Big),$$

2. Consider $t_{\vec{k}}$ as a pairing,
$$\langle \cdot, \cdot \rangle : M(D_{k_1 \oplus k_2}(X^2)) \otimes M(D_{k_3}(Y)) \longrightarrow \mathbb{Q}_p,$$
with respect to which $p^{k_3^*} U_p^{\iota}$ is right adjoint to U_p:
$$\begin{aligned}\langle (\psi_1 \otimes \psi_2)|U_p, \psi_3|W_p \rangle &= \langle \psi_1 \otimes \psi_2, \psi_3|W_p p^{k_3^*} U_p^{\iota} \rangle \\ &= p^{k_3^*} \langle \psi_1 \otimes \psi_2, \psi_3|U_p W_p \rangle \\ &= p^{k_3^*} a_3 \langle \psi_1 \otimes \psi_2, \psi_3|W_p \rangle\end{aligned}$$

Theorem: If $\vec{k} \in \Omega_{\mathrm{cl}}$, then
$$\frac{t_{\vec{k}}^{\circ}(\varphi_{k_1} \otimes \varphi_{k_2} \otimes \varphi_{k_3}|W_p)^2}{\langle \varphi_{\vec{k}}, \varphi_{\vec{k}}|W_p \rangle} = \mathcal{E}_{p,\vec{k}}^2 \cdot \left(1 - p^{k_3^*} \frac{a_p(f_{3,k_3})}{a_p(f_{1,k_1}) a_p(f_{2,k_2})}\right)^2 \cdot T(f_{\vec{k}}^{\sharp}).$$

Lunch! Free afternoon!