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Shin Hattori — Canonical subgroups via Breuil-Kisin modules

§ 1. Introduction

p prime, $K/\mathbb{Q}_p$ ext of cdvf w valuation $v_p(p)=1$ residue field k, $\bar{K}$ alg closure.

$A/\mathcal{O}_K$ abelian scheme of rel dim g. A ordinary $\Rightarrow A[p^n]^\circ(\mathcal{O}_{\bar{K}}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$

Overconvergence: if A is sufficiently near to ordinary, then $\exists\, C_n \subseteq A/\mathcal{O}_K$ fin flat subgp scheme s.t. $C_n(\mathcal{O}_{\bar{K}}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$, "level-$n$ canonical subgp" of A.

This is useful for e.g.

- defining overconvergent Siegel MF of arbitrary weights
- studying U_p operator
- constructing families of overconv Siegel MF
- (possibly) proving a case of the Artin conjecture

Finer statements are important. E.g.

- give explicitly region for C_n to exist and take it as large as possible
- prove properties of C_n (Frob kernel, Hodge-Tate kernel, cont'd on anti-can isogenies)

Today: To obtain such stronger statements via Breuil-Kisin classification of fin flat gp schemes$/\mathcal{O}_K$

§ 2. Main theorem

Let $m_{\bar{K}}^{\geq i} = \{ a \in \mathcal{O}_{\bar{K}} \mid v_p(a) \geq i \}$. $\mathcal{O}_{\bar{K},i} := \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}$

For a fin flat gp scheme $\mathcal{G}/\mathcal{O}_K$ with $p^n\mathcal{G}=0$ define $\mathcal{G}_i :=$ closure in $\mathcal{G}$ of $\mathrm{Ker}(\mathcal{G}(\mathcal{O}_{\bar{K}}) \to \mathcal{G}(\mathcal{O}_{\bar{K},i}))$ — the i-th lower ramific subgp.

$\mathcal{G}^j :=$ an upper variant — the j th ram subgp

$$\mathcal{G}_{i+} = \bigcup_{i'>i} \mathcal{G}_{i'} \qquad \mathcal{G}^{j+} = \bigcup_{j'>j} \mathcal{G}^{j'}$$

Remark: If $p\mathcal{G}=0$, then $\mathcal{G}^j = \left((\mathcal{G}^\vee)_{(\frac{1}{p-1}-\frac{j}{p})+} \right)^\perp$ $\qquad \left(\begin{matrix} \mathcal{H} \subset \mathcal{G} \\ \mathcal{H}^\perp = (\mathcal{G}/\mathcal{H})^\vee \end{matrix} \right)$

Hodge-Tate map: $HT_i : \mathcal{G}(\mathcal{O}_{\bar{K}}) = \mathrm{Hom}(\mathcal{G}^\vee_{\mathcal{O}_{\bar{K}}}, \mu_{p^n}) \xrightarrow{\text{pull-back}} \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},i}$

Degree: $\deg(\mathcal{G}) = \sum_i v_p(a_i)$ with $\omega_{\mathcal{G}} \simeq \bigoplus_i \mathcal{O}_K/(a_i)$

Hodge height: $\mathrm{Hdg}(\mathcal{G}) := v_p\left(\det\left(V_{\mathcal{G}^\vee[p]\times \mathrm{Spec}(\mathcal{O}_K/p)} ; \mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathrm{Spec}(\mathcal{O}_K/p))\right)\right) \in [0,1]$

Remark: $\mathrm{Hdg}(\mathcal{G}) = 0 \iff \mathcal{G}$ ordinary

$\mathrm{Hdg}(\mathcal{G}^\vee) = \mathrm{Hdg}(\mathcal{G})$.

<u>Theorem</u>: Let $\mathcal{G}$ be a BT_n of height $= h$, dim d, $\mathrm{Hdg}(\mathcal{G}) = w$ $/\mathcal{O}_K$

1) If $w < \frac{1}{p^{n-2}(p+1)}$, then $\exists\, C_n \subseteq \mathcal{G}$ fin flat subgp scheme of order p^{nd} s.t $C_n \times \mathrm{Spec}(\mathcal{O}_{K,1-p^{n-1}w}) = \mathrm{Ker}\, F^n$ of $\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_{K,1-p^{n-1}w})$

"level $= n$ can subgp of $\mathcal{G}$".

Moreover,

a. $\deg(\mathcal{G}/C_n) = \frac{w(p^n-1)}{p-1}$

b. $C_n' =$ level n can subgp of $\mathcal{G}^\vee$

$\Rightarrow C_n' = C_n^\perp$

c. For $n=1$ $\mathcal{G} = \mathcal{G}_{\frac{1-w}{p-1}} = \mathcal{G}^{\frac{pw}{p-1}+}$ & C_1 as in 1) is unique.

2) If $w < \frac{p-1}{p^n-1}$, then

d. $C_n(\mathcal{O}_{\bar{K}}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^d$

e. the closure in C_n of $C_n(\mathcal{O}_{\bar{K}})[p^i] = C_i$

3) If $w < \frac{p(p-1)}{p^{n+1}-1}$, then C_n satisfying (1) + d is unique

4) If $w < \frac{p-1}{p^n}$, then

f. $C_n(\mathcal{O}_{\bar{K}}) = \mathrm{Ker}\, HT_{n-\frac{p^n-1}{p-1}w}$

g. $C_n = \mathcal{G}_i$ for some specified range of i

5) If $w < \frac{1}{2p^{n-1}}$ then $C_n = \mathcal{G}^j$ for some specified range of j

6) For a BT $\Gamma/_{\mathcal{O}_K}$ w/ $\mathrm{Hdg}(\Gamma) < \frac{1}{2}$ & $D \subseteq \Gamma[p]$ st $C_1(\mathcal{O}_{\bar{K}}) \oplus D(\mathcal{O}_{\bar{K}}) = \Gamma[p](\mathcal{O}_{\bar{K}})$ then we have

$\mathrm{Hdg}(\Gamma/D) = \frac{w}{p}$ & $\Gamma[p]/_D \subseteq (\Gamma/D)[p]$ □

<u>Remark</u>: g is difficult. $\exists$ family version using c) + family construction of upper ram gps.

§3. Construction of C_n

May assume $n=1$ by induction & then may assume k is perfect by B.C

<u>Breuil-Kisin classifn</u>: (due to Kim, Liu, Lau)

Fix uniformizer π of K. $\pi_n \in \bar{K}$ s.t $\pi_0 = \pi$ & $\pi_{n+1}^p = \pi_n$

$K_\infty = \bigcup K(\pi_n)$ $W := W(k)$, $E(u) \in W[u]$ monic Eisenstein poly for π deg $= e$.

$\mathfrak{S}_1 = k[[u]] \curvearrowleft \varphi$: p^{th} power Frob

$\underline{\mathrm{Mod}}^{1,\varphi}_{\mathfrak{S}_1}$ = category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where $\mathfrak{M}$ is a fin free $\mathfrak{S}_1$-module and $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ φ-semi lin endom st $u^e\, \mathrm{Coker}(\varphi^*\mathfrak{M} \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \mathfrak{M}) = 0$

$R = \varprojlim \big(\mathcal{O}_{\bar K}/p \xleftarrow[x^p \leftarrow\!\shortmid\, x]{} \mathcal{O}_{\bar K}/p \leftarrow \cdots\big) \circlearrowleft G_K$ $\mathfrak{S}_1$-alg by $u \mapsto \underline{\pi} = (\pi_0, \pi_1, \pi_2, \cdots)$

Thm ∃ an exact anti-equivalence

$$\mathfrak{M}^* : (\text{fin flat gp sheme}/\mathcal{O}_K \text{ killed by } p) \xrightarrow{\sim} \underline{\mathrm{Mod}}^{1,\varphi}_{\mathfrak{S}_1} \quad \text{w/ exact quasi-inverse } \mathcal{G}^*$$

s.t. ① $\mathcal{G}(\mathcal{O}_{\bar K}) \simeq \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, R)$ w/ $\mathfrak{M} = \mathfrak{M}^*(\mathcal{G})$ as G_{K_∞}-modules

② $\mathbb{D}^*(\mathcal{G})(S \to \mathcal{O}_K) \simeq S_{\varphi,\mathfrak{S}_1} \otimes \mathfrak{M} =: \mathcal{M}(\mathfrak{M})$ compatible w/ Frob, Hodge filtration.

where

$$\begin{array}{ccc} S = W[u]\left[\frac{E(u)^m}{m!} \,\middle|\, m \geq 0\right]^\wedge & & \\ \downarrow \quad\; \overset{u}{\updownarrow} & \quad \& \quad & \mathrm{Fil}^1 \mathcal{M}(\mathfrak{M}) = \mathrm{Ker}\big(S \otimes_{\varphi,\mathfrak{S}_1} \mathfrak{M} \to \mathfrak{M}/u^e\mathfrak{M}\big) \\ \mathcal{O}_K \quad \pi & & \end{array}$$

③ $\mathrm{Lie}(\mathcal{G}^\vee) \simeq \mathcal{M}(\mathfrak{M})/\mathrm{Fil}^1\mathcal{M}(\mathfrak{M})$ & $\omega_{\mathcal{G}} = \mathrm{Fil}^1\mathcal{M}(\mathfrak{M})/\mathrm{Fil}^1 S.\mathcal{M}(\mathfrak{M})$

put $\mathfrak{M}' = \mathfrak{M}^*(\mathcal{G}^\vee)$, $\mathfrak{M}'_1 := \mathfrak{M}'/u^e\mathfrak{M}' \circlearrowleft \varphi_{\mathfrak{M}'_1}$ $\mathrm{Fil}^1\mathfrak{M}'_1 := \mathrm{Im}(1\otimes\varphi_{\mathfrak{M}'_1})$

$\Longrightarrow v_u(\det(\varphi_{\mathfrak{M}'_1};\, \mathrm{Fil}^1\mathfrak{M}'_1)) = ew$

Key lemma 1 ∃! $\mathfrak{L}' \subseteq \mathfrak{M}'$ $\varphi_{\mathfrak{M}'}$-stable direct summand as a $\mathfrak{S}_1$-module lifting $\dfrac{\mathrm{Fil}^1\mathfrak{M}'_1}{u^{e(1-w)}\mathrm{Fil}^1\mathfrak{M}'_1} \subset \dfrac{\mathfrak{M}'_1}{u^{e(1-w)}\mathfrak{M}'_1}$. Moreover, $\mathfrak{L}' \in \underline{\mathrm{Mod}}^{1,\varphi}_{\mathfrak{S}_1}$

Key lemma 2 $\mathcal{G}^*(\mathfrak{M}'/\mathfrak{L}') \hookrightarrow \mathcal{G}^\vee$ is the unique fin flat subgp scheme s.t. $\mathcal{G}^*(\mathfrak{M}'/\mathfrak{L}')^\perp \times \mathrm{Spec}(\mathcal{O}_{K,1-w}) = \mathrm{Ker}\, F$ of $\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_{K,1-w})$

We put $C_1 = \mathcal{G}^*(\mathfrak{M}'/\mathfrak{L}')^\perp$, uniqueness follows from Key lemma 2

It follows that $C_1' = C_1^\perp$ and C_1 is compatible w/ ∀ fin BC

"C_1 = ram subgps" ⇐ the case of $\mathcal{G}(\mathcal{O}_{\bar K}) = \mathcal{G}(\mathcal{O}_K)$ ⇐ $\mathcal{H}^*(\mathfrak{M}'/\mathfrak{L}') = \mathcal{H}^*(\mathfrak{M}')^{\frac{1-w}{p-1}}$

where $\mathcal{H}^*(\mathfrak{M})/_{k[[u]]}$ $B \mapsto \mathrm{Hom}_{\varphi,\mathfrak{S}_1}(\mathfrak{M}, B)$