A. Iovita An overconvergent Eichler-Shimura isomorphism (with Andreatta & Stevens)

Main goal: Geometrically relate 2 different things:

Modular symbols $\longleftrightarrow$ Modular forms

Classical, overconv., families Classical, overconv., families

We want geometrically defined maps.

Modular symbols are (cfr Pollack) relevant for special values of (p-adic) L-functions, also Galois rep^s.

① Classical Eichler-Shimura

(arithmetic form) $N \geq 3$, $p \geq 3$ prime, $p \nmid N$, $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$, $k \geq 0$

$V_k = \mathrm{Sym}^k(\mathbb{Q}_p^2) \circlearrowleft \Gamma$, $H^1(\Gamma, V_k)$ = modular symbols of wt k.

(⚠ In Pollack's talk: $MS_\Gamma(k) = H^1_c(\Gamma, V_k)$)

Modular symbols ~~define~~ are fin. dim vectorspace, action Hecke operators (T_ℓ, $\ell \nmid Np$, U_p)

Now let $Y(N,p)_{\mathbb{Q}}$ be affine scheme representing (E, ψ_N, C) cyclic order p

$\Gamma = \pi_1(Y(N,p)_{\mathbb{C}}, \text{base}) \longleftrightarrow$ local system V_k on $Y(N,p)_{\mathbb{C}}$

$$\begin{tikzcd} \overline{\mathbb{Q}} \arrow[r, hook] \arrow[dr] & \mathbb{C} \\ & \overline{\mathbb{Q}}_p \end{tikzcd}$$

$\overset{\text{GAGA}}{\longleftrightarrow}$ V_k étale local system on $Y(N,p)_{\mathbb{C}}$ $\longleftrightarrow$ V_k étale local system on $Y(N,p)_{\overline{\mathbb{Q}}}$

$\longleftrightarrow$ V_k étale local system on $Y(N,p)_{\overline{\mathbb{Q}}_p}$, so

$$H^1(\Gamma, V_k) \cong H^1(Y(N,p)_{\mathbb{C}}, V_k) \cong H^1(Y(N,p)_{\overline{\mathbb{Q}}}, V_k) \cong H^1(Y(N,p)_{\overline{\mathbb{Q}}_p}, V_k)$$

$G_{\mathbb{Q}} \circlearrowleft$ $G_{\mathbb{Q}_p} \circlearrowleft$

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ Y(N,p)_{\mathbb{Q}_p} \arrow[r, hook] & X(N,p)_{\mathbb{Q}_p} \arrow[u, bend right, dashed, "e"'] \end{tikzcd}$$

$\underline{\omega} = e^*(\Omega^1_{\mathcal{E}/X(N,p)})$

Theorem (Faltings)

There is a natural isomorphism

$$H^1(\Gamma, V_k) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \left(H^0(X(N,p), \underline{\omega}^{k+2}) \otimes \mathbb{C}_p\right) \oplus \left(H^1(X(N,p), \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1)\right)$$

which is compatible with Hecke + $G_{\mathbb{Q}_p}$

② Overconvergence and families

a) modular forms: $\mathcal{W}$ = weight space; it is the rigid analytic space$/\mathbb{Q}_p$ assoc. to completed Noeth. alg.: $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \cong \mathbb{Z}_p[\mathbb{F}_p^\times][[T]]$

$(1+p) \longmapsto T$

There is a universal character: k^{univ} :

$$\begin{tikzcd} \mathbb{Z}_p^\times \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathcal{W}}(\mathcal{W})^\times \\ & (\mathbb{Z}_p[[\mathbb{Z}_p^\times]])^\times \arrow[ur] & \end{tikzcd}$$

$t \in \mathbb{Z}_p^\times$, $\kappa \in \mathcal{W}$: $k^{univ}(t)(\kappa) = \kappa(t)$

$k_0 \in U \subseteq \mathcal{W}$ a connected affinoid (with good red), $k \in U(K)$

$$\begin{tikzcd} \mathbb{Z}_p^\times \arrow[r, "k^{univ}"] \arrow[rr, bend right, "k_U"'] \arrow[rrr, bend right=40, "k_0"'] & \mathcal{O}_{\mathcal{W}}(\mathcal{W})^\times \arrow[r] & A_U^\times \arrow[r] & K^\times \end{tikzcd}$$

A_U = affinoid alg of U (

$A_U \to K$

accessible (Coleman)

(assume k_U, k_0 are analytic on $1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$)

Let $w \in \mathbb{Q}$, $0 < w < \frac{p}{p+1}$; $X(w) = \{ x \in X_1(N)_{\mathbb{Q}_p} \mid |\mathrm{Hasse}(x)| \geq p^{-w} \}$

$K/\mathbb{Q}_p$ finite s.t. $p^{\frac{w}{p-1}} \in K$

$$\begin{tikzcd} \text{affinoid } X(w) \arrow[r, phantom, "\subseteq"] \arrow[rd, dashed] & X_1(N)^{rig} \\ & X(N,p)^{rig} \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \overset{\psi}{(E, \psi_N)} \arrow[r, mapsto] & \underset{\text{canonical subgp.}}{(E, \psi_N, C)} \end{tikzcd}$$

Glenn Stevens defined • $\underline{\omega}_{A_U}^{\dagger, k_0} =: \omega_U^\dagger$ evalustion of the functor on the universal object (

loc. free of rk 1 as $A_U \hat{\otimes} \mathcal{O}_{X(w)}$.

$$\begin{tikzcd} E \arrow[d] \\ X(w) \end{tikzcd}$$

ψ_N, $Y = \frac{p^w}{E_{p-1}}$ (if $p \geq 5$)

• $\underline{\omega}^{\dagger, k_0}$ sheaves on $X(w)$

loc free of rk 1 as $\mathcal{O}_{X(w)}$-module.

Now $k_0 \in U$, we have exact $0 \longrightarrow A_U \xrightarrow{t_{k_0}} A_U \longrightarrow K \longrightarrow 0$

so also $0 \longrightarrow \underline{\omega}_U^\dagger \longrightarrow \underline{\omega}_U^\dagger \longrightarrow \underline{\omega}^{\dagger, k_0} \longrightarrow 0$

and hence $0 \longrightarrow H^0(X(w), \underline{\omega}_U^\dagger) \xrightarrow{t_{k_0}} H^0(X(w), \underline{\omega}_U^\dagger) \longrightarrow H^0(X(w), \underline{\omega}^{\dagger, k_0}) \longrightarrow 0$

"families of oc mod. forms" "oc mod forms of wt k_0" (

b) Modular symbols:

$k_0 \in U \subseteq \mathcal{W}$ $D_U \longrightarrow D_{k_0}$; T-modules

$\mathbb{Z}_p^\times \curvearrowright T_0 = \mathbb{Z}_p^\times \times \mathbb{Z}_p \subseteq \mathbb{Z}_p^2$

$\circlearrowleft$

$I_W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid p | c \right\}$

k_0: $A_{k_0} = \left\{ f : T_0 \longrightarrow K \;\middle|\; \begin{array}{l} 1) \text{ f analytic} \\ 2) \text{ hom of deg } k_0 \text{ for } \mathbb{Z}_p^\times \end{array} \right\}$

$\circlearrowleft$ I_W

$\hookrightarrow$ i.e $f(\lambda x) = k_0(\lambda) f(x)$ $\forall \lambda \in \mathbb{Z}_p^\times$, $x \in T_0$ (

Now we have Banach module $D_{k_0} = \mathrm{Hom}_{cont}(A_{k_0}, K) \supseteq I_w \supseteq \Gamma$

$$\begin{tikzcd}
D_U & \text{same def using } k_0, A_U \\
H^1(\Gamma, D_{k_0}) \arrow[u] & : \text{ oc mod symbols of wt } k_0 \\
H^1(\Gamma, D_U) \arrow[u] & : \text{ families over } U \text{ of wt } k_0
\end{tikzcd}$$

$D_{k_0}, D_U \rightsquigarrow \mathcal{D}_{k_0}, \mathcal{D}_U$ étale sheaves on $Y(N,p)_{\mathbb{Q}}$ or $Y(N,p)_{\mathbb{Q}_p}$

Results:

Fix $h \in \mathbb{Q}$, $h \geq 0$, $U \subseteq W$ connected affinoid (k_U accessible)

s.t. 1) $\exists\, k \in U$, $k \in \mathbb{Z}$

2) $H^1(\Gamma, D_U)$ and $H^0(X(w), \underline{\omega}_U^{\dagger})$ have h-decompositions.

$H^1(\Gamma, D_U)^{(\leq h)}$ $\quad H^0(X(w), \underline{\omega}_U^{\dagger})^{(\leq h)}$

Theorem: There is a geometric map $\Psi_U^{(h)} : H^1(\Gamma, D_U)^{(\leq h)} \otimes \mathbb{C}_p \to H^0(X(w), \underline{\omega}_U^{\dagger})^{(\leq h)} \otimes_K \mathbb{C}_p$ that commutes with Hecke, G_K-action, with specialization, having the property:

$\exists$ finite set $Z \subseteq U(\mathbb{C}_p)$ s.t. $\forall$ affinoid $V \subseteq U$ defined$/K$, $V(\mathbb{C}_p) \cap Z = \phi$:

$$H^1(\Gamma, D_V)^{(\leq h)} \otimes \mathbb{C}_p \cong \Big(H^0(X(w), \underset{\otimes \underline{\omega}^2}{\underline{\omega}_V^{\dagger}})^{(\leq h)} \otimes \mathbb{C}_p \Big) \oplus P_V^h \otimes \mathbb{C}_p(\chi \cdot \chi_V^{univ})$$

G_K, Hecke.

χ cycl. $\quad$ $$\begin{tikzcd} G_K \arrow[r, "\chi"] \arrow[rr, bend right, "\chi_V^{univ}"'] & \mathbb{Z}_p^\times \arrow[r, "k_V"] & A_V^\times \end{tikzcd}$$

P_V^h is a fin free A_V-mod Hecke module T_ℓ.

Corollary: $\forall\, k \in U(K) - Z$, accessible, then

$$H^1(\Gamma, D_k)^{(\leq h)} \otimes \mathbb{C}_p \cong H^0(X(w), \underline{\omega}^{\dagger, k+2})^{(\leq h)} \otimes \mathbb{C}_p \oplus \left(P_{k+2}^h \otimes \mathbb{C}_p(k+1) \right) \quad \square$$

Corollary: $k \in U(K) - Z$, accessible, $f \in H^0(X(w), \underline{\omega}^{\dagger, k})$ eigenform then

$H^1(\Gamma, D_k)_f^{(\leq h)} \circlearrowleft G_{\mathbb{Q}}$: this is the global Galois rep attached to f by pseudo-rep?