

A. Iovita An overconvergent Eichler-Shimura isomorphism (with Andreatta & Stevens)

Main goal: Geometrically relate 2 different things:



We want geometrically defined maps.

Modular symbols are (cf. Pollack) relevant for special values of (p -adic) L -functions, also Galois rep's.

① Classical Eichler-Shimura

(arithmetic form) $N \geq 3, p \geq 3$ prime, $p \nmid N, \Gamma = \Gamma_1(N) \cap \Gamma_0(p), k \geq 0$

$V_k = \text{Sym}^k(\mathbb{Q}_p^2) \rtimes \Gamma, H^1(\Gamma, V_k) =$ modular symbols of wt. k .

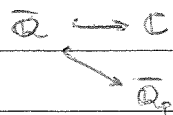
(Δ In Pollack's talk: $MS_p(k) = H^1_c(\Gamma, V_k)$)

Modular symbols ~~define~~ are fin dim vector space, action Hecke operators (T_p, U_p)

Now let $Y(N, p)_{\mathbb{C}}$ be affine scheme representing (E, η_N, C)

$\Gamma = \Pi_1(Y(N, p)_{\mathbb{C}}, \text{base}) \longleftrightarrow$ local system V_k on $Y(N, p)_{\mathbb{C}}$ cyclic order p

$\xleftrightarrow{\text{GAGA}} V_k$ étale local system on $Y(N, p)_{\mathbb{C}} \longleftrightarrow V_k$ étale local system on $Y(N, p)_{\mathbb{C}}$



$\longleftrightarrow V_k$ étale local system on $Y(N, p)_{\mathbb{Q}_p}$ \cong

$$H^1(\Gamma, V_k) \cong H^1(Y(N, p)_{\mathbb{C}}, V_k) \cong H^1(Y(N, p)_{\mathbb{Q}_p}, V_k) \cong H^1(Y(N, p)_{\mathbb{Q}_p}, V_k)$$



Theorem (Faltings)

There is a natural isomorphism

$$H^1(\Gamma, V_k) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \left(H^0(X(N, p), \omega^{k+1}) \otimes \mathbb{C}_p \right) \oplus \left(H^1(X(N, p), \omega^k) \otimes \mathbb{C}_p(k+1) \right)$$

which is compatible with Hecke + $G_{\mathbb{Q}_p}$

② Overconvergence and families

a) modular forms: $\mathcal{W} =$ weight space; it is the rigid analytic space / \mathbb{Q}_p assoc to completed Noeth alg: $\mathbb{Z}_p[[Z_p^*]] \cong \mathbb{Z}_p[[F_p^*]][[T]]$

$$(1+p) \mapsto T$$

There is a universal character: $k^{\text{univ}}: \mathbb{Z}_p^{\times} \longrightarrow \mathcal{O}_{\mathcal{W}}(\mathcal{W})^{\times}$
 \searrow
 $(\mathbb{Z}_p[[Z_p^*]])^{\times}$

$$t \in \mathbb{Z}_p^{\times}, \kappa \in \mathcal{W}: k^{\text{univ}}(t)(\kappa) = \kappa(t)$$

$b_p \in U \subseteq \mathcal{M}$ a connected affinoid (with good red), $k \in U(K)$

$$\mathbb{Z}_p^x \xrightarrow{k} \mathcal{O}_U(\mathcal{M})^x \xrightarrow{k_0} A_U^x \xrightarrow{k_0} K^x$$

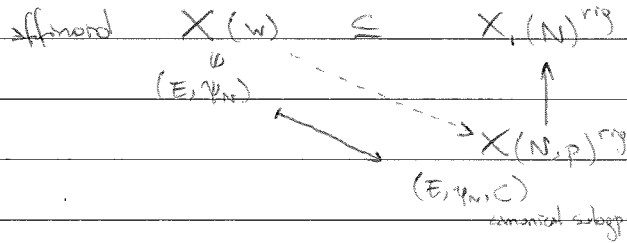
$A_U =$ affinoid alg of U ($A_U \rightarrow K$)

accessible (Coleman)

(assume k_U, k_0 are analytic on $1+p\mathbb{Z}_p \subseteq \mathbb{Z}_p^x$)

Let $w \in \mathbb{Q}$, $0 < w < \frac{p}{p+1}$; $X(w) = \{x \in X_1(N)_{\mathbb{Q}_p} \mid |H_{\text{loc}}(x)| \geq p^{-w}\}$

K/\mathbb{Q}_p finite s.t. $p^{\frac{w}{p-1}} \in K$



Coleman-Stevens defined $\omega_{A_U}^{t, k_0} =: \omega_U^t$ evaluation of the functor on the universal object

loc. free of rk 1 as $A_U \hat{\otimes} \mathcal{O}_{X(w)}$.

$$E, \psi_N, \gamma = \frac{p^w}{E_{p-1}} \quad (p \geq 5)$$

\downarrow
 $X(w)$

ω_U^{t, k_0} sheaves on $X(w)$

loc free of rk 1 as $\mathcal{O}_{X(w)}$ -module.

Now $k_0 \in U$, we have exact

$$0 \rightarrow A_U \xrightarrow{t_{k_0}} A_U \rightarrow K \rightarrow 0$$

so also

$$0 \rightarrow \omega_U^t \rightarrow \omega_U^t \rightarrow \omega_U^{t, k_0} \rightarrow 0$$

and hence

$$0 \rightarrow H^0(X(w), \omega_U^t) \xrightarrow{t_{k_0}} H^0(X(w), \omega_U^t) \rightarrow H^0(X(w), \omega_U^{t, k_0}) \rightarrow 0$$

"families of \mathcal{O}_X mod forms" " \mathcal{O}_X mod forms of wt k_0 "

b) Modular symbols:

$k_0 \in U \subseteq \mathcal{M}$ $D_U \rightarrow D_{k_0}$; Γ -modules

$$\mathbb{Z}_p^x \hookrightarrow T_U = \mathbb{Z}_p^x \times \mathbb{Z}_p \subseteq \mathbb{Z}_p^c$$

\cup

$$I_U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid p|c \right\}$$

k_0 :

$$A_{k_0} = \left\{ f: T_U \rightarrow K \mid \begin{array}{l} 1) f \text{ analytic} \\ 2) \text{ hom of deg } k_0 \text{ for } \mathbb{Z}_p^x \end{array} \right\}$$

\hookrightarrow
 I_U

\hookrightarrow i.e. $f(\lambda x) = \kappa_0(\lambda) f(x)$ $\forall \lambda \in \mathbb{Z}_p^x, x \in T_U$

Now we have Brieskorn module $D_{k_0} = \text{Hom}_{\text{cont}}(A_{k_0}, K) \supseteq I_w \supseteq \Gamma$

D_0 same def using k_0, A_0



$H^1(\Gamma, D_{k_0})$: \mathcal{O}_C mod symbols of wt k_0



$H^1(\Gamma, D_0)$: families over V of wt k_0

$D_{k_0}, D_0 \rightsquigarrow D_{k_0}, D_0$ étale sheaves on $Y(N, p)_{\mathbb{Q}}^*$ or $Y(N, p)_{\mathbb{C}_p}$

Results:

Fix $h \in \mathbb{Q}, h \geq 0, U \subseteq W$ connected affinoid (k_0 accessible)

st 1) $\exists k \in U, k \in \mathbb{Z}$

2) $H^1(\Gamma, D_0)$ and $H^0(X(w), \omega_0^+)$ have h -decompositions.

$$H^1(\Gamma, D_0)^{(sh)} \quad H^0(X(w), \omega_0^+)^{(sh)}$$

Theorem: There is a geometric map $\gamma_U^{(h)}: H^1(\Gamma, D_0)^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p \rightarrow H^0(X(w), \omega_0^+)^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p$ that commutes with Hecke, G_K -action, with specialization, having the property:

\exists finite set $Z \subseteq U(\mathbb{C}_p)$ st \forall affinoid $V \in U$ defined $/k, V(\mathbb{C}_p) \cap Z = \emptyset$:

$$H^1(\Gamma, D_0)^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p \cong \left(H^0(X(w), \omega_0^+)^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p \right) \oplus P_r^h \otimes_{\mathbb{C}_p} (X \cdot X_V^{univ})$$

G_K, Hecke

$$X \text{ cycl.} \quad G_K \xrightarrow{\alpha} Z^X \xrightarrow{h_V} A_V^X \quad P_r^h \text{ is a fin free } A_0\text{-mod}$$

$\xrightarrow{X_V^{univ}}$

Hecke module T_0 .

Corollary: $\forall k \in U(K) - Z$, accessible, then

$$H^1(\Gamma, D_k)^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p \cong H^0(X(w), \omega^{T, k, 2})^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p \oplus (P_{r, k}^h \otimes_{\mathbb{C}_p} \mathbb{C}_p(k+1)) \quad \square$$

Corollary: $k \in U(K) - Z$, accessible, $f \in H^0(X(w), \omega^{T, k})$ eigenform then

$H^1(\Gamma, D_k)^{(sh)} \otimes_{\mathbb{C}_p} \mathbb{C}_p \supseteq G_{\mathbb{Q}}$: this is the global Galois rep attached to f by pseudo-rep?

