The $p$-adic Gross-Zagier formula
at supersingular primes

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What is the \( p \)-adic Gross-Zagier formula?

(Settings and explicit formulas are given later.)

The classical Gross-Zagier formula

\[
L'(E/K, 1) \sim \text{the height of the Heegner point in } E(K)
\]

(Gross-Zagier (1986))

The \( p \)-adic Gross-Zagier formula

\[
L'_p(E/K, 1) \sim \text{the } p\text{-adic height of the Heegner point in } E(K)
\]

(good ordinary \( p \): Perrin-Riou (1987), supersingular \( p \): K., (2012).)

(∃ Bertolini-Darmon’s works in different settings.)
Suppose that $\text{ord}_{s=1} L(E/\mathbb{Q}, s) = 1$.

- The classical G-Z formula $\Rightarrow$ the weak BSD ($\text{rank } E(\mathbb{Q}) = 1$) the full BSD (lead. coeff.) up to $\mathbb{Q}^\times$

- The $p$-adic G-Z formula $\Rightarrow$ the full BSD up to $\mathbb{Z}_{(p)}^\times$

**Theorem (K. etc)**

*If $E$ has CM and of analytic rank 1, then the full BSD conjecture is true up to a power of bad primes and 2. Namely,*

$$
\frac{L'(E/\mathbb{Q}, 1)}{\Omega_{E/\mathbb{Q}}^+ \text{Reg}_\infty(E/\mathbb{Q})} = \nu \frac{\#\text{III}(E/\mathbb{Q}) \prod_{\ell | N} \text{Tam}_{\ell}(E/\mathbb{Q})}{\#E(\mathbb{Q})_{\text{tor}}^2}
$$

*for some $\nu \in \mathbb{Z}[1/2N]^\times$.***
Let \( \mathcal{L}_p(E/\mathbb{Q}, \alpha, s) \) be the \( p \)-adic \( L \)-function. As a corollary of \( p \)-GZ, we show

**Theorem**

Suppose \( E \) has CM or good supersingular reduction at \( p \).

1. \( \text{ord}_{s=1} L(E/\mathbb{Q}, s) = 1 \iff \text{ord}_{s=1} \mathcal{L}_p(E/\mathbb{Q}, \alpha, s) = 1. \)

2. If \( \text{ord}_{s=1} L(E/\mathbb{Q}, s) = 1 \), we have

\[
\frac{L'(E/\mathbb{Q}, 1)}{\Omega_E^+ \text{Reg}_\infty(E/\mathbb{Q})} = \left(1 - \frac{1}{\alpha}\right)^{-2} \frac{\mathcal{L}'_p(E/\mathbb{Q}, \alpha, 1)}{\text{Reg}_{p, \alpha}(E/\mathbb{Q})}.
\]

Namely, if the analytic rank is 1, the complex and \( p \)-adic BSD are equivalent.

- If \( E \) has CM or good supersingular reduction at \( p \), the \( p \)-adic height pairing is known to be non-trivial. In particular, \( \text{Reg}_{p, \alpha}(E/\mathbb{Q}) \neq 0 \) if \( \text{rank} E(\mathbb{Q}) = 1 \).

- Main conjecture (for CM) and Perrin-Riou and Schneider’s result on the \( p \)-adic BSD up to a \( p \)-adic unit \( \Rightarrow \) BSD for CM elliptic curve of rank 1.
The classical Gross-Zagier formula

- $E$: an elliptic curve over $\mathbb{Q}$ of conductor $N$.
- $K$: an imaginary quadratic field with discriminant $d_K$.
- $L(E/K, s)$: the Hasse-Weil $L$-function of $E/K$.

**Heegner condition:** prime $\ell|N \Rightarrow \ell$ splits in $K$.

This condition implies that

- $L(E/K, 1) = 0$. (The sign of fun. eq. is $-1$.)
- $\exists$ Heegner points $z_H \in X_0(N)(H)$ over the Hilbert class field $H/K$. Take $\pi : X_0(N) \to E$ and for Manin constant $c_\pi$ we put
  $$z_{K,E} = \text{Tr}_{H/K} \pi(z_H) \otimes c_\pi^{-1} \in E(K) \otimes \mathbb{Q}.$$
Theorem (Gross-Zagier (1986))

Assume the Heegner condition and \((2N, d_K) = 1\). Then

\[
L'(E/K, 1) = u^{-2} \Omega_{E/K} \langle z_{K,E}, z_{K,E} \rangle_{\infty,K}.
\]

- \(u = \#O_K^\times / 2\)
- \(\langle , \rangle_{\infty,K}: \text{the Néron-Tate height pairing of } E/K\)
- \(\Omega_{E/K} = \frac{1}{\sqrt{|d_K|}} \int_{E(\mathbb{C})} \omega_E \wedge i \overline{\omega_E} \sim \text{the area of } E(\mathbb{C})\)

Recall by using Waldspruger’s non-vanishing result and Kolyvagin’s result,

\[
\text{ord}_{s=1} L(E/\mathbb{Q}, s) = 1 \implies \frac{L'(E/\mathbb{Q}, 1)}{\Omega_E^+ \text{Reg}_\infty(E/\mathbb{Q})} \in \mathbb{Q}, \quad \#\text{III}(E/\mathbb{Q}) < \infty.
\]
The $p$-adic $L$-function of $E/K$

- $p$: a good prime of $E/\mathbb{Q}$ prime to $d_K$.
- $\alpha$: an admissible root of the $p$-Euler factor $X^2 - a_p X + p$.
  (ordinary: $\alpha$ is a unit root, supersingular: any root.)

$\exists!$ a $p$-adic distribution $d\mu_{E/\mathbb{Q},\alpha}$ on $\mathbb{Z}_p$ of order $< 1$ characterized by

$$
\int_{\mathbb{Z}_p^\times} \chi d\mu_{E/\mathbb{Q},\alpha} = \frac{\tau(\chi) L(E/\mathbb{Q}, \overline{\chi}, 1)}{\alpha^n \Omega_{E/\mathbb{Q}}^\pm} \in \overline{\mathbb{Q}} \subset \mathbb{C}_p
$$

($\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}^\times$, $\Omega_{E/\mathbb{Q}}^\pm$: real or imaginary period, $\tau(\chi)$: Gauss sum.)

The cyclotomic $p$-adic $L$-function of $E/\mathbb{Q}$ is defined by

$$
\mathcal{L}_p(E/\mathbb{Q}, \alpha, s) = \int_{\mathbb{Z}_p^\times} \langle \chi \rangle^{s-1} d\mu_{E/\mathbb{Q},\alpha}
$$

as a $p$-adic analytic function of $s$ on $\mathbb{Z}_p$. 

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The cyclotomic $p$-adic $L$-function of $E$ over $K$ is defined by

$$
\mathcal{L}_p(E/K, \alpha, s) := \mathcal{L}_p(E/\mathbb{Q}, \alpha, s) \mathcal{L}_p(E^\varepsilon/\mathbb{Q}, \varepsilon(p)\alpha, s) \frac{\Omega_{E/Q}^+ \Omega_{E^\varepsilon/Q}^+}{\Omega_{E/K}^+}.
$$

($\varepsilon$: the character of $K/\mathbb{Q}$, $E^\varepsilon$: the twist of $E$ by $\varepsilon$.)

**Theorem (ordinary: Perrin-Riou, supersingular: K.)**

Assume the Heegner cond., $(2N, d_K) = 1$ and $p$ splits in $K$. Then

$$
\mathcal{L}_p'(E/K, \alpha, 1) = u^{-2} \left(1 - \frac{1}{\alpha}\right)^2 \left(1 - \frac{1}{\varepsilon(p)\alpha}\right)^2 \langle z_K, E, z_K, E \rangle_{p, K, \alpha}.
$$

$\langle \ , \ \rangle_{p, K, \alpha}$: the cyclotomic $p$-adic height pairing of $E/K$ corresponding to the $\alpha$-eigen space of the Frobenius of the Dieudonné module of $E$.

- For supersingular $p$, the inert case is also OK.
The basic strategy of the proof of our GZ formula is almost the same as the classical case and the $p$-adic ordinary case.

We construct two $p$-adic modular forms.

- \( F \): a $p$-adic modular form knowing the $p$-adic height of \( z_H \).
- \( G \): a $p$-adic modular form knowing the value \( L'_p(E/K, \alpha, 1) \).

Calculate their Fourier coefficients independently.
Then it turns out they are equal. (We don’t know the reason.)
We define a $p$-adic modular form $F$ as

$$F := \sum_{\sigma \in \text{Gal}(H/K)} \sum_{n=1}^{\infty} \langle z_H, T_n z_H^\sigma \rangle_{p,H,\alpha} q^n.$$ 

where $T_n$’s are Hecke operators for $\Gamma_0(N)$.

Use $\langle \cdot, \cdot \rangle_{p,H,\alpha} = \sum_v \langle \cdot, \cdot \rangle_{p,v,\alpha}$ to compute of the $p$-adic height.

- $v \nmid p$: the computation of $\langle \cdot, \cdot \rangle_{p,v,\alpha}$ is the same as Néron-Tate case. It is the intersection pairing and reduced to the computation of Gross-Zagier.

- $v | p$: we show that $\langle z_H, T_n z_H^\sigma \rangle_{p,v,\alpha}$ is essentially zero. ($p$: split)
The strategy to show the vanishing of the local height is the same as that in the ordinary case by Perrin-Riou. Write as \( \langle z_H^\sigma, T_m z_H^\sigma \rangle_{p, \alpha, \nu} = \log_p x_m \) with \( x_m \in \mathbb{Z}_p^\times \).

**The basic principle**

\( Q_{p,\infty} \): the cyclotomic \( \mathbb{Z}_p \)-extension of \( Q_p \), \( Q_{p,n} \): the \( n \)-th layer.

For \( x \in Q_p^\times \),

\[
\log_p x \equiv 0 \mod p^n \iff x \in N_{Q_{p,n}/Q_p} Q_{p,n}^\times
\]

(Class field theory, \( p \in N_{Q_{p,\infty}/Q_p} Q_{p,\infty}^\times \) (universal norm), \( \log_p p = 0 \).)

Try the above for \( x = x_m \). Actually, in the supersingular case, numerous denominators appear and the principle is not directly applied. Need the splitting of Hodge filtration by \( \alpha \)-eigen space to control denominators.
Key ingredients for the vanishing

We want to show that $x_m$ is a norm from the $n$-th layer.

**Key**

The norm construction of the $p$-adic height pairing.
(ordinary: Schneider, supersingular: Perrin-Riou improved by K.)

- **ordinary case**: Use the universal norm group $N_{\mathbb{Q}_p,\infty}/\mathbb{Q}_p E(\mathbb{Q}_p,\infty)$.

- **supersingular case**: no universal norm but use

  $\exists$ norm systems $(c_n) \in \prod_{n} E(\mathbb{Q}_p,n)$ satisfying

  $$\text{Tr}_{n+1/n} c_{n+1} - a_p c_n + c_{n-1} = 0.$$  \hspace{1cm} (1)

- $\exists$ A system of Heegner points satisfies the same relation in the
  anti-cyclotomic direction. (For split $p$, cyclotomic = anti-cyclotomic over $\hat{\mathbb{Q}_p}^{ur}$. However, we need to care the uniformizer: $\log_p p = 0$.)
To apply Perrin-Riou’s norm construction to Heegner pts, we need Coleman power series for $\hat{E}$ of height 2.

**Coleman Power Series for $\hat{E}$ of Height 2**

Given a system $(P_n)_n \in \prod_n \hat{E}(\mathbb{Q}_p,n)$ satisfying

\[
\text{Tr}_{n+1/n} P_{n+1} - a_p P_n + P_{n-1} = 0,
\]

(2)

\[\exists f(T) \in \mathbb{Z}_p[[T]] \text{ such that } f(\zeta_p^n - 1) = P_n \text{ for all } n \text{ if and only if}
\]

\[(P_{n+1})^p \equiv P_n \mod p\mathcal{O}_{\mathbb{C}_p}
\]

(3)

where $\hat{E}(\mathbb{C}_p) = \mathfrak{m}_{\mathbb{C}_p}$. Moreover, $(\varphi - a_p + \psi) \log \hat{E} f = 0$ and $f(0) \in p\mathbb{Z}_p$.

Conversely, such $f \in \mathbb{Z}_p[[T]]$ gives a system satisfying (2), (3).

- A system of Heegner points of higher order satisfies (2) and (3).
- The last part is due to Perrin-Riou.
- K. Ota generalized this to higher dimensional formal groups with general height over unramified rings. (Use Knospe’s result.)
Construct the Eisenstein measure $d\Phi_\sigma$ on $\mathbb{Z}_p$ valued in the space of $p$-adic modular form of level $Np^\infty$. For this, we use convolutions of Eisenstein series and theta functions attached to $K$ of level $p$-power.

Consider

$$G^\sigma := \frac{d}{ds} \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} d\Phi_\sigma \big|_{s=1} = \int_{\mathbb{Z}_p^\times} \log_p \langle x \rangle d\Phi_\sigma.$$ 

Put $G = \sum_\sigma G^\sigma \in \overline{M}_2(\Gamma_0(Np^\infty), \mathbb{Z}_p) := \bigcup_n M_2(\Gamma_0(Np^n), \mathbb{Z}_p)$.

Suppose that $E$ corresponds to a new form $f$.

$$\text{Rankin-Selberg} \implies (G, f) = \text{the derivative of the } p\text{-adic } L\text{-fun}?$$

How do we take the Petersson inner product $p$-adically?
Relation between $G$ and $\mathcal{L}_p'(E/K, 1)$

The ordinary case

Consider Hida’s ordinary projection (which kills the supersingular part)

$$e := \lim_{n \to \infty} U_p^n : \overline{M}_2(\Gamma_0(Np^\infty), \mathbb{Z}_p) \to M_2(\Gamma_0(Np), \mathbb{Z}_p).$$

("holomorphication". cf. Strum’s holomorphic projection.)

On $M_2(\Gamma_0(Np), \mathbb{Z}_p) = M_2(\Gamma_0(Np), \mathbb{Z}) \otimes \mathbb{Z}_p$, the Petersson inner product is extended linearly.

If $f$ is ordinary at $p$, then

$$\langle f, e \sum_\sigma \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} d\Phi_\sigma \rangle \sim \mathcal{L}_p(E/K, \alpha, s)$$

In particular, the $f$-part of $G^o := e G$ is $\mathcal{L}_p'(E/K, \alpha, 1)$. 
Relation between $G$ and $\mathcal{L}_p'(E/K, 1)$

The supersingular case

Eisenstein measure $\sum_{\sigma} d\Phi_{\sigma} \quad \overset{\text{big gap}}{\longrightarrow} \quad p$-adic $L$-fun. of $E/K$

Crucial fact

$\mathcal{L}_p(E/K, \alpha, s)$ cannot be constructed as an 1-admissible distribution!

(1-admissible $\iff$ of order $< 1 \iff$ determined by values at step functions.)

It is not characterized by the interpolation property at critical values.

- This is why we define $\mathcal{L}_p(E/K, \alpha, s)$ as a product of $p$-adic $L$'s $/\mathbb{Q}$, and treat only the cyclotomic $p$-adic $L$ in the supersingular case.
- Another construction (e.g. Rankin-Selberg) seems difficult.
Consider the naive two-variable cyclotomic $p$-adic $L$-function:

$$L_p(E, \varepsilon, \alpha, s, t) := L_p(E/\mathbb{Q}, \alpha, s)L_p(E^\varepsilon/\mathbb{Q}, \varepsilon(p)\alpha, t)\frac{\Omega_E^{+}/\Omega_{E/\mathbb{Q}}^{+}}{\Omega_{E/\mathbb{K}}^{+}}.$$ 

This function is 1-admissible on horizontal and vertical directions, and we can capture the diagonal direction by these directions.

**Solution**

Make everything two variables: construct two-variable Eisenstein measure $d\Phi(x, y)$ and recover $L_p(E, \varepsilon, \alpha, s, t)$ by the Rankin-Selberg method.

- Use Kato’s zeta element in the space of modular forms.
- The diagonal direction of $d\Phi(x, y)$ is the one-variable $\sum d\Phi_\sigma$. 
\[
\int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(x, y) \, d\Phi(x, y) \xrightarrow{f\text{-part}} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(x, y) \, d\mu \mathcal{L}_p(x, y).
\]

The ordinary projection \( e \) does not commute the operation to take \( f\text{-part} \). However, the diagonal and the horizontal directions are good.

\[
G = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \log_p \langle xy \rangle \, d\Phi(x, y) (\sum_{\sigma} \int_{\mathbb{Z}_p^\times} \log_p \langle x \rangle \, d\Phi_{\sigma}(x))
\xrightarrow{f\text{-part}} \mathcal{L}_p'(E/K, \alpha, 1) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \log_p \langle xy \rangle \, d\mu \mathcal{L}_p(x, y).
\]

- \( f(x, y) = \log_p \langle xy \rangle = \log_p \langle x \rangle + \log_p \langle y \rangle \). (horizontal & vertical.)
- \( G \) is finally shown to be essentially \( F \), of finite level \( N \).

\( G \) is finally related to \( \mathcal{L}_p'(E/K, \alpha, 1) \)!
Thank you very much!