The p-adic Gross-Zagier formula at supersingular primes

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What is the p-adic Gross-Zagier formula ?

(Settings and explicit formulas are given later.)

The classical Gross-Zagier formula

$$L'(E/K,1) \sim \text{the height of the Heegner point in } E(K)$$

(Gross-Zagier (1986))

The p-adic Gross-Zagier formula

$$\mathscr{L}_p'(E/K,1) \sim \text{the } p\text{-adic height of the Heegner point in } E(K)$$

(good ordinary p: Perrin-Riou (1987), supersingular p: K., (2012).)

($\exists$ Bertolini-Darmon's works in different settings.)

Application for the BSD conj.

Suppose that $\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s) = 1$.

- The classical G-Z formula $\Rightarrow$ the weak BSD ($\mathrm{rank}\, E(\mathbb{Q}) = 1$)
  the full BSD (lead. coeff.) up to $\mathbb{Q}^\times$
- The p-adic G-Z formula $\Rightarrow$ the full BSD up to $\mathbb{Z}_{(p)}^\times$

Theorem (K. etc)

If E has CM and of analytic rank 1, then the full BSD conjecture is true up to a power of bad primes and 2. Namely,

$$\frac{L'(E/\mathbb{Q}, 1)}{\Omega^+_{E/\mathbb{Q}} \mathrm{Reg}_\infty(E/\mathbb{Q})} = v \frac{\sharp Ш(E/\mathbb{Q}) \prod_{\ell | N} \mathrm{Tam}_\ell(E/\mathbb{Q})}{\sharp E(\mathbb{Q})^2_{\mathrm{tor}}}$$

for some $v \in \mathbb{Z}[1/2N]^\times$.

Let $\mathscr{L}_p(E/\mathbb{Q},\alpha,s)$ be the p-adic L-function. As a corollary of p-GZ, we show

Theorem

Suppose E has CM or good supersingular reduction at p.

1. $\mathrm{ord}_{s=1} L(E/\mathbb{Q},s) = 1 \iff \mathrm{ord}_{s=1}\mathscr{L}_p(E/\mathbb{Q},\alpha,s) = 1.$
2. *If* $\mathrm{ord}_{s=1} L(E/\mathbb{Q},s) = 1$, *we have*

$$\frac{L'(E/\mathbb{Q},1)}{\Omega_E^+ \operatorname{Reg}_\infty(E/\mathbb{Q})} = \left(1-\frac{1}{\alpha}\right)^{-2} \frac{\mathscr{L}_p'(E/\mathbb{Q},\alpha,1)}{\operatorname{Reg}_{p,\alpha}(E/\mathbb{Q})}.$$

Namely, if the analytic rank is 1, *the complex and p-adic BSD are equivalent.*

- If E has CM or good supersingular reduction at p, the p-adic height pairing is known to be non-trivial. In particular, $\operatorname{Reg}_{p,\alpha}(E/\mathbb{Q}) \neq 0$ if $\operatorname{rank} E(\mathbb{Q}) = 1$.
- Main conjecture (for CM) and Perrin-Riou and Schneider's result on the p-adic BSD up to a p-adic unit $\Rightarrow$ BSD for CM elliptic curve of rank 1.

The classical Gross-Zagier formula

- E: an elliptic curve over $\mathbb{Q}$ of conductor N.
- K: an imaginary quadratic field with discriminant d_K.
- $L(E/K, s)$: the Hasse-Weil L-function of E/K.

<u>Heegner condition</u>: prime $\ell | N \Rightarrow \ell$ splits in K.

This condition implies that

- $L(E/K, 1) = 0$. (The sign of fun. eq. is -1.)
- $\exists$ Heegner points $z_H \in X_0(N)(H)$ over the Hilbert class field H/K. Take $\pi : X_0(N) \to E$ and for Manin constant c_π we put

$$z_{K,E} = \mathrm{Tr}_{H/K}\, \pi(z_H) \otimes c_\pi^{-1} \in E(K) \otimes \mathbb{Q}.$$

Theorem (Gross-Zagier (1986))

Assume the Heegner condition and $(2N, d_K) = 1$*. Then*

$$L'(E/K, 1) = u^{-2}\,\Omega_{E/K}\,\langle z_{K,E}, z_{K,E}\rangle_{\infty,K}.$$

- $u = \sharp\mathcal{O}_K^\times/2$
- $\langle\ ,\ \rangle_{\infty,K}$*: the Néron-Tate height pairing of* E/K
- $\Omega_{E/K} = \frac{1}{\sqrt{|d_K|}}\int_{E(\mathbb{C})}\omega_E \wedge i\,\overline{\omega_E} \sim$ *the area of* $E(\mathbb{C})$

Recall by using Waldspruger's non-vanishing result and Kolyvagin's result,

$$\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s) = 1 \quad \Longrightarrow \quad \frac{L'(E/\mathbb{Q}, 1)}{\Omega_E^+\,\mathrm{Reg}_\infty(E/\mathbb{Q})} \in \mathbb{Q}, \quad \sharp\text{Ш}(E/\mathbb{Q}) < \infty.$$

THE p-ADIC L-FUNCTION OF E/K

- p: a good prime of $E/\mathbb{Q}$ prime to d_K.
- α: an admissible root of the p-Euler factor $X^2 - a_p X + p$.
  (ordinary: α is a unit root, supersingular: any root.)

$\exists!$ a p-adic distribution $d\mu_{E/\mathbb{Q},\alpha}$ on $\mathbb{Z}_p$ of order < 1 characterized by

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_{E/\mathbb{Q},\alpha} = \frac{\tau(\chi)}{\alpha^n} \frac{L(E/\mathbb{Q}, \overline{\chi}, 1)}{\Omega^{\pm}_{E/\mathbb{Q}}} \in \overline{\mathbb{Q}} \subset \mathbb{C}_p$$

($\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$, $\Omega^{\pm}_{E/\mathbb{Q}}$: real or imaginary period, $\tau(\chi)$: Gauss sum.)
The cyclotomic p-adic L-function of $E/\mathbb{Q}$ is defined by

$$\mathscr{L}_p(E/\mathbb{Q}, \alpha, s) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} d\mu_{E/\mathbb{Q},\alpha}$$

as a p-adic analytic function of s on $\mathbb{Z}_p$.

The cyclotomic p-adic L-function of E over K is defined by

$$\mathscr{L}_p(E/K,\alpha,s) := \mathscr{L}_p(E/\mathbb{Q},\alpha,s)\mathscr{L}_p(E^{\varepsilon}/\mathbb{Q},\varepsilon(p)\alpha,s)\frac{\Omega^+_{E/\mathbb{Q}}\Omega^+_{E^{\varepsilon}/\mathbb{Q}}}{\Omega_{E/K}}.$$

(ε: the character of $K/\mathbb{Q}$, E^{ε}: the twist of E by ε.)

Theorem (ordinary: Perrin-Riou, supersingular: K.)

Assume the Heegner cond., $(2N, d_K) = 1$ and p splits in K. Then

$$\mathscr{L}'_p(E/K,\alpha,1) = u^{-2}\left(1-\frac{1}{\alpha}\right)^2\left(1-\frac{1}{\varepsilon(p)\alpha}\right)^2\langle z_{K,E}, z_{K,E}\rangle_{p,K,\alpha}.$$

$\langle\ ,\ \rangle_{p,K,\alpha}$*: the cyclotomic p-adic height pairing of E/K corresponding to the α-eigen space of the Frobenius of the Dieudonné module of E.*

- For supersingular p, the inert case is also OK.

The strategy of the proof

The basic strategy of the proof of our GZ formula is almost the same as the classical case and the p-adic ordinary case.

We construct two p-adic modular forms.

- F: a p-adic modular form knowing the p-adic height of z_H.
- G: a p-adic modular form knowing the value $\mathscr{L}'_p(E/K, \alpha, 1)$.

Calculate their Fourier coefficients independently.

Then it turns out they are equal. (We don't know the reason.)

The modular form F

We define a p-adic modular form F as

$$F := \sum_{\sigma \in \mathrm{Gal}(H/K)} \sum_{n=1}^{\infty} \langle z_H, T_n z_H^{\sigma} \rangle_{p,H,\alpha} \, q^n.$$

where T_n's are Hecke operators for $\Gamma_0(N)$.
Use $\langle \ , \ \rangle_{p,H,\alpha} = \sum_v \langle \ , \ \rangle_{p,v,\alpha}$ to compute of the p-adic height.

- $v \nmid p$: the computation of $\langle \ , \ \rangle_{p,v,\alpha}$ is the same as Néron-Tate case. It is the intersection pairing and reduced to the computation of Gross-Zagier.
- $v \mid p$: we show that $\langle z_H, T_n z_H^{\sigma} \rangle_{p,v,\alpha}$ is essentially zero. (p: split)

Local height at $v \mid p$

The strategy to show the vanishing of the local height is the same as that in the ordinary case by Perrin-Riou.

Write as $\langle z_H^\sigma, T_m z_H^\sigma \rangle_{p,\alpha,v} = \log_p x_m$ with $x_m \in \mathbb{Z}_p^\times$.

The basic principle

$\mathbb{Q}_{p,\infty}$: the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$, $\mathbb{Q}_{p,n}$: the n-th layer.

For $x \in \mathbb{Q}_p^\times$,

$$\log_p x \equiv 0 \mod p^n \iff x \in N_{\mathbb{Q}_{p,n}/\mathbb{Q}_p} \mathbb{Q}_{p,n}^\times$$

(Class field theory, $p \in N_{\mathbb{Q}_{p,\infty}/\mathbb{Q}_p} \mathbb{Q}_{p,\infty}^\times$ (universal norm), $\log_p p = 0$.)

Try the above for $x = x_m$. Actually, in the supersingular case, numerous denominators appear and the principle is not directly applied. Need the splitting of Hodge filtration by α-eigen space to control denominators.

Key ingredients for the vanishing

We want to show that x_m is a norm from the n-th layer.

Key

The norm construction of the p-adic height pairing.

(ordinary: Schneider, supersingular: Perrin-Riou improved by K.)

- ordinary case: Use the universal norm group $N_{\mathbb{Q}_{p,\infty}/\mathbb{Q}_p} E(\mathbb{Q}_{p,\infty})$.
- supersingular case: no universal norm but use $\exists$ norm systems $(c_n) \in \prod_n E(\mathbb{Q}_{p,n})$ satisfying

$$\mathrm{Tr}_{n+1/n}\, c_{n+1} - a_p c_n + c_{n-1} = 0. \tag{1}$$

- $\exists$ A system of Heegner points satisfies the same relation in the anti-cyclotomic direction. (For split p, cyclotomic $=$ anti-cyclotomic over $\hat{\mathbb{Q}}_p^{\mathrm{ur}}$. However, we need to care the uniformizer: $\log_p p = 0$.)

To apply Perrin-Riou's norm construction to Heegner pts, we need

Coleman power series for $\hat{E}$ of height 2

Given a system $(P_n)_n \in \prod_n \hat{E}(\mathbb{Q}_{p,n})$ satisfying

$$\mathrm{Tr}_{n+1/n} P_{n+1} - a_p P_n + P_{n-1} = 0, \tag{2}$$

$\exists\ f(T) \in \mathbb{Z}_p[[T]]$ such that $f(\zeta_{p^n} - 1) = P_n$ for all n if and only if

$$(P_{n+1})^p \equiv P_n \mod p\mathcal{O}_{\mathbb{C}_p} \tag{3}$$

where $\hat{E}(\mathbb{C}_p) = \mathfrak{m}_{\mathbb{C}_p}$. Moreover, $(\varphi - a_p + \psi) \log_{\hat{E}} f = 0$ and $f(0) \in p\mathbb{Z}_p$.
Conversely, such $f \in \mathbb{Z}_p[[T]]$ gives a system satisfying (2), (3).

- A system of Heegner points of higher order satisfies (2) and (3).
- The last part is due to Perrin-Riou.
- K. Ota generalized this to higher dimensional formal groups with general height over unramified rings. (Use Knospe's result.)

The modular form G

1. Construct the Eisenstein measure $d\Phi_\sigma$ on $\mathbb{Z}_p$ valued in the space of p-adic modular form of level Np^∞. For this, we use convolutions of Eisenstein series and theta functions attached to K of level p-power.
2. Consider

$$G^\sigma := \frac{d}{ds} \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} d\Phi_\sigma \mid_{s=1} = \int_{\mathbb{Z}_p^\times} \log_p \langle x \rangle d\Phi_\sigma.$$

Put $G = \sum_\sigma G^\sigma \in \overline{M}_2(\Gamma_0(Np^\infty), \mathbb{Z}_p) := \overline{\cup_n M_2(\Gamma_0(Np^n), \mathbb{Z}_p)}$.

Suppose that E corresponds to a new form f.

Rankin-Selberg $\Longrightarrow$ (G, f) = the derivative of the p-adic L-fun ?

How do we take the Petersson inner prodct p-adically ?

Relation between G and $\mathscr{L}'_p(E/K,1)$

The ordinary case

Consider Hida's ordinary projection (which kills the supersingular part)

$$e := \lim_{n\to\infty} U_p^{n!} \;:\; \overline{M}_2(\Gamma_0(Np^\infty),\mathbb{Z}_p) \to M_2(\Gamma_0(Np),\mathbb{Z}_p).$$

("holomorphication". cf. Strum's holomorphic projection.)

On $M_2(\Gamma_0(Np),\mathbb{Z}_p) = M_2(\Gamma_0(Np),\mathbb{Z}) \otimes \mathbb{Z}_p$, the Petersson inner product is extended linearly.

If f is ordinary at p, then

$$(f,\; e\sum_\sigma \int_{\mathbb{Z}_p^\times} \langle x\rangle^{s-1} d\Phi_\sigma) \;\sim\; \mathscr{L}_p(E/K,\alpha,s)$$

In particular, the f-part of $G^o := e\,G$ is $\mathscr{L}'_p(E/K,\alpha,1)$.

Relation between G and $\mathscr{L}'_p(E/K,1)$

The supersingular case

$$\text{Eisenstein measure } \textstyle\sum_\sigma d\Phi_\sigma \quad \overset{\text{big gap}}{\longleftrightarrow} \quad p\text{-adic } L\text{-fun. of } E/K$$

Crucial fact

$\mathscr{L}_p(E/K,\alpha,s)$ can not be constructed as an 1-admissible distribution !

(1-admissible $\leftrightarrow$ of order < 1 $\leftrightarrow$ determined by values at step functions.)

It is not characterized by the interpolation property at critical values.

- This is why we define $\mathscr{L}_p(E/K,\alpha,s)$ as a product of p-adic L's $/\mathbb{Q}$, and treat only the cyclotomic p-adic L in the supersingular case.
- Another construction (e.g. Rankin-Selberg) seems difficult.

Key Observation

Consider the naive two-variable cyclotomic p-adic L-function:

$$\mathscr{L}_p(E,\varepsilon,\alpha,s,t) := \mathscr{L}_p(E/\mathbb{Q},\alpha,s)\mathscr{L}_p(E^\varepsilon/\mathbb{Q},\varepsilon(p)\alpha,t)\frac{\Omega^+_{E/\mathbb{Q}}\Omega^+_{E^\varepsilon/\mathbb{Q}}}{\Omega_{E/K}}.$$

This function is 1-admissible on horizontal and vertical directions, and we can capture the diagonal direction by these directions.

Solution

Make everything two variables: construct two-variable Eisenstein measure $d\Phi(x,y)$ and recover $\mathscr{L}_p(E,\varepsilon,\alpha,s,t)$ by the Rankin-Selberg method.

- Use Kato's zeta element in the space of modular forms.
- The diagonal direction of $d\Phi(x,y)$ is the one-variable $\sum d\Phi_\sigma$.

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(x,y)\, d\Phi(x,y) \quad \xrightarrow[f\text{-part ?}]{\text{relation ?}} \quad \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(x,y)\, d\mu_{\mathscr{L}_p}(x,y).$$

The ordinary projection e does not commute the operation to take f-part.

However, the diagonal and the horizontal directions are good.

$$G = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \log_p \langle xy \rangle \, d\Phi(x,y) \; (= \sum_\sigma \int_{\mathbb{Z}_p^\times} \log_p \langle x \rangle \, d\Phi_\sigma(x))$$

$$\xrightarrow{f\text{-part}} \quad \mathscr{L}_p'(E/K, \alpha, 1) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \log_p \langle xy \rangle \, d\mu_{\mathscr{L}_p}(x,y).$$

- $f(x,y) = \log_p \langle xy \rangle = \log_p \langle x \rangle + \log_p \langle y \rangle$. (horizontal & vertical.)
- G is finally shown to be essentially F, of finite level N.

G is finally related to $\mathscr{L}_p'(E/K, \alpha, 1)$!

Thank you very much !