Monday 24/09

## Kartik Prasanna: p-adic L-functions and the coniveau filtration on Chow groups.

$k$ field, $X$ smooth proj $/k$, $Z^j(X)$ alg cycles of codim $j$ on $X$, dof $/k$

$$Z^j(X)_0 \supseteq Z^j(X)_{alg} \supseteq Z^j(X)_{rat}$$

$CH^j(X) = Z^j(X)/Z^j(X)_{rat}$

$CH^j(X)_0 = Z^j(X)_0/Z^j(X)_{rat}$ $\qquad CH^j(X)_{\mathbb{Q}} := CH^j(X)\otimes\mathbb{Q}$ etc.

$CH^j(X)_{alg} = Z^j(X)_{alg}/Z^j(X)_{rat}$

Bloch - Beilinson conj: $k = \#$fld, then

a) $CH^j(X)_0$ is a f.g ab group

b) $\operatorname{rk} CH^j(X)_0 = \operatorname{ord}_{s=j} L(H^{2j-1}(\bar{X}), s) = \operatorname{ord}_{s=0} L(H^{2j-1}(\bar{X})(j), s)$

$M$ "motive", $L(M,s)$, central point is a critical integer $L(M,\text{center}) = 0$

Question: Can we find a variety $X_{/k}$ st $M \subseteq H^{2j-1}(\bar{X})$; $\operatorname{rk} CH^j(X) > 0$

Motives: simplest nontrivial ex's

$K = \mathbb{Q}(\sqrt{-D})$, $Cl_K = 1$, $D$ odd, $w_K = 2$ $\qquad (D = 7, 11, 19, 43, 67, 163)$

$A$ ell curve $/\mathbb{Q}$, with CM by $\mathcal{O}_K$ min conductor $D^2$

$\psi$ Hecke char of $K$ ass to $A$, type $(1,0)$ $\psi|_{A_{\mathbb{Q}}^\times} = \varepsilon_K$, $\psi^2$ unram.

Powers of $\psi$: $\quad\psi,\ \psi^3,\ \psi^5,\ \psi^7, \ldots$

$+ \quad - \quad + \quad - \qquad D = 7$

$- \quad + \quad - \qquad D = 11, 19, \ldots$

Now look at $X_1(D)$, $W_{2r-1}$ = Kuga-Sato variety of dim $2r$ $/X_1(D)$

$\theta_{\psi^{2r}} \in S_{2r+1}(\Gamma_0(D), \varepsilon_K)$ define $M_{\psi^{2r}} \subseteq H^{2r}(W_{2r-1})$

$X_{2r-1,1} := W_{2r-1} \times A$ $\quad H^{2r+1}(X_{2r-1,1}) \supseteq M_{\psi^{2r}} \otimes M_{\psi} = M_{\psi^{2r+1}} \oplus M_{\psi^{2r-1}}(\cdot)$

Generalised Heegner cycle

$$\begin{tikzcd} W_{2r-1} \supseteq A^{2r-1} \arrow[d] \\ X_1(D) \end{tikzcd}$$

$A \times A^{2r-1}$

$= \underbrace{(A\times A)}_{A} \times \underbrace{(A\times A)}_{\Gamma_{\sqrt{-D}}}^{r-1}$

- Projectors, to make non-triv
- Take trace to make field of dof $= \mathbb{Q}$.

Main theorem: (Weak version) Suppose $r$ is odd if $D=7$, $r$ even if $D \neq 7$

Then $\Delta_{2r-1,1} \in CH^{r+1}(X_{2r-1,1})_0 \otimes \mathbb{Q}$ is not zero.

Conjecture: In all these cases, $L'(\psi^{2r+1}, r+1) \neq 0$

**Remark:** There should be a Gross-Zagier formula, relating $L'(\psi^{2r+1}, r+1) \Longleftrightarrow \langle \Delta_{2r+1,0}, \Delta_{2r+1,1} \rangle$

+ non-deg of ht pairing $\Longrightarrow L' \neq 0$

**Stronger version:** (filtrations)

$k$ any field of char $=0$, $\dim X = n$. $Z^j(X) \supseteq Z^j(X)_0 \supseteq Z^j(X)_{alg} \supseteq Z^j(X)_{rat}$

If $j=1$ or $j=n$, then $Z^j(X)_0 = Z^j(X)_{alg}$

Question: Do they always coincide?

Answer: (Griffiths, Annals ~ 69) $X_{/\mathbb{C}}$ generic hypersurface of deg 5 in $\mathbb{P}^4$ ($\dim X = 3$)

$Gr^2(X)$ is not torsion.

(Ceresa) $C$ generic curve of genus $g$ $/\mathbb{C}$. $X = \mathrm{Jac}(C)$. $\Delta = C - [-1]^* C$

$\Delta \in CH^{g-1}(X)_0$ non-torsion in $Gr^{g-1}(X)$.

But what does "generic" mean? $\longrightarrow$ Outside a countable union of locally closed subsets of the parameter space.

**Question:** What happens over # fields?

(Bruno Harris ~ 82) $C: x^4 + y^4 = z^4$, $X = \mathrm{Jac}(C)$

$g=3$. $\Delta = C - [-1]^* C \in CH^2(X)_0$.

$\Delta$ is non-trivial in $Gr^2(X)$

Abel-Jacobi: $X/\mathbb{C}$: $CH^j(X)_0 \longrightarrow J^j(X) = \dfrac{H^{2j-1}(X, \mathbb{C})}{Fil^j H^{2j-1}(X,\mathbb{C}) + \mathrm{Im}\, H_{2j-1}(X, \mathbb{Z})}$

$$\begin{tikzcd} CH^j(X)_0 \arrow[r] & J^j(X) \\ CH^j(X)_{alg} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \text{"max. ab. subvar"}\ J^j(X)_{alg} \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

Example of B. H $CH^2(X)_0 \longrightarrow J^2(X) = \dfrac{(H^{2,1}(X) \oplus H^{3,0}(X))^\vee}{\mathrm{Im}\, H_3(X, \mathbb{Z})} \longrightarrow H^{3,0}(X)^\vee / \mathrm{Im}\, H_3(X, \mathbb{Z})$

$X$ is isogenous to $E \times E \times E'$, $E: y^2 = x^3 - x$ $\psi$

$E': y^2 = x^3 + x$ $\psi'$ $\chi = \psi^2 \psi'$

$H^3(X) = \underbrace{M_{\psi^2\psi'}}_{(3,0)+(0,3)} \oplus \underbrace{\phantom{M}}_{(2,1)+(1,2)}$

Show that $2 \cdot \int_0^1 \int_0^x \dfrac{dt}{(1-t^4)^{3/2}} \cdot \dfrac{dx}{(1-x^4)^{3/4}} \Big/ \int_0^1 \dfrac{dt}{(1-t^4)^{1/2}} \cdot \int_0^1 \dfrac{dx}{(1-x^4)^{3/4}} \notin \mathbb{Z}$.

It turns out to be $1.2417\ldots$

Is it also not rational?

**Bloch:** use p-adic AJ map

$$\begin{tikzcd} CH^2(X)_0 \arrow[r] & H^1(\mathbb{Q}, H^3(\bar{X}, \mathbb{Q}_p)(2)) \arrow[r] & H^1(\mathbb{Q}, M_X) \arrow[d, "\ell \neq p"] \\ & & H^1(\mathbb{Q}_\ell, M_X) \end{tikzcd}$$

**Refined conjecture:** Coniveau filtration

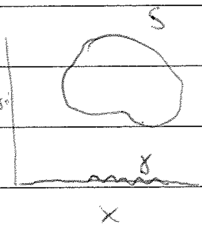
Grothendieck: $N^i H^*(X_{\bar{k}}, \mathbb{Q}_\ell) = \lim\limits_{\substack{Y \subseteq X_{\bar{k}} \\ \mathrm{codim}\, Y \geq i}} \ker(H^*(X_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow H^*(X_{\bar{k}} - Y, \mathbb{Q}_\ell))$

$= \lim\limits_{Y \subseteq X_{\bar{k}}} \mathrm{Im}(H^*_{(Y)}(X_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow H^*(X_{\bar{k}}, \mathbb{Q}_\ell)).$

Bloch-Ogus: $N^i CH^j(X)_{\mathbb{Q}} = \{\gamma \in CH^j(X)_{\mathbb{Q}} :$ ∃ a cycle $C$ representing $\gamma$, & a closed subscheme $Y \subseteq X_{\bar{k}}$ of codim $\geq i$ s.t. $\mathrm{supp}(C) \subseteq Y$ and is homologically triv on $Y\}$

$$\begin{array}{lcl} N^0 CH^j(X)_{\mathbb{Q}} & = & CH^j(X)_{0,\mathbb{Q}} \\ \vdots & & \vdots \\ \underline{\cup} & & \underline{\cup} \\ N^{j-1} CH^j(X)_{\mathbb{Q}} & = & CH^j(X)_{alg,\mathbb{Q}} \ni \gamma \\ \underline{\cup} & & \\ N^j CH^j(X)_{\mathbb{Q}} = 0 & & \end{array}$$

(Bloch)



Refined conj: $\mathrm{rk}\ gr^i_N CH^j(X) = \mathrm{ord}_{s=j} L(gr^j_N H^{2j-1}(X_{\bar{k}}, s))$

Main theorem (strong version) Suppose that $r$ is odd if $D = 7$, even if $D \neq 7$, then $[\Delta_{2r-1,1}] \in N^0 CH^{r+1}(X_{2r-1,1})_{\mathbb{Q}}/N^1$ is not zero

**Key idea:** $X/k$ # field

$v$ place of $k$ above $p$.

If $X$ has good red at $v$,

$$\begin{tikzcd} CH^j(X)_0 \arrow[r] \arrow[rrd, bend right] \arrow[rrddd, "\alpha"] \arrow[rrdddd, "\beta"'] & H^1(k, H^{2j-1}(\bar{X}_{\bar{\mathbb{Q}}_p}(j))) \arrow[r] & H^1(\overset{K}{\overset{\shortparallel}{k_v}}, V) \\ & & H^1_f(v, V) \arrow[d, shift left, "\log"] \\ & & D_{dR}(V)/\mathrm{Fil}^0 D_{dR}(V) \arrow[u, shift left, "\exp"] \\ & & = H^{2j-1}_{dR}(X)/\mathrm{Fil}^j H^{2j-1} \\ & & \overset{P.D}{=} (\mathrm{Fil}^j H^{2j-1}_{dR}(X))^\vee \end{tikzcd}$$

**Key prop:** $\alpha(N^i CH^j(X)) \subseteq \mathrm{Fil}^i H^{2j-1}_{dR}(X)/\mathrm{Fil}^j$

Cor $(\beta(N^i CH^j(X))$ annihilates $\mathrm{Fil}^{2j-i} H^{2j-1}_{dR}(X)$

In particular, $\beta(N^j CH^j(X))$ annihilates $\mathrm{Fil}^{2j-1} H^{2j-1}_{dR}(X) = H^0(X, \Omega^{2j-1})$

Thm: $(\beta(\Delta_{2r-1,1}))[\omega_f \wedge \omega_A])^2 = \mathcal{L}_p(\psi^r \cdot (\psi^*)^{1-r}) \cdot \mathcal{L}_p(\psi^{-r} \cdot (\psi^*)^{r+1})$

$X_{2r-1,1} = W_{2r-1} \times A \qquad f = \theta_{\psi^{2r}}$