Glenn Stevens — Hodge-Tate sequence and sheaves of overconvergent modular forms. (with Andreatta & Iovita)

Let $N \geq 3$, $p \nmid N$, $p \geq 3$. $\Gamma = \Gamma_1(N)$, $\Gamma_0 = \Gamma \cap \Gamma_0(p)$

Theorem (Hida, Coleman)

If $f \in M_{k_0+2}(\Gamma_0)$ with $k_0 \geq 0$ eigenform, $f|U_p = \alpha_p f$. Suppose

a) $\alpha_p^2 \neq p^{k_0+1}$

b) $\mathrm{ord}_p(\alpha_p) < k_0+1$

Then f belongs to a p-adic family (analytic)

Explanations

a) $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \supseteq \mathbb{Z}$ $(k \longmapsto (t \mapsto t^k))$

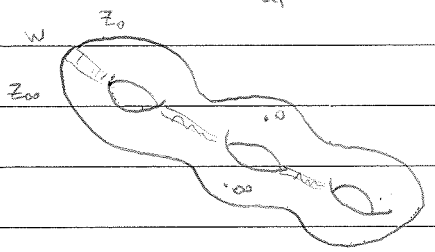
b) Eisenstein family: $G_k^* = \frac{1}{2}\zeta_p(1-k) + \sum_{n\geq 1} \sigma_{k+1}^*(n) q^n$ $\qquad \sigma_{k+1}^*(n) = \sum_{\substack{d|n \\ p\nmid d}} d^{k+1}$

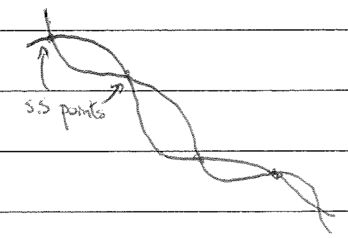
k varies over $\mathcal{W}$, usual Eisenstein series are interpolated.

Modular forms are sections of line bundles. Let $\underline{\omega} = \pi_*(\Omega^1_{E/X(N,p)})$ where $X(N,p) = X_{\Gamma_0}$

For $k \geq 0$, we have $M_k(\Gamma_0) = H^0(X(N,p), \underline{\omega}^k)$

$X(N,p)^{an}_{\mathbb{Q}_p}$ $\qquad\qquad$ $/\mathbb{F}_p$





$X_1(N)^{ord} = Z := Z_\infty$

Katz defined $\underline{\omega}^k$, $k \in \mathcal{W}(\mathbb{Q}_p)$, but only over Z. G_k^* is a section of $\underline{\omega}^k$ over Z.

In fact, if $k_0 \in \mathbb{Z}$ then $G_{k_0}^*$ is overconvergent.

We know that $E_{p-1} \in H^0(X(N,p), \underline{\omega}^{p-1})$ and $E_{p-1} \equiv h$ (p) (Hasse invariant) $\qquad (p > 3)$

Choose $w \in \mathbb{Q}^+$, $0 < w < \frac{p}{p+1}$

$$\{x \in X(N,p) \mid |E_{p-1}(x)| \geq p^{-w}\} = Z_\infty(w) \cup Z_0(w)$$

$X(N)(w) = Z_\infty(w) = Z(w)$. $\qquad$ $\exists$ canonical subgroup $C \subseteq E/_{Z(w)}$

Definition: $M_k(w) := H^0(X(N)(w), \underline{\omega}^k)$ $\qquad k \in \mathbb{Z}$.

(Fact $G_k^* \in M_k(w)$ $k \in \mathbb{Z}$)

There is a natural map $M_{k+2} \longrightarrow H^1(X, \mathrm{Symm}^k(\mathbb{C}^2))$ $\qquad$ (Eichler-Shimura)

Can we instead make families on the right? $_{RHS}$

Cohomological approach: $\exists$ an analytic family of Verma modules

$$\mathcal{D}_k \xrightarrow{\text{if } k \geq 0} \mathrm{Symm}^k(\mathbb{Q}_p^2)$$
$$k \in \mathcal{W}$$

Notation: $K/\mathbb{Q}_p$ finite, $\frac{w}{p-1} = v$ s.t. $\exists\, p^v \in \mathcal{O}_K$ with $\mathrm{ord}(p^v) = v$

Let $U_K \subseteq \mathcal{W}_K$ affinoid subdomain (think of a disk), $U_K = \mathrm{Spm}(A_K)$, $A = A_K^\circ$, $U = \mathrm{Spf}(A)$

We have: $k : \mathbb{Z}_p^\times \xrightarrow{\text{cont}} A_K^\times$: $k \in \mathcal{W}(A_K)$ universal weight over U_K

Assumptions: a) $\mathbb{Z} \cap U_K \neq \emptyset$

b) k as a function on $\mathbb{Z}_p^\times$ is defined locally by a power series of radius $\frac{1}{p}$.

Goal: Construct a p-adic family of line bundles $\omega_A^{\dagger,k}$ on $\mathfrak{X}(N)(w)$

We do a little more: we construct a family of overconvergent modular sheaves.

An overconvergent modular sheaf is a Cartesian functor

$$\mathrm{Ell}(N)_A^w \longrightarrow \mathrm{Sheaves}_K, \quad \text{but let's be more precise:}$$

Overconvergent modular sheaves:

$\mathrm{FSchemes}_A$: p-adic formal schemes $/A$ $\qquad \mathcal{U} \to U$ object

morphisms of F. schemes $/A$

$\mathrm{Ell}(N)_A^w$: $(E/\mathcal{U},\ \psi_N,\ Y)$ $\qquad Y \cdot \mathrm{Ha}(E/\mathcal{U}) = p^w$

ell. curve, level N, growth cond.

morphisms: (φ, α)

$(E/\mathcal{U} \longrightarrow E'/\mathcal{U}')$

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[dr] & E'_{\mathcal{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & E' \arrow[d] \\ & \mathcal{U} \arrow[r, "\varphi"] & \mathcal{U}' \end{tikzcd}$$

α an isogeny s.t. α is an isom on canonical subgps.

$\mathrm{Sheaves}_K$: $(\mathcal{U}, \mathcal{F})$ $\qquad \mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{U}_K}$-modules on $\mathcal{U}$

s.t. $\mathcal{F} \cong \mathcal{F}_0 \otimes_{\mathcal{O}_K} K$ $\qquad \mathcal{F}_0$ coherent on $\mathcal{U}$

morphisms: $(\varphi, \varphi^\dagger)$

$(\mathcal{U}, \mathcal{F}) \longrightarrow (\mathcal{U}', \mathcal{F}')$; $\qquad \mathcal{U} \xrightarrow{\varphi} \mathcal{U}'$

$\varphi^\dagger : \varphi^*(\mathcal{F}') \to \mathcal{F}$

Definition: An oc modular sheaf is a cartesian functor

$$\begin{tikzcd} \mathrm{Ell}(N)_A^w \arrow[rr, "\mathcal{F}"] \arrow[dr] & & \mathrm{Sheaves}_K \arrow[dl] \\ & \mathrm{FSchemes}_A & \end{tikzcd}$$

Cartesian means
$$\begin{tikzcd} \varphi^*(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & E' \arrow[d] \\ \mathcal{U} \arrow[r, "\varphi"] & \mathcal{U}' \end{tikzcd}$$

$$\mathcal{F}(\varphi^*(E')/\mathcal{U}) = \varphi^*(\mathcal{F}(E'/\mathcal{U}'))$$

Results:

We construct an OC modular sheaf $\omega_A^{\dagger,k}$ on $\mathrm{Ell}(N)_A^w$ with the following properties: Let $E/\mathcal{O}_K \in \mathrm{Ell}(N)^w_{\mathcal{O}_K}$

if $k_0 \in U_K \rightsquigarrow A \xrightarrow{k_0} \mathcal{O}_K \to \mathcal{O}_u$

$$E/\mathcal{U}_{k_0} \quad k_0 \in U_K \qquad \begin{tikzcd} & k_0 \arrow[dr] & \\ \mathcal{U}_{k_0} \arrow[ur] \arrow[rr] & & U \end{tikzcd}$$

ⓐ $\omega_A^{\dagger,k}(E/\mathcal{U}_{k_0})$ is an analytic family of OC sheaves

ⓑ If $k_0 \in \mathbb{Z}$ $\quad \omega_A^{\dagger,k}(E/\mathcal{U}_{k_0}) \cong \omega^{k_0}/\mathcal{U}$

ⓒ The family $G^*_{k_0}$ is a section of $\omega_A^{\dagger,k}(E/\mathcal{U}_{k_0})$

The construction:

Let $E/\mathcal{U} \in \mathrm{Ell}(N)^w_A$ $\qquad \mathcal{U} = \mathrm{Spf}(R)$, some technical conditions on R.

We have the Hodge-Tate sequence:

$$0 \to \omega^{-1}_{E/R} \otimes_R \widehat{\overline{R}}(1) \to T_p(E/R) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \xrightarrow{d\log} \omega_{E/R} \otimes \widehat{\overline{R}} \to 0$$

which is almost exact. It is a complex, the homology is annihilated by p^v

Define: F^0, F^1 s.t.

$$0 \to F^1 \xrightarrow{a} T_p(E) \otimes \widehat{\overline{R}} \xrightarrow{d\log} F^0 \to 0 \quad \text{is exact}$$

Theorem:

$$\begin{tikzcd} 0 \arrow[r] & F^1/p^{1-v}F^1 \arrow[r] \arrow[d, "\wr"] & T_p(E) \otimes \overline{R}/p^{1-v}\overline{R} \arrow[r] \arrow[d, "\wr"] & F^0/p^{1-v}F^0 \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & C \otimes_{\mathbb{Z}_p} \overline{R}/p^{1-v} \arrow[r, hook] & E[p] \otimes \overline{R}/p^{1-v} \arrow[r] & C^\vee \otimes \arrow[r] & 0 \end{tikzcd}$$

For simplicity, assume $C^\vee(\overline{R}) = C^\vee(R)$

$F_0 = (F^0)^g \qquad g = \mathrm{Gal}(\overline{R}/R)$

$$\to E[p] \to C^\vee \otimes R/p^{1-v} \to 0$$

$\mathcal{F}'_E$ defined to be pre-image in $\mathcal{F}_0$ of $C^\vee \setminus \{0\}$

$\mathcal{F}'_E \subset \omega_{E/R}$ $\qquad$ (This is the key)

$\mathbb{Z}_p^\times(1+p^{1-v}R)$ a torsor $\quad \longrightarrow \quad \mathcal{H}om(\mathcal{F}_{E'}, \mathcal{O}_{\mathcal{U}}^{\times(-k)})$

$T_{\mathcal{U}}$ - torsor

$$\mathcal{H}om(\mathcal{F}_{E'}, \mathcal{O}_{\mathcal{U}}^{\times(-k)}) = \omega_A^{\dagger,k}(E/\mathcal{U})$$