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joint with Y KUBO

Thm (H Imai, 1975)

$K/\mathbb{Q}_p$: finite ext

$L := K(\mu)$, $\mu = 1^{1/p^\infty} = \{$ p-power roots of 1 $\}$

A/K : abelian var with potentially good reduction

$\Rightarrow A(L)_{tors}$ is finite.

today's theorem:

Thm K: complete discrete valuation field of mixed char $(0,p)$

k residue field, essentially of finite type

$M := K(K^{1/p^\infty})$

k
) finite
$\mathbb{F}$) purely insep
) purely trans
$\mathbb{F}_p$

X proper smooth variety$/K$ with pot. good. reduction

$i \in \mathbb{Z}_{>0}$ odd

$V = \prod_\ell V_\ell := \prod_\ell H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$

$\cup$

$T = \prod_\ell T_\ell :=$ G_K stable $\hat{\mathbb{Z}}$ lattice $G_K = Gal(\bar{K}/K)$

then $\Longrightarrow$

$$(V/T)^{\mathcal{G}_M} \text{ is finite}$$

$\uparrow$ $\mathcal{G}_M$ fixed points.

<ins>Note</ins> (V_ℓ/T_ℓ) is finite $\Longleftrightarrow V_\ell^{\mathcal{G}_M} = 0$

cf. <ins>Thm</ins> (Coates - Sujatha - Wintenberger, 2001)

K, L as in Imai's theorem

X, i, V_p as in our theorem

$H := \mathrm{Im}(\mathcal{G}_L \longrightarrow GL_{\mathbb{Q}_p}(V_p))$

$\Longrightarrow H^j(H, V_p) = 0$ for any $j \geq 0$.

<ins>Applications</ins>

<ins>ex.</ins> - Control thm (Greenberg)

- Zerbes (Euler char) (removes an assumption)
- Kurihara's on rank of $H^1(F_{S/L}, T_pE)$
- Hachimori - Venjakob's on non existence of pseudo null submodules.

<ins>Proof</ins> (of today's thm) in 3 steps:

① $(V_p/T_p)^{\mathcal{G}_M}$ is finite - (for this k can be arbitrary)

② $(V_\ell/T_\ell)^{\mathcal{G}_M}$ is finite for all ℓ - (for this $i > 0$ can be arbitrary)

③ $(V_\ell/T_\ell)^{\mathcal{G}_M} = 0$ for almost all ℓ.

Methods

Specialization + wt argument

(+ p-adic Hodge theory for ①)

first:

Key observation:

$$\begin{tikzcd}[column sep=small]
M = K(K^{1/p^\infty}) \arrow[d, dash] \arrow[d, phantom, "\quad\quad\quad\quad\quad\quad H"] \\
L = K(1^{1/p^\infty}) \arrow[d, dash] \arrow[d, phantom, "\quad\quad\quad\quad\quad\quad\quad\quad \Gamma = \mathcal{G}/H"] \\
K
\end{tikzcd}
\qquad \mathcal{G} \text{ (from } K \text{ to } M\text{)}$$

$\chi : \mathcal{G} \to \mathbb{Z}_p^\times$ cyclotomic character

$$\Rightarrow \quad \sigma \tau \sigma^{-1} = \tau^{\chi(\sigma)} \text{ for } \sigma \in \mathcal{G},\ \tau \in H. \quad (*)$$

Lem If $\rho : \mathcal{G} \to GL_E(W)$ is a continuous rep then H acts potentially unipotently on W

(☺ $(*) \Rightarrow \forall\, \tau$ has e.v. $\in \mu$)).
eigenvalue.

① By Lem

$W = V_p^{G_M}$ is a rep of $\mathcal{G} = G_K/G_M$ of the form:

$$\begin{pmatrix} \chi^{\dagger} & & * \\ & \ddots & \\ & & \chi^{\dagger} \end{pmatrix} \quad (\dagger = \text{some power})$$

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By assumption on i, X, + (specialization + wt)

$V = H^i_{et}(X_{\bar{k}}, \mathbb{Q}_p)$ ~~som~~ cannot contain

dR

Use (Nakkajima), (Nakkayima-Shiho), to reduce to (Kato-Messing), (Chiarellotto-Le Stum)

2, 3 By lem + assumptions

$W = V_\ell^{\mathcal{G}_M}$ is a rep of $\mathcal{G}$ of the form

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$V_\ell = H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell) \simeq H^i_{et}(X_{\bar{h}}, \mathbb{Q}_\ell)$$

$$\wr|$$

$k \supset R$ $\quad$ $\mathrm{Spec}(R) \ni s$ $\quad$ $H^i_{et}(X_{\overline{k(s)}}, \mathbb{Q}_\ell)$ $\circlearrowleft$ Frob$_s$

subring f.g. / $\mathbb{F}_p$

e.v. of Frob$_s \not\equiv 1$ mod ℓ

$$\mathcal{G}_K \to \mathcal{G}_h \to \pi_1(S) \to \pi_1(s) = \langle \mathrm{Frob}_s \rangle$$

finite index

so $(T_\ell/\ell T_\ell)^{\mathcal{G}_M} = 0$ $\checkmark$