Local heights on hyperelliptic curves

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Notation

- C: genus g hyperelliptic curve of the form $y^2 = f(x)$, with $\deg f(x) = 2g+1$
- K: number field
- k: local field (char 0) with valuation ring $\mathcal{O}$, uniformizer π
- F: residue field, $\mathcal{O}/\pi\mathcal{O}$, with $|F| = q$.
- J: Jacobian of C over k

We'll assume C has good ordinary reduction at π.

Definition

The Coleman-Gross height pairing is a symmetric bilinear pairing

$$h : \mathrm{Div}^0(C) \times \mathrm{Div}^0(C) \to \mathbf{Q}_p,$$

which can be written as a sum of local height pairings

$$h = \sum_v h_v$$

over all finite places v of K.

Techniques to compute h_v depend on $\mathrm{char}(F)$:

- $\mathrm{char}(F) \neq p$: intersection theory (Prop 1.2 in C-G)
- $\mathrm{char}(F) = p$: logarithms, normalized differentials, Coleman integration

Local height: residue characteristic p

So for the rest of the talk, we'll assume $\mathrm{char}(F) = p$, and that $k = K_v$, the completion at $v|p$.

Definition

Let $D_1, D_2 \in \mathrm{Div}^0(C)$ have disjoint support and ω_{D_1} a normalized differential associated to D_1. The local height pairing at v above p is given by

$$h_v(D_1, D_2) = \mathrm{tr}_{k/\mathbf{Q}_p}\left(\int_{D_2} \omega_{D_1}\right).$$

We will describe how to construct ω_{D_1}, using the analytic methods of Coleman and Gross.

Differentials

Definition

A differential on C over k is

- of the first kind (denoted $H^{1,0}_{dR}(C/k)$): regular everywhere
- of the second kind: residue 0 everywhere
- of the third kind (denoted $T(k)$): simple poles and integer residues

By the residual divisor homomorphism, $T(k)$ fits into the following exact sequence:

$$0 \longrightarrow H^{1,0}_{dR}(C/k) \longrightarrow T(k) \xrightarrow{\text{Res}} \text{Div}^0(C) \longrightarrow 0.$$

Let $T_l(k) \subset T(k)$ be the logarithmic differentials ($\frac{df}{f}$ for $f \in k(C)^*$). Since $T_l(k) \cap H^{1,0}_{dR}(C/k) = \{0\}$ and $\text{Res}(\frac{df}{f}) = (f)$,

$$0 \longrightarrow H^{1,0}_{dR}(C/k) \longrightarrow T(k)/T_l(k) \longrightarrow J(k) \longrightarrow 0.$$

Cohomology

Recall the exact sequence

$$0 \longrightarrow H^{1,0}_{dR}(C/k) \longrightarrow H^1_{dR}(C/k) \longrightarrow H^1_{dR}(C, \mathcal{O}_{C/k}) \longrightarrow 0, \qquad (1)$$

where

- $H^{1,0}_{dR}(C/k)$ has dimension g
- $H^1_{dR}(C, \mathcal{O}_{C/k})$ also has dimension g and may be canonically identified with the tangent space at the origin of J.
- $H^1_{dR}(C/k)$ has a canonical non-degenerate alternating form given by the algebraic cup product pairing

$$\begin{aligned} H^1_{dR}(C/k) \times H^1_{dR}(C/k) &\longrightarrow k \\ ([\nu_1], [\nu_2]) &\mapsto [\nu_1] \cup [\nu_2] = \sum_x \mathrm{Res}_x\left(\nu_2 \int \nu_1\right), \end{aligned}$$

for ν_i differentials of the second kind.

Theorem (Coleman-Gross)

There is a canonical homomorphism

$$\Psi : T(k)/T_l(k) \longrightarrow H^1_{dR}(C/k)$$

which is the identity on differentials of the first kind and makes the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{1,0}(C/k) & \longrightarrow & T(k)/T_l(k) & \longrightarrow & J(k) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \Psi=\log_T} & & \downarrow{\scriptstyle \log_J} & & \\
0 & \longrightarrow & H^{1,0}_{dR}(C/k) & \longrightarrow & H^1_{dR}(C/k) & \longrightarrow & H^1_{dR}(C, \mathcal{O}_{C/k}) & \longrightarrow & 0.
\end{array}$$

To compute with Ψ, we use the following

Theorem (Besser)

Let ω be a meromorphic form and ρ a form of the second kind. Then $\Psi(\omega) \cup [\rho] = \langle \omega, \rho \rangle$.

Local and global symbols

Definition

For ω a meromorphic form and ρ a form of the second kind, we define the *global symbol* $\langle \omega, \rho \rangle$ as a sum of *local symbols* $\langle \omega, \rho \rangle_A$. We have

$$\langle \omega, \rho \rangle = \sum_A \langle \omega, \rho \rangle_A = \sum_A \mathrm{Res}_A \left(\omega \left(\int \rho + \int_Z^A \rho \right) \right),$$

where

- $A \in \{$Weierstrass points of C, poles of $\omega\}$,
- each $\langle \omega, \rho \rangle_A$ is computed via local coordinates at A,
- the first integral is a formal antiderivative, and
- the second (Coleman) integral sets the constant of integration (for a fixed Z).

Global symbols and Ψ

We compute Ψ of a meromorphic differential via cup products and global symbols.
Given a basis $\{\omega_i\}_{i=0}^{2g-1}$ for $H^1_{dR}(C/k)$, we write

$$\Psi(\omega) = c_0\omega_0 + \cdots + c_{2g-1}\omega_{2g-1}.$$

We solve for the coefficients c_i by considering a linear system involving global symbols and cup products:

$$\langle \omega, \omega_j \rangle = \Psi(\omega) \cup [\omega_j] = \sum_{i=0}^{2g-1} c_i([\omega_i] \cup [\omega_j]).$$

Splitting of $H^1_{dR}(C/k)$: getting ω_{D_1}

We fix a direct sum decomposition

$$H^1_{dR}(C/k) = H^{1,0}_{dR}(C/k) \oplus W,$$

where W is the unit root subspace for the action of Frobenius.

Definition

Let $D_1 \in \mathrm{Div}^0(C/k)$. We define ω_{D_1} to be a differential of the third kind with residue divisor D_1 such that $\Psi(\omega_{D_1}) \in W$.

Lemma

ω_{D_1} is unique.

The normalized differential ω_{D_1}

Thus choosing ω of the third kind with $\mathrm{Res}(\omega) = D_1$, by the splitting

$$H^1_{dR}(C/k) = H^{1,0}_{dR}(C/k) \oplus W,$$

we have

$$\Psi(\omega) = \eta + \Psi(\omega_{D_1}),$$

for η holomorphic and some element $\Psi(\omega_{D_1}) \in W$. Then taking

$$\omega_{D_1} := \omega - \eta,$$

we have $\Psi(\omega_{D_1}) = \Psi(\omega - \eta) = \Psi(\omega) - \Psi(\eta) = \Psi(\omega) - \eta$.

Algorithm: computing $h_p(D_1, D_2)$

Input: C hyperelliptic curve over $\mathbf{Q}_p$ with p a prime of good ordinary reduction, $D_1 = (P) - (Q), D_2 = (R) - (S) \in \mathrm{Div}^0(C)$ with disjoint support

Output: $h_p(D_1, D_2) = \int_{D_2} \omega_{D_1}$

(1) **From** D_1 **to** ω. Choose ω a differential of the third kind with $\mathrm{Res}(\omega) = D_1$.

(2) **The map** Ψ. Compute $\log(\omega) = \Psi(\omega)$ for ω.

(3) **From** ω **to** ω_{D_1} **and** η. Via the decomposition

$$H^1_{dR}(C/k) \simeq H^{1,0}_{dR}(C/k) \oplus W,$$

write

$$\log(\omega) = \eta + \log(\omega_{D_1}),$$

where η is holomorphic, and $\log(\omega_{D_1}) \in W$. This gives $\omega_{D_1} = \omega - \eta$.

Algorithm, continued

(4) **Coleman integration: holomorphic differential.** Compute $\int_{D_2} \eta$.

(5) **Coleman integration: meromorphic differential.** Let ϕ be a p-power lift of Frobenius and set $\alpha := \phi^*\omega - p\omega$. Then for β a differential with residue divisor $D_2 = (R) - (S)$, we compute

$$\begin{aligned}\int_{D_2} \omega &= \int_S^R \omega \\ &= \frac{1}{1-p}\left(\Psi(\alpha) \cup \Psi(\beta)\right) + \sum \mathrm{Res}\left(\alpha \int \beta\right) \\ &\quad - \frac{1}{1-p}\left(\int_{\phi(S)}^{S} \omega + \int_R^{\phi(R)} \omega\right).\end{aligned}$$

(6) **Height pairing**. Subtract the integrals to recover the pairing:

$$h_p(D_1, D_2) = \int_S^R \omega_{(P)-(Q)} = \int_S^R \omega - \int_S^R \eta.$$