Local heights on hyperelliptic curves

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Effective Methods in $p$-adic Cohomology
Mathematical Institute, University of Oxford
Tuesday, March 16, 2010
Outline

1. Introduction
   - Notation
   - Coleman-Gross height pairing
   - Local height: residue characteristic $p$

2. Differentials and cohomology
   - Differentials
   - Cohomology
   - The map $\Psi$

3. Computing with $\Psi$
   - Local and global symbols
   - Splitting of $H^1_{dR}(C/k)$
   - The normalized differential

4. Algorithm
Notation

- $C$: genus $g$ hyperelliptic curve of the form $y^2 = f(x)$, with $\deg f(x) = 2g + 1$
- $K$: number field
- $k$: local field (char 0) with valuation ring $\mathcal{O}$, uniformizer $\pi$
- $F$: residue field, $\mathcal{O}/\pi\mathcal{O}$, with $|F| = q$.
- $J$: Jacobian of $C$ over $k$

We’ll assume $C$ has good ordinary reduction at $\pi$. 
Definition

The Coleman-Gross height pairing is a symmetric bilinear pairing

\[ h : \text{Div}^0(C) \times \text{Div}^0(C) \rightarrow \mathbb{Q}_p, \]

which can be written as a sum of local height pairings

\[ h = \sum_v h_v \]

over all finite places \( v \) of \( K \).

Techniques to compute \( h_v \) depend on \( \text{char}(F) \):

- \( \text{char}(F) \neq p \): intersection theory (Prop 1.2 in C-G)
- \( \text{char}(F) = p \): logarithms, normalized differentials, Coleman integration
Local height: residue characteristic $p$

So for the rest of the talk, we’ll assume $\text{char}(F) = p$, and that $k = K_v$, the completion at $v|p$.

**Definition**

Let $D_1, D_2 \in \text{Div}^0(C)$ have disjoint support and $\omega_{D_1}$ a normalized differential associated to $D_1$. The local height pairing at $v$ above $p$ is given by

$$h_v(D_1, D_2) = \text{tr}_{k/Q_p} \left( \int_{D_2} \omega_{D_1} \right).$$

We will describe how to construct $\omega_{D_1}$, using the analytic methods of Coleman and Gross.
**Differentials**

**Definition**
A differential on $C$ over $k$ is
- of the first kind (denoted $H^{1,0}_{dR}(C/k)$): regular everywhere
- of the second kind: residue 0 everywhere
- of the third kind (denoted $T(k)$): simple poles and integer residues

By the residual divisor homomorphism, $T(k)$ fits into the following exact sequence:

$$0 \rightarrow H^{1,0}_{dR}(C/k) \rightarrow T(k) \xrightarrow{\text{Res}} \text{Div}^0(C) \rightarrow 0.$$ 

Let $T_l(k) \subset T(k)$ be the logarithmic differentials ($\frac{df}{f}$ for $f \in k(C)^*$). Since $T_l(k) \cap H^{1,0}_{dR}(C/k) = \{0\}$ and $\text{Res}(\frac{df}{f}) = (f)$,

$$0 \rightarrow H^{1,0}_{dR}(C/k) \rightarrow T(k)/T_l(k) \rightarrow J(k) \rightarrow 0.$$
Cohomology

Recall the exact sequence

$$0 \longrightarrow H^{1,0}_{dR}(C/k) \longrightarrow H^1_{dR}(C/k) \longrightarrow H^1_{dR}(C, \mathcal{O}_C/k) \longrightarrow 0,$$

(1)

where

- $H^{1,0}_{dR}(C/k)$ has dimension $g$
- $H^1_{dR}(C, \mathcal{O}_C/k)$ also has dimension $g$ and may be canonically identified with the tangent space at the origin of $J$.
- $H^1_{dR}(C/k)$ has a canonical non-degenerate alternating form given by the algebraic cup product pairing

$$H^1_{dR}(C/k) \times H^1_{dR}(C/k) \longrightarrow k$$

$$([\nu_1], [\nu_2]) \mapsto [\nu_1] \cup [\nu_2] = \sum_x \text{Res}_x(\nu_2 \int \nu_1),$$

for $\nu_i$ differentials of the second kind.
Theorem (Coleman-Gross)

There is a canonical homomorphism

$$\Psi : T(k)/T_1(k) \longrightarrow H^1_{dR}(C/k)$$

which is the identity on differentials of the first kind and makes the following diagram commute:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^{1,0}(C/k) & \longrightarrow & T(k)/T_1(k) & \longrightarrow & J(k) & \longrightarrow & 0 \\
\| & & & & \Psi = \log_T & & \log_J & \\
0 & \longrightarrow & H^1_{dR}(C/k) & \longrightarrow & H^1_{dR}(C/k) & \longrightarrow & H^1_{dR}(C, \mathcal{O}_{C/k}) & \longrightarrow & 0.
\end{array}
$$

To compute with \(\Psi\), we use the following

Theorem (Besser)

Let \(\omega\) be a meromorphic form and \(\rho\) a form of the second kind. Then

$$\Psi(\omega) \cup [\rho] = \langle \omega, \rho \rangle.$$
Local and global symbols

Definition

For $\omega$ a meromorphic form and $\rho$ a form of the second kind, we define the global symbol $\langle \omega, \rho \rangle$ as a sum of local symbols $\langle \omega, \rho \rangle_A$. We have

$$\langle \omega, \rho \rangle = \sum_A \langle \omega, \rho \rangle_A = \sum_A \text{Res}_A \left( \omega \left( \int \rho + \int_{A}^{Z} \rho \right) \right),$$

where

- $A \in \{\text{Weierstrass points of } C, \text{ poles of } \omega\}$,
- each $\langle \omega, \rho \rangle_A$ is computed via local coordinates at $A$,
- the first integral is a formal antiderivative, and
- the second (Coleman) integral sets the constant of integration (for a fixed $Z$).
Global symbols and $\Psi$

We compute $\Psi$ of a meromorphic differential via cup products and global symbols. Given a basis $\{\omega_i\}_{i=0}^{2g-1}$ for $H_d^1(C/k)$, we write

$$\Psi(\omega) = c_0 \omega_0 + \cdots + c_{2g-1} \omega_{2g-1}.$$ 

We solve for the coefficients $c_i$ by considering a linear system involving global symbols and cup products:

$$\langle \omega, \omega_j \rangle = \Psi(\omega) \cup [\omega_j] = \sum_{i=0}^{2g-1} c_i ([\omega_i] \cup [\omega_j]).$$
Splitting of $H^1_{dR}(C/k)$: getting $\omega_{D_1}$

We fix a direct sum decomposition

$$H^1_{dR}(C/k) = H^1_{dR}^{1,0}(C/k) \oplus W,$$

where $W$ is the unit root subspace for the action of Frobenius.

**Definition**

Let $D_1 \in \text{Div}^0(C/k)$. We define $\omega_{D_1}$ to be a differential of the third kind with residue divisor $D_1$ such that $\Psi(\omega_{D_1}) \in W$.

**Lemma**

$\omega_{D_1}$ is unique.
The normalized differential $\omega_{D_1}$

Thus choosing $\omega$ of the third kind with $\text{Res}(\omega) = D_1$, by the splitting

$$H^1_{dR}(C/k) = H^{1,0}_{dR}(C/k) \oplus W,$$

we have

$$\Psi(\omega) = \eta + \Psi(\omega_{D_1}),$$

for $\eta$ holomorphic and some element $\Psi(\omega_{D_1}) \in W$. Then taking

$$\omega_{D_1} := \omega - \eta,$$

we have $\Psi(\omega_{D_1}) = \Psi(\omega - \eta) = \Psi(\omega) - \Psi(\eta) = \Psi(\omega) - \eta$. 
**Algorithm: computing** $h_p(D_1, D_2)$

**Input:** $C$ hyperelliptic curve over $\mathbb{Q}_p$ with $p$ a prime of good ordinary reduction, $D_1 = (P) - (Q), D_2 = (R) - (S) \in \text{Div}^0(C)$ with disjoint support

**Output:** $h_p(D_1, D_2) = \int_{D_2} \omega_{D_1}$

1. **From $D_1$ to $\omega$.** Choose $\omega$ a differential of the third kind with $\text{Res}(\omega) = D_1$.

2. **The map $\Psi$.** Compute $\log(\omega) = \Psi(\omega)$ for $\omega$.

3. **From $\omega$ to $\omega_{D_1}$ and $\eta$.** Via the decomposition

   $$H^1_{dR}(C/k) \cong H^{1,0}_{dR}(C/k) \oplus W,$$

   write

   $$\log(\omega) = \eta + \log(\omega_{D_1}),$$

   where $\eta$ is holomorphic, and $\log(\omega_{D_1}) \in W$. This gives

   $$\omega_{D_1} = \omega - \eta.$$
Algorithm, continued

(4) Coleman integration: holomorphic differential. Compute $\int_{D_2} \eta$.

(5) Coleman integration: meromorphic differential. Let $\phi$ be a $p$-power lift of Frobenius and set $\alpha := \phi^* \omega - p \omega$. Then for $\beta$ a differential with residue divisor $D_2 = (R) - (S)$, we compute

$$\int_{D_2} \omega = \int_S^R \omega$$

$$= \frac{1}{1 - p} (\Psi(\alpha) \cup \Psi(\beta)) + \sum \text{Res} \left( \alpha \int \beta \right)$$

$$- \frac{1}{1 - p} \left( \int_{\phi(S)}^S \omega + \int_{\phi(R)}^R \omega \right).$$

(6) Height pairing. Subtract the integrals to recover the pairing:

$$h_p(D_1, D_2) = \int_S^R \omega_{(P) - (Q)} = \int_S^R \omega - \int_S^R \eta.$$