Convergence polygon of a connection and differential Grothendieck-Ogg-Shafarevich formula for coverings of p-adic analytic curves

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Let k be a non-archimedean field extension of $\mathbb{Q}_p$, $|p| = p^{-1}$, and let k° be its ring of integers. We use analytic geometry in the sense of Berkovich. Let X be a rig-smooth compact strictly k-analytic curve. We deal with connections $(\mathcal{E}, \nabla)$ on X/k. We use a formal semistable model $\mathfrak{X}$ of X over k° to measure radii of disks globally on X; we assume that $\mathcal{E}$ extends to a locally free $\mathcal{O}_{\mathfrak{X}}$-module.

We attach to each point $x \in X$, a *normalized convergence polygon* $\mathcal{N}(x) = \mathcal{N}_{\mathfrak{X}}(x; (\mathcal{E}, \nabla))$ of $(\mathcal{E}, \nabla)$ at x, a convex polygon connecting $(0, 0)$ with $(\mathrm{rk}\,\mathcal{E}, H(x))$ in the first quadrant, in such a way that

1. The highest slope of $\mathcal{N}(x)$ is $-\log_p \mathcal{R}(x)$, where $\mathcal{R}(x) = \mathcal{R}_{\mathfrak{X}}(x, (\mathcal{E}, \nabla))$ is our $\mathfrak{X}$-*normalized radius of convergence* of $(\mathcal{E}, \nabla)$ at x.
2. $\mathcal{N}(x)$ is a global intrinsic form of the polygon defined by (P. Th. Young and) Kedlaya when x is a point of Berkovich type (2) or (3).
3. the function $x \mapsto \mathcal{N}(x)$ is continuous on X.

Of special interest is the function $x \mapsto H(x)$, called the *total height* of $\mathcal{N}(x)$.

The skeleton $\Gamma = \mathbf{S}(\mathfrak{Y})$ of a formal semistable model $\mathfrak{Y}$ of X is a complete graph in X and there is a continuous retraction $\tau_{\mathfrak{Y}} = \tau_\Gamma : X \to \Gamma$. We say that the complete graph Γ *controls* the connection $(\mathcal{E}, \nabla)$ on X if there exists an admissible blow-up $\mathfrak{Y} \to \mathfrak{X}$ such that

$$\mathcal{N}_{\mathfrak{X}}(x, (\mathcal{E}, \nabla)) = \mathcal{N}_{\mathfrak{X}}(\tau_{\mathfrak{Y}}(x), (\mathcal{E}, \nabla)) \ . \qquad (0.0.0.1)$$

All these notions extend to the case when $(\mathcal{E}, \nabla)$ has meromorphic singularities along a divisor $Z = \mathfrak{Z}_\eta \subset X$, where $\mathfrak{Z}$ is a Cartier divisor on $\mathfrak{X}/k^\circ$. Some open half-lines, ending in Z, must be added as edges to graphs. Then, Γ determines a partition $\mathscr{P}_\Gamma$ of $X \setminus Z$ into a finite family of affinoids with good canonical reduction, open annuli and pointed open disks. For example, the connection with solution $\exp \pi(x - x^p)$ on $\mathcal{O}_{\mathbb{P}^1}(*\infty)$ is controlled by a star-shaped graph consisting of $p-1$ segments and one half-line (ending at ∞) all departing from the generic point of the disk $D(0, (p^{\frac{1}{p-1} - \frac{1}{p^2}})^+)$.

We interpret and globalize results of Robba, Christol-Mebkhout, and Kedlaya as follows.

1. Every connection on X is controlled by a finite graph Γ.

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2. Assume X is projective Z is as above, and Γ controls $(\mathcal{E},\nabla)$. Assume moreover that for any disk, open annulus, pointed open annulus or affinoid $U \subset X \setminus Z$, the De Rham cohomology of U with coefficients in $(\mathcal{E},\nabla)$ (resp. in $j^{\dagger}_{U,X}(\mathcal{E},\nabla)$ if U is affinoid) is finite-dimensional. Then the function $x \mapsto H(x)$ is harmonic on $X \setminus Z$, except possibly at the vertices of Γ, *i.e.* at the maximal points η_U of affinoids $U \in \mathscr{P}_{\Gamma}$. In that case, either $U = X$ is a projective curve with good reduction, $\Gamma = \{\eta_X\}$, and then H is also harmonic at η_X or U is an affinoid with good canonical reduction $\mathcal{U}$ and the laplacian of H at η_U is

$$\Delta_{\eta_U}(H) = (\mathrm{rk}\,\mathcal{E}) \cdot \chi_{\mathrm{rig}}(\mathfrak{U}_s/k) - \chi(X, j^{\dagger}_{U\subset X}\mathcal{DR}_{X/k}((\mathcal{E},\nabla))\ . \tag{0.0.0.2}$$

This is our differential formula of Grothendieck-Ogg-Shafarevich in rigid cohomology (due to Christol-Mebkhout in the special case of overconvergent isocrystals satisfying the Robba condition and the **NL** assumption.)

3. Consider the special case of a generically finite étale covering $f : Y \to X$ of projective curves and of $(\mathcal{E},\nabla) = (f_*\mathcal{O}_Y, f_*(d_{Y/k}))$ (in which case the finiteness assumption follows from a result of Grosse-Klönne). Let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be a semistable model of X (resp. Y) such that f extends to a finite flat morphism $\mathfrak{f} : \mathfrak{Y} \to \mathfrak{X}$ with discriminant the Cartier divisor $\mathfrak{Z}$ of $\mathfrak{X}/k^{\circ}$ (this is possible after replacing k by a finite extension). Let $Z = \mathfrak{Z}_{\eta}$; then for any $x \in X \setminus Z$, let the integral closure A of $\mathscr{H}(x)^{\circ}$ in the affine algebra of the fiber Y_x is a finite flat $\mathscr{H}(x)^{\circ}$-algebra. The morphism $\mathfrak{f}$ offers a logarithmic presentation of A, and, therefore, for any $a \in \mathbb{Q}$, the logarithmic Abbes-Saito ramification of $A/\mathscr{H}(x)^{\circ}$ is $\leq a$ iff $\mathcal{R}_{\mathfrak{X}}(x,(\mathcal{E},\nabla)) \geq p^{-a}$.

4. Under the assumptions of the previous point, if $\vec{v}$ is any tangent vector to X at η_U, represented by either a residue class of U or by the inverse image by specialization of a double point of $\mathfrak{X}_s$ in the closure of $\mathfrak{U}_s$, then $d_{\vec{v}}H$ is the Swan conductor of Y/X introduced by Huber and discussed by Ramero, at the point $\vec{v} \in X_{III}$ (Huber's notation).