Laurent Berger 17/03/10 12:10

Computing the reduction of some p-adic representations

V = p-adic rep. of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

= E-v.s., $E/\mathbb{Q}_p$ finite

$T \subset V$ = $G_{\mathbb{Q}_p}$-stable O_E-lattice

$\overline{V} = (T/m_E T)^{ss}$

Q: Given V, what is $\overline{V}$?

Possible $\overline{V}$, $n \geq 1$ $\quad \pi_n = (-p)^{\frac{1}{p^n-1}}$

$\omega_n : G_{\mathbb{Q}_p} \to \mathbb{F}_{p^n}^\times$

$g \mapsto g(\pi_n)/\pi_n$

eg. $n=1$, π_1 = Dwork's π

$\omega_1 = \omega_{cycl}$

$\overline{V}$ irreducible $\Rightarrow \exists\, n \geq 1$: $\overline{V}$ is a twist of

$\mathrm{ind}_{\mathbb{Q}_{p^n}}^{\mathbb{Q}_p} \omega_n^h$

($\mathrm{ind}(\omega_n^h)$ = the one with $\det = \omega_1^h$)

Fontaine's theory:

V is crystalline if $\exists$ a filtered φ-module D of $\dim = \dim V$ such that

$$V = \mathrm{Fil}^0\left(B_{cris} \otimes_{\mathbb{Q}_p} D\right)^{\varphi = 1}$$

For the rest of the talk, $k \geq 2$, $a_p \in \mathcal{O}_E$,

$D_{k,a_p} = Ee_1 \oplus Ee_2$ with

$$\mathrm{Mat}(f) = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}, \quad \mathrm{Fil}^j = \begin{cases} D_{k,a_p} & j \leq 0 \\ Ee_1 & 1 \leq j \leq k-1 \\ \{0\} & j \leq k \end{cases}$$

$$V_{k,a_p} = V(D_{k,a_p})^*$$

Remarks 1) f = mod. form level $p \nmid N$, wt $k \geq 2$

$\longrightarrow V_p f$ = E-lin. rep of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

$V_p f \,|\, \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) = V_{k,a_p}$

2) $v_p(a_p) = 0 \Rightarrow V_{k,a_p}$ is reducible

3) $a_p = 0 \Rightarrow V_{k,0} = \mathrm{ind}(\chi_2^{k-1})$

$\Rightarrow \overline{V}_{k,0} = \mathrm{ind}(\omega_2^{k-1})$

4) $\overline{V}_{k,a_p}$ is a locally constant fn of a_p

$(k, a_p) \longmapsto \overline{V}_{k,a_p}$? $[\, v_p(a_p) > 0 \,]$

$k \leq p$: Fontaine-Laffaille $\longrightarrow \overline{V}_{k,a_p} \simeq \mathrm{ind}(\omega_2^{k-1})$

Breuil ($\sim$2000) Π_{k,a_p} = F-rep. of $GL_2(\mathbb{Q}_p)$

"associated" to V_{k,a_p}

$\hookrightarrow \overline{\Pi}_{k,a_p} \longleftrightarrow \overline{V}_{k,a_p}$

$$\overline{\Pi}_{k,a_p} = \frac{\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} \mathrm{Sym}_v^{k-2} \overline{E}^2}{T - a_p}$$

Hecke operator

Savitt-Stein : if $a_p = a_p(f)$, $\exists$ $g \underset{\mathrm{mod}\, p}{\equiv} f$ s.t. $wt(g) \leq p$, $p \mid N_g$

$k = p+1$, $\overline{V}_{k,a_p} = \mathrm{ind}(\omega_2^p)$

$p+2 \leq k \leq 2p$ $\begin{cases} 0 < v(a_p) < 1 & \mathrm{ind}(\omega_2^{k-p}) \\ v(a_p) = 1 & \text{reducible} \\ v(a_p) > 1 & \mathrm{ind}(\omega_2^{k-1}) \end{cases}$

$k = 2p+1$ $v(a_p^2 + p) < 3/2 \longrightarrow \mathrm{ind}(\omega_2^L)$

$> 3/2 \longrightarrow \omega \oplus \omega$

$= 3/2 \longrightarrow$ reducible

Theory of (φ, Γ)-modules

Wach modules

N = free $\mathcal{O}_E[[X]]$-modules $+$ $\varphi : (\varphi f)(x) = f((1+x)^p - 1)$

$+$ action of $\mathbb{Z}_p^\times = \Gamma$

$$\cancel{N:=} \; E \otimes_{\mathcal{O}_E} N/X = \text{filtered } \varphi\text{-module}$$

$$\begin{tikzcd} N \arrow[r] \arrow[dr] & D_{\mathrm{cris}}(V) \\ & \overline{V} \end{tikzcd}$$

Using Wach modules

1) $\overline{V}_{k,a_p} \to \overline{V}_{k,a_p'}$ for a_p' close to a_p

$\Rightarrow \overline{V}_{k,a_p} = \mathrm{ind}(\omega_2^{k-1})$ if $v_p(a_p) > \left\lfloor \frac{k-2}{p-1} \right\rfloor$

2) we prove that $\overline{\Pi}_{k,a_p} \Rightarrow \overline{V}_{k,a_p}$

$\bmod p-1$, $0 \le [\;] \le p-2$

If $0 < v_p(a_p) < 1$ (Buzzard - Gee) $\Rightarrow \begin{cases} p-1 \nmid k-3 \to \mathrm{ind}(\omega_2^{[k-2]+1}) \\ p-1 \mid k-3 \quad \left\{ \begin{array}{l} \text{same} \\ \text{or} \\ \text{reducible} \end{array} \right. \end{cases}$

3) Wach modules are determined by their truncations $\Rightarrow$ algorithm

$$\begin{tikzcd} \boxed{k, a_p \bmod p^m} \arrow[loop right, "m \mapsto m+1"] \arrow[d] \\ \boxed{\overline{V}_{k,a_p}} \end{tikzcd}$$

Conj: If k is even and $\overline{V}_{k,a_p}$ is reducible

$\Rightarrow v_p(a_p) \in \mathbb{Z}$.