

Computing the reduction of some p -adic representations

$V = p$ -adic rep. of $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

$= E$ -v.s., E/\mathbb{Q}_p finite

$T \subset V = G_{\mathbb{Q}_p}$ -stable \mathcal{O}_E -lattice

$$\overline{V} = (T/m_E T)^{ss}$$

Q: Given V , what is \overline{V} ?

Possible \overline{V} , $n \geq 1$ $\pi_n = (-\rho)^{\frac{1}{p^n-1}}$

$$\begin{aligned} \omega_n : G_{\mathbb{Q}_p} &\rightarrow \mathbb{F}_{p^n}^\times \\ g &\mapsto g(\pi_n) / \pi_n \end{aligned}$$

eg. $n=1$, $\pi_1 =$ Dwork's π

$$\omega_1 = \omega_{\text{cycl}}$$

\overline{V} irreducible $\Rightarrow \exists n \geq 1$: \overline{V} is a twist of

$$\text{ind}_{\mathbb{Q}_p}^{\mathbb{Q}_p} \omega_n^h$$

($\text{ind}(\omega_n^h) =$ the one with $\text{det} = \omega_1^h$)

Fontaine's theory;

V is crystalline if \exists a filtered ϕ -module D of $\dim = \dim V$ such that

$$V = \text{Fil}^0(\mathbb{B}_{\text{cris}} \otimes_{\mathbb{Q}_p} D)^{\phi=1}$$

For the rest of the talk, $k \geq 2$, $a_p \in \mathcal{O}_E$,

$$D_{k, a_p} = Ee_1 \oplus Ee_2 \quad \text{with}$$

$$\text{Mat}(f) = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}, \quad \text{Fil}_k^{0, \phi} = \begin{cases} D_{k, a_p} & j \leq 0 \\ Ee_1 & 1 \leq j \leq k-1 \\ \{0\} & j \leq k \end{cases}$$

$$V_{k, a_p} = V(D_{k, a_p})^{\phi=1}$$

Remarks 1) $f = \text{mod. form level } p \times N$, wt $k \geq 2$

$\rightarrow V_p f = E$ -lin. rep of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

$$V_p f | \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) = V_{k, a_p}$$

2) $v_p(a_p) = 0 \Rightarrow V_{k, a_p}$ is reducible

3) $a_p = 0 \Rightarrow V_{k, 0} = \text{ind}(\chi_2^{k-1})$

$$\Rightarrow \bar{V}_{k, 0} = \text{ind}(\omega_2^{k-1})$$

4) \bar{V}_{k, a_p} is a locally constant f^n of a_p

$$(k, a_p) \mapsto \overline{V}_{k, a_p} ? \quad [v_p(a_p) > 0]$$

$$k \leq p : \text{Fontaine-Laffaille} \rightarrow \overline{V}_{k, a_p} \cong \text{ind}(w_2^{k-1})$$

Breuil (~2000) $\overline{\Pi}_{k, a_p}$ = F-rep. of $GL_2(\mathbb{Q}_p)$
 "associated" to V_{k, a_p}

$$\hookrightarrow \overline{\Pi}_{k, a_p} \leftrightarrow \overline{V}_{k, a_p}$$

$$\overline{\Pi}_{k, a_p} = \frac{\text{ind}_{GL_2(\mathbb{Z}_p)}^{GL_2(\mathbb{Q}_p)} \otimes_{\mathbb{F}_p} \text{Sym}_v^{k-2} E^2}{T - a_p}$$

↖ Hecke operator

Savitt-Stein : if $a_p = a_p(f)$, $\exists g \in f$ st. $\text{wt}(g) \leq p$ $p \mid N_g$

$$k = p+1, \overline{V}_{k, a_p} = \text{ind}(w_2^p)$$

$$p+2 \leq k \leq 2p \quad \begin{cases} 0 < v(a_p) < 1 & \text{ind}(w_2^{k-p}) \\ v(a_p) = 1 & \text{reducible} \\ v(a_p) > 1 & \text{ind}(w_2^{k-1}) \end{cases}$$

$$k = 2p+1 \quad v(a_p^2 + p) < \frac{3}{2} \rightarrow \text{ind}(w_2^L)$$

$$> \frac{3}{2} \rightarrow w \oplus w$$

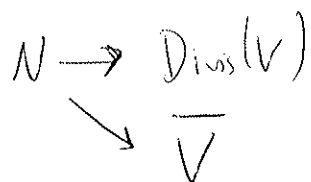
$$= \frac{3}{2} \rightarrow \text{reducible}$$

Theory of (q, P) -modules

Wach modules

$N = \text{free } \mathcal{O}_E[[X]]\text{-modules} + \phi: (\phi f)(x) = f((1+x)^{p-1}x)$
 + action of $\mathbb{Z}_p^* = \Gamma$

$\mathbb{A} = \mathbb{Z}_p \otimes_{\mathcal{O}_E} \frac{N}{X} = \text{filtered } \phi\text{-module}$



Using Wach modules

1) $\bar{V}_{k, a_p} \rightarrow \bar{V}_{k, a'_p}$ for a'_p close to a_p

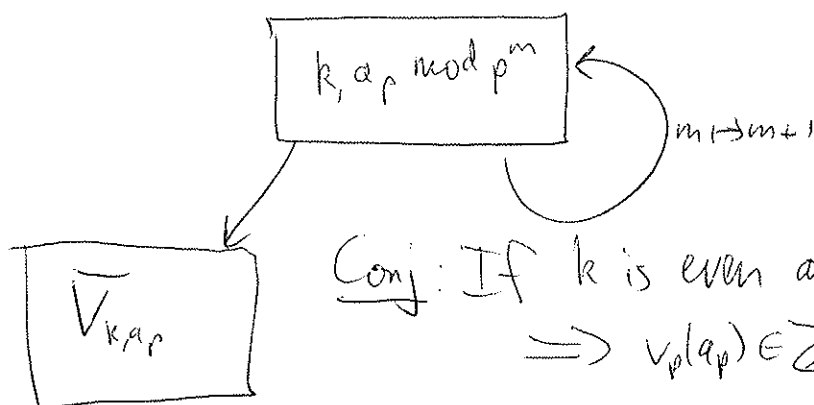
$\Rightarrow \bar{V}_{k, a_p} = \text{ind}(W_2^{R-1})$ if $v_p(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$

2) we prove that $\overline{\mathbb{T}}_{k, a_p} \Rightarrow \bar{V}_{k, a_p}$ $0 \leq i \leq p-2$
 $\text{mod } p-1$

If $0 < v_p(a_p) < 1$ \Rightarrow $\begin{cases} p-1 \nmid k-3 & \rightarrow \text{ind}(W_2^{\lfloor \frac{k-2}{p-1} \rfloor + 1}) \\ p-1 \mid k-3 & \begin{cases} \text{same} \\ \text{or} \\ \text{reducible} \end{cases} \end{cases}$

(Beuzard-Gee)

3) Wach modules are determined by their truncations \Rightarrow algorithm



Conj: If k is even and \bar{V}_{k, a_p} is reducible $\Rightarrow v_p(a_p) \in \mathbb{Z}$.