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Rational points for regular models of algebraic varieties of Hodge level ≥ 1

Joint with H. Esnault & K. Rülling.

1. Introduction.

$[K:\mathbb{Q}_p]<\infty$, $R=\mathcal{O}_K$, k res. field, $q=|k|$

$\forall r\geq 1$, $[k_r:k]=r$.

$W=W(k)$, $K_0=\mathrm{Frac}(W)$.

X proper flat R-scheme, X_K, X_k the generic, special fibres.

Thm 1: Assume further that

i) X_K geometrically connected

ii) X regular

iii) $H^i(X_K, \mathcal{O}_{X_K})=0 \quad \forall i\geq 1$

Then $\forall r\geq 1$, $|X(k_r)|\equiv 1 \mod q^r$

Thm (Esnault, 2006): Same conclusion with iii) replaced by

iii)' $\forall i\geq 1$, $\forall \xi\in H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell)$ $\exists\, U\subset X_K$, $U\neq\emptyset$ s.t.

$\xi\mapsto 0$ in $H^i_{ét}(U_{\bar{K}}, \mathbb{Q}_\ell)$ ("coniveau 1")

iii)' $\Rightarrow$ iii) (Deligne Hodge II)

iii) $\Rightarrow$ iii)' conjectured by Hodge's conj. (Groth.)

1.2 Reformulation in terms of Witt vectors

$Z(X_k, t)$: zeta function of X_k

$$Z(X_k, t) = \frac{\prod_i (1-\alpha_i t)}{\prod_j (1-\beta_j t)} \qquad \alpha_i, \beta_j \in \overline{\mathbb{Q}}_p \quad \alpha_i \neq \beta_j \ \forall i,j$$

$$Z^{<1}(X_k, t) = \frac{\prod_{v(\alpha_i)<1} (1-\alpha_i t)}{\prod_{v(\beta_j)<1} (1-\beta_j t)} \qquad v \text{ valuation s.t. } v(q) = 1$$

(part of Z of slope <1)

<u>Thm</u> (Berthelot, Bloch, Esnault) (X_k proper)

$$Z^{<1}(X_k, t) = \prod_i \det\left(1 - tF \,\middle|\, H^i(X_k, W\mathcal{O}_{X_k,\mathbb{Q}})\right)^{(-1)^{n+1}}$$

where $\forall n$, $W_n\mathcal{O}_{X_k} = \{(a_0, \dots, a_{n-1}) \mid a_i \in \mathcal{O}_{X_k}\}$

$W\mathcal{O}_{X_k} = \varprojlim_n W_n\mathcal{O}_{X_k}$, $W\mathcal{O}_{X_k,\mathbb{Q}} = W\mathcal{O}_{X_k} \otimes \mathbb{Q}$

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Then Thm 1 follows from:

<u>Thm 2</u>: Same assumptions as thm 1, then

1) $H^0(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) \cong K_0$

2) $\forall\ i \geq 1$, $H^i(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) = 0$

<u>Rmk</u>: $\exists$ trivial case: [Note that ii) $\Leftrightarrow$ $H^i(X, \mathcal{O}_X)$ is p-torsion if $i \geq 1$]

if $H^i(X, \mathcal{O}_X) = 0 \ \forall\ i \geq 1$

$\Rightarrow H^i(X_k, \mathcal{O}_{X_k}) = 0 \ \forall\ i \geq 1$

$\Rightarrow H^i(X_k, W\mathcal{O}_{X_k,\mathbb{Q}}) = 0 \ \forall\ i \geq 1$

2. p-adic Hodge theory

Prop: Thm 1 holds if X/R is semi-stable.

Proof: 1) Endow X, X_k with log-structures coming from the special fibre of X.

$\rightsquigarrow$ log-crystalline groups $H^i_{log\text{-}crys}(X_k/W)$.

We have a log-de-Rham-Witt complex $W\omega^\bullet_{X_k}$

with $W\omega^0_{X_k} = W\mathcal{O}_{X_k}$,

$$H^i_{log\text{-}crys}(X_k/W) \cong H^i(X_k, \underbrace{W\omega^\bullet_{X_k}})$$

filtration $\Rightarrow$ slope filtration on $H^i_{log\text{-}crys} \otimes \mathbb{Q}$

2) $\exists$ Hyodo-Kato isom.:

$$H^i_{log\text{-}crys}(X_k/W) \cong H^i(X_K, \Omega^\bullet_{X_K})$$

3) C_{st}:

$$B_{st} \otimes H^i_{log\text{-}crys}(X_k/W) \cong B_{st} \otimes H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_p)$$

$\Longrightarrow$ $H^i_{log\text{-}crys}(X_k/W) \otimes K_0$ + Frob. + filt^n via 2) is weakly admissable (sense of Fontaine)

$\Longrightarrow$ Newton polygon of $H^i_{log\text{-}crys}(X_k/W) \otimes K_0 \geq$ Hodge polygon of $H^i(X_K, \Omega^\bullet_{X_K})$

$\Rightarrow$ first slope of Newton is ≥ 1

$\Rightarrow$ $H^i(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) = 0$.

__General case__: Use de Jong's alteration:

Can find Y s.t. $\begin{array}{c} Y \\ f\downarrow \\ X \\ \downarrow \\ R \end{array}$ with f projective, Y integral, semistable $/R'$ ← finite ext. of R

X integral and $Y \to X$ generically finite étale.

$\Rightarrow$ For any N, we can find a finite extension K'/K, $R' \subset K'$ integers,

and an N-truncated simplicial hypercovering of X_k,

such that
1) __proper__ N-truncated hypercover
2) all Y_q's are pull-backs of semi-stable schemes over sub-extensions of K in K'.
3) hypercovering is an étale hypercovering over a nonempty open $\subset X$.

__Then__ (choose $K' \overset{\iota}{\hookrightarrow} \mathbb{C}$) $N > 2\dim(X_{K'})$

Proper coh. description for $\mathbb{C}$-coh. $\Rightarrow H^i(Y_{\bullet K'}, \Omega^\bullet_{Y_\bullet K'}) \cong H^i(X_{K'}, \Omega^\bullet_{X_{K'}})$ isom. of Hodge structures

$$\Rightarrow H^i(Y_{\bullet K}, \mathcal{O}_{Y_\bullet K}) = 0 \quad \forall\, i \geq 1$$

one gets:

$$\begin{tikzcd}[column sep=small]
 & H^i(Y_{\bullet k'}, W\mathcal{O}_{Y_{\bullet k'}, \mathbb{Q}}) \arrow[dr] & \\
H^i(X_k, W\mathcal{O}_{X_k, \mathbb{Q}}) \arrow[ur] \arrow[rr, dashed] \arrow[dr] & & H^i(Y_{0k'}, W\mathcal{O}_{Y_{0k'}, \mathbb{Q}}) \\
 & H^i(\underbrace{Y_{0,k}}_{\substack{Y_0 \text{ reduced} \\ \text{mod } m \subset R}}, W\mathcal{O}_{Y_{0,k}, \mathbb{Q}}) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Tsuji : the main thm of p-adic Hodge theory extend to trunc. simp. schemes

$\Rightarrow$ - - - - -> is the zero map.

Thm 3: Let $Y \xrightarrow{f} X$ be a projective, generically finite and flat morphism between two finitely generated flat regular R-schemes of the same dimension.

Then $H^i(X_k, WO_{X_k,\mathbb{Q}}) \hookrightarrow H^i(Y_k, WO_{Y_k,\mathbb{Q}})$

Key point: f is a local complete intersection morphism of virtual relative dim 0.

l.c.i. : loc. on Y, $\exists$

$$\begin{tikzcd} Y \arrow[r, hook, "\text{reg. imm.}"] \arrow[dr, "f"'] & P \arrow[d, "\text{smooth}"] \\ & X \end{tikzcd}$$

virtual relative dim $= \dim_{P/X} - \operatorname{codim}_P(Y)$

Thm 3 can be generalised to
- X integral
- $Y \to X$ l.c.i of virt. rel. dim. 0

$\sim$

3. The trace morphism τ_f

Prop: $f: Y \to X$ l.c.i, v.r.d 0, X noeth. with dual complex

Then $\exists\ \tau_f : Rf_*O_Y \to O_X$ s.t.
- i) trans. w.r.t. f
- ii) compatible with base change
- iii) If f finite flat, $Rf_*O_Y = f_*O_Y$
  $\Rightarrow \tau_f(b) = \mathrm{Tr}_{O_X}(\cdot b)$

$$f^! \mathcal{O}_X \cong \omega_Y = \left(\mathcal{I}/\mathcal{I}^2\right)^\vee \otimes_{\mathcal{O}_P} \Omega^d_{P/X}$$

$d = \dim(P/X)$

$\mathcal{I}$ = ideal of Y in P

$\exists$ canonical section of $f^! \mathcal{O}_Y$:

$(t_1, \dots, t_d) \mapsto dt_1 \wedge \cdots \wedge dt_d$.

Can be extended to the de Rham-Witt relative theory

$\Rightarrow$ using the trace map you get Thm 3.