Amnon Besser 16/03/10 9:30

Deformation techniques for computation of Coleman integrals

Coleman integration:

$[K:\mathbb{Q}_p] < \infty$, $R = \mathcal{O}_K$, $k = R/\pi$, $\#k = q = p^r$

$T_n^{\dagger} = R\langle t_1, \dots, t_n\rangle^{\dagger} = \left\{ \sum a_I t^I \,\middle|\, a_I \in R,\ \lim_{I\to\infty} |a_I| r^I = 0 \text{ for some } r > 1 \right\}$

$T_n^{\dagger} \twoheadrightarrow \mathcal{A}/R \iff \mathcal{A}$ wcfg (weakly complete finitely gen.) algebra

$A = \mathcal{A} \otimes K$, $\bar{A} = \mathcal{A}/\pi$, so $\bar{A}$ f.g. k-algebra, assumed smooth

$\phi : \mathcal{A} \circlearrowleft$, $\bar{\phi} : \bar{A} \circlearrowleft$ s.t. $\bar{\phi}(a) = a^q$, $\phi = \phi \otimes K$.

Module of differentials $\Omega^1_{A/K}$

$H^1_{dR}(A) = \left(\operatorname{coker}(d : A \to \Omega^1_{A/K})\right)^{d=0}$ has action of ϕ, where eigenvalues are Weil numbers of weight 1, 2,

Let $X = \operatorname{spm}(A)$.

Coleman theory: $\omega \in {\Omega^1_{A/K}}^{d=0} \xrightarrow{\text{linear}} F_\omega$ = loc. analytic function on X $+C$

s.t.
1) $dF_\omega = \omega$
2) $F_{dg} = g + C$, $g \in A$
3) $F_{\phi^*\omega} = \phi^* F_\omega + C$

This association is <u>unique</u> & <u>indep. of choice of ϕ</u>.

Constructions

$\omega_1, \dots, \omega_n$ basis for $H^1_{dR}(A)$, $\underline{\omega} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$

$\Omega^1_{A/K}$ $d\omega=0$

$$\phi^* \underline{\omega} = M \underline{\omega} + dg, \quad M \in M_{n\times n}(K), \quad g \in A^n$$

$$\phi^* F_{\underline{\omega}} = M F_{\underline{\omega}} + g + \cancel{c}$$

Now take Teichmüller pt. $\phi(x) = x$,

$$F_{\underline{\omega}}(x) = F_{\underline{\omega}}(\phi(x)) = M F_{\underline{\omega}}(x) + g(x)$$

$$\therefore F_{\underline{\omega}}(x) = (I - M)^{-1} g(x) \qquad \text{(nb. e-values of } M \text{ are not 1)}$$

Use 1) to determine $F_{\underline{\omega}}$ at non-Teich. pts,

$\rightsquigarrow F_{\underline{\omega}}$ determined.

If $\omega = \sum a_i \omega_i + dg$, then $F_\omega = \sum a_i F_{\omega_i} + g$

(Frobenius-equivariant paths)

(M, ∇) overconvergent F-isocrystal on A,
(under some conditions) can integrate elements of $\Omega^1(M)^{\nabla=0}$

$F_{\phi^*\omega} = \phi^* F_\omega$ - solve in a similar way

(M, ∇) universal n-step unipotent isocrystal - is actually an F-isocrystal.
For any other (M_1, ∇_1) unipotent isocrystal, can project integrals $(M, \nabla) \longrightarrow (M_1, \nabla_1)$

Small deformation

$$\begin{array}{c} X \\ \downarrow \pi \\ \{|z|<1\} \end{array}$$

$\omega \in \Omega^1(X_0)$, $d\omega = 0$

rigid cohomology $\rightarrow$ $H^1_{dR}(X) = H^1_{dR}(X_0)$

$\tilde{\omega} \in \Omega^1(X)^{d=0}$: $\omega = \tilde{\omega}|_{X_0} + \cancel{dg}$

so $F_\omega = F_{\tilde{\omega}}|_{X_0}$.

If $\underset{x_0}{\underset{\cup}{X_0}}$ good, $\underset{x_p}{\underset{\cup}{X_p}}$ not so good s.t. $x_0 \equiv x_p \pmod{\pi}$

so. $F_{\tilde{\omega}|_{X_p}}(x_p) = F_{\tilde{\omega}|_{X_0}}(x_0) + \int_{x_0}^{x_p} \tilde{\omega}$

eg. poly. $f(x,t)$, $X_t = \{y^2 = f(x,t)\} - \{|y|<1\}$

Could have X_0 defined over unramified ext,
X_p ——— highly ramified ext.

$$X = \{(x,y,t),\ y^2 = f(x,t),\ |t|<1\} = X_p \times \{|t|<1\}$$

$$(x,y,t) \longmapsto \left(x,\ y\sqrt{\frac{f(x,p)}{f(x,t)}},\ t\right)$$

$$\omega = \frac{x^i\,dx}{y} \rightsquigarrow \tilde{\omega} = \frac{x^i\,dx}{y\sqrt{\frac{f(x,p)}{f(x,t)}}}$$

Bigger deformations

$$\begin{array}{c} X = \mathrm{spm}(A) \\ \downarrow \\ S = \mathrm{spm}(B) \end{array}$$

$\omega \in \Omega^1_{X/S}$ s.t. $d_r\omega = 0$ (closed relative differential)

Naively, want
$s \longmapsto F_{\omega_s}$ but the "g" varies over fibres.

"Idea"

$$F_{\frac{\partial}{\partial s}\omega} = \frac{\partial}{\partial s} F_\omega \qquad \text{(only works if family is trivial)}$$

Modification

$$\omega \rightsquigarrow \tilde{\omega} \in \Omega^1_X \ , \quad d\tilde{\omega} \in \Gamma(X, \pi^*\Omega^1_S \otimes \Omega^1_{X/S})$$

same as computing Gauss-Manin connection

$$F_{d\tilde{\omega}} \in \Gamma(X, \pi^*\Omega^1_S \otimes \mathcal{O}_{col}(X/S))$$

$$F_\omega \in \Gamma(X, \mathcal{O}_{col}(X/S))$$

$\boxed{dF_\omega - \tilde{\omega} = F_{d\tilde{\omega}}}$ is the condition we need.

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$$\underline{\omega} \in (\Omega^1_{X/S})^n \quad \text{basis for} \quad \mathcal{H}^1_{dR}(X/S)$$

$$\rightsquigarrow \underline{\tilde{\omega}} \in (\Omega^1_X)^n, \quad d\tilde{\omega} = \Theta(s) \otimes \underline{\tilde{\omega}} + d_r \underline{g}$$

where $\Theta(s) \in M_{n\times n}(\Omega^1_S)$, $\underline{g} \in \Gamma(X, \pi^*\Omega^1_S)$

$$\therefore \ dF_{\underline{\omega}} = \underline{\tilde{\omega}} + \Theta(s) F_{\underline{\omega}} + \underline{g}$$

ie. $\underbrace{(d - \Theta(s))}_{\text{pullback of Gauss-Manin connection to } X} F_{\underline{\omega}} = \underline{\tilde{\omega}} + \underline{g}$