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Relative Dwork cohomology of nondegenerate exponential sums

1. σ-modules

$\Delta \subseteq \mathbb{R}^s$ polytope, $0 \in \Delta$, $\dim \Delta = s$

$u \in \mathbb{Z}^s$, $w(u) = \min\{\gamma \mid u \in \gamma\Delta\}$

$[K:\mathbb{Q}_p] < \infty$, R integers in K, res. field $\mathbb{F}_q$

$$A_\Delta^\dagger := \left\{ \sum_{u \in \mathbb{Z}^s} a_u t^u \;\middle|\; \liminf \frac{\mathrm{ord}_p(a_u)}{w(u)} > 0 \right\}$$

$\mathcal{H}$ = free, finite rank $A_\Delta^\dagger$-module

$\varphi : \mathcal{H} \to \mathcal{H}$ σ-linear $\quad (\sigma : t \mapsto t^q)$

$(\varphi, \mathcal{H})$ is a σ-module

$$L(\varphi, \mathbb{G}_m^s, T) := \prod_{\bar{t} \in |\mathbb{G}_m^s|} \det\left(1 - \varphi_t^{\deg(\bar{t})} T^{\deg(\bar{t})} \mid \mathcal{H}_t\right)^{-1}$$

$t = \mathrm{Teich}(\bar{t})$

Thm: $L(\varphi, \mathbb{G}_m^s, T)$ is p-adic mero. $|T|_p < \infty$

Thm: (Haessig, Sperber)

a) Assume $L(\varphi, \mathbb{G}_m^s, T) = \dfrac{\prod_{i=1}^{N}(1-\alpha_i T)}{\prod_{j=1}^{M}(1-\beta_j T)}$

Then, $\deg L(\varphi, \mathbb{G}_m^s, T) = N - M \leq c_1^s \, s! \, \mathrm{vol}(\Delta) . \mathrm{rank}(\mathcal{H})$

total degree $= N+M \leq$ Messy $\leq s!\,\mathrm{vol}(\Delta)\,\mathrm{rank}(\mathcal{H})\cdot 2^{s+(1+\frac{1}{s})c_2 \left(1+2^{c_1(1+\frac{s}{s})}\right)^s}$

b) $\Delta \subseteq \mathbb{R}^s_{\geq 0}$. Write $L(\varphi, \mathbb{A}^s, T) = 1 + a_1 T + O(T^2)$

$$\mathrm{ord}_p(a_1) \geq \underset{\uparrow \text{depends on } \varphi}{b}(q-1)\min\{w(u) \mid u \in \mathbb{Z}^s_{\geq 0},\, w(u) > 0\} + h$$

where $q \equiv 0 \bmod p^h$

2. Examples

$g \in \mathbb{F}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

$\square(g) =$ Newton polytope of g

$\dim \square = n$, g nondegenerate wrt $\square$.

$S \subseteq (\text{Interior } \square \cap \mathbb{Z}^n)$

family: $G(t,x) = g(x) + \sum_{u \in S} t_u x^u$

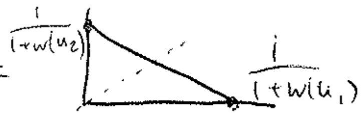
$t \in \mathbb{F}_q^{|S|}$: $S_k(t) = \sum_{x \in (\mathbb{F}_{q^k}^*)^n} \zeta_p^{\mathrm{Tr}(G(t,x))}$

$$L(G,t,T) := \exp\left(\sum_{k=1}^{\infty} S_k(t) \frac{T^k}{k}\right)$$
$$= \det(1 - \mathrm{Frob}_{q,t} T \mid \mathcal{H}_t)^{(-1)^{n-1}}$$

$(\mathrm{Frob}_{q,t}, \mathcal{H}_t) = (\mathrm{Frob}_q, \mathcal{H})\big|_t$

where $\mathcal{H} =$ rel. Dwork cohom.

$=$ rank $n!\,\mathrm{vol}(\square)$, $A_\Delta^\dagger$-module

$\Delta =$ 

etc $\subset \mathbb{R}^{|S|}$

$(\varphi, \mathcal{H}) := (\mathrm{Frob}_q, \mathcal{H})$

sharp

Thm $\Rightarrow \deg L(\mathrm{Sym}^k \varphi/_{\mathbb{G}_m^n}, \mathbb{G}_m^{|S|}, T) \leq |S|!\,\mathrm{vol}(\Delta) \binom{k + n!\,\mathrm{vol}(\square) - 1}{n!\,\mathrm{vol}(\square) - 1}$

& $L(\mathrm{Sym}^k \varphi/_{\mathbb{G}_m^n}, \mathbb{A}^{|S|}, T) = 1 + a_1 T + O(T^2)$, then

sharp

$\mathrm{ord}_p(a_1) \geq \sum_{u \in S} (1 - w_\square(u))$

3. Unit root L-functions

$(\varphi, \mathcal{H})$ finite rank, ordinary σ-module over $A_\Delta^\dagger$.

$(\varphi_i, \mathcal{H}_i)$: i^{th} ~~sl~~ slope unit root σ-module

<u>Thm</u>: Assume $\Delta \subseteq \mathbb{R}^s_{\geq 0}$, $\dim \Delta = s$

Write $L(\varphi_0, \mathbb{A}^s, T) = 1 + a_1 T + O(T^2)$.

then $\mathrm{ord}_p(a_1) \geq b(q-1)\min\{w(u) \mid u \in \mathbb{Z}^s_{\geq 0}, w(u) \neq 0\}$

<u>e.g.</u> $G(t,x) = x^d + tx \in \mathbb{F}_p[t,x]$. $(d,p) = 1$

$p \equiv 1 \bmod (d)$

$(\varphi, \mathcal{H})$ satisfies $\varphi \equiv 0 \bmod (p^{1/d})$

so let $\tilde{\varphi} = p^{-1/d} \varphi$. Then write

$L(\tilde{\varphi}_0, \mathbb{A}^1, T) = 1 + a_1 T + O(T^2)$,

$\Rightarrow \mathrm{ord}_p(a_1) \geq \frac{d-1}{d}$