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Average ranks of elliptic 3-folds and zeta functions of singular hypersurfaces

Motivation: $E/\mathbb{Q}$ elliptic curve

$y^2 = x^3 + Ax + B$ minimal Weierstass equation

$A, B \in \mathbb{Z}$, $\nexists u \notin \{\pm 1\} : u^4 | A, u^6 | B$

$\Delta_{min} = 4A^3 + 27B^2$

- $\mathcal{E}_N = \{ E \mid$ elliptic curves with $|\Delta_{min}| \leq N \}$

$$av = \lim_{N \to \infty} \frac{\sum_{E \in \mathcal{E}_N} \text{rank}(E(\mathbb{Q}))}{\# \mathcal{E}_N}$$

little known : Rumour : $\limsup \leq \frac{3}{2}$

- $$\mu_r = \lim_{n \to \infty} \frac{\# \{ E \in \mathcal{E}_N \mid \text{rank}\, E(\mathbb{Q}) = r \}}{\# \mathcal{E}_N}$$

Conj: $\mu_0 = \mu_1 = \frac{1}{2}$ $\quad av = \frac{1}{2}$

Evidence: BSD : $\text{rank}\, E(\mathbb{Q}) = \text{ord}_{s=1} L(E,s)$
$= \text{ord}_{s=1} L(f,s)$

modular form

Functional equation: $L(E, 2-s) = (-1)^w L(E,s)$

Idea : $w = 0$ with prob $\frac{1}{2}$ $\Rightarrow$ even parity
$w = 1$ with prob $\frac{1}{2}$ $\Rightarrow$ odd parity

Idea : $w = 0 \overset{\text{Prob. 1}}{\Longrightarrow} r = 0$
$w = 1 \overset{\text{Prob 1}}{\Longrightarrow} r = 1$

Numerical evidence : Problem : $\mu_2 > 0$

Case $\mathbb{F}_q(C)$, $C/\mathbb{F}_q$ sm. proj. curve

v_p place, $\mathbb{F}_q[C] = \{f \in \mathbb{F}_q(C) \mid v_{p'}(f) > 0 \ \forall p' \neq p\}$

$[\deg = -v_p]$

$$\mathcal{E}_N = \{E/\mathbb{F}_q(C) \mid \deg \Delta_{min} \leq N\}$$

Can define av, μ_r similarly for case $K = \mathbb{Q}$

Conj : $av = \frac{1}{2}$, $\mu_0 = \mu_1 = \frac{1}{2}$

Evidence : $E/\mathbb{F}_q(C)$ $\quad S \downarrow C$ smooth proj. minimal surface

$P_2(T)$ char. poly of Frob. on $H^2_{\acute{e}t}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$

$$\mathrm{rank}\left(\mathrm{Div}(S)/\sim_{alg}\right) =: \rho(S) \leq \mathrm{ord}_{T=\frac{1}{q}} P_2(T)$$

$=$ under Tate conjecture

Shioda-Tate formula relates $\rho(S)$ to rank $E(\mathbb{F}_q(C))$

$\exists$. Functional eqn. for $P_2(T)$ with sign

Expect prob. $\frac{1}{2}$ even rank

$\frac{1}{2}$ odd rank

Numerical evidence : $q = 7$ (Lauder 2008)

What happens for E.C. $/\mathbb{F}_q(V)$, $\dim V > 1$

$V/\mathbb{F}_q$ sm. proj. variety

$\mathcal{L}$ ample line bundle on V

$$E/\mathbb{F}_q(V) \qquad \begin{tikzcd} \widetilde{W} \arrow[d] \\ V \arrow[u, bend left, "\sigma_0"] \end{tikzcd} \quad \text{elliptic } n\text{-fold}, \quad n = \dim V + 1$$

zero section from base pt.

$$\exists\, k \geq 0 \leadsto W \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{-2k} \oplus \mathcal{L}^{-3k})$$

Assume $q = p^r$, $p > 3$, then $W = \{Y^2 Z = X^3 + AXZ^2 + BZ^3\}$

with $A \in H^0(\mathcal{L}^{4k})$, $B \in H^0(\mathcal{L}^{6k})$

$$\begin{tikzcd} W \arrow[d, "\pi"] & \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{-2k} \oplus \mathcal{L}^{-3k}) \arrow[dl] \\ V & \end{tikzcd}$$

- W might be singular

$P \in E(\mathbb{F}_q(V)) \sim \sigma_P : V \dashrightarrow W$

$\pi \circ \sigma_P = \mathrm{id}_V$ (as a rational map)

$$\mathcal{E}_P = \overline{\mathrm{Im}(\sigma_P)}$$

~~Proof~~ <u>Prop. 1</u>: $\dim V \geq 2$

$\mathrm{rk}(\mathrm{Pic}(W)) = \mathrm{rk}(\mathrm{Pic}(V)) + 1$

$[-1]^* : \mathrm{Pic}(W) \to \mathrm{Pic}(W)$ is the identity

<u>Sketch proof</u>: First contract $\{X = Z = 0\}$ in $\mathbb{P}$ to a line, $E =$ excep. divisor

$W_0 = \mathrm{Image}(W)$, ample HS

Apply Lefschetz hyperplane theorem:

$\Longrightarrow \mathrm{rk}(\mathrm{Pic}(W_0)) = 1$

Then use exact

$$\mathrm{Pic}(W_0) \to \mathrm{Pic}(W) \hookrightarrow \underset{\mathrm{Pic}(V)}{\overset{\|}{\mathrm{Pic}(E)}} \implies \text{“}\le\text{”}$$

"$\ge$" is easy.

Prop. 2: $P \in E(\mathbb{F}_q(V))$ $\quad [2]P \neq 0$

$$[-1]^* \mathcal{E}_P \not\sim_{alg} \mathcal{E}_P$$

Moreover, if $P \notin E(\mathbb{F}_q(V))[2]$ then $\mathcal{E}_P$ has infinite order in $\mathrm{Div}(W)/\mathrm{Pic}(W)$

$\mathrm{rank}(E(\mathbb{F}_q(V))) > 0 \implies W$ contains a divisor that is not $\mathbb{Q}$-Cartier.

Thm (SGA): Suppose S is a proj. var.. If S has only hypersurface singularities and S contains a divisor that is not $\mathbb{Q}$-Cartier, then

$$\dim S_{sing} \ge \dim S - 3.$$

Rmk: $W \subset \mathbb{P}$ has only hypersurface singularities.

To bound the number of curves with positive rank, we need to bound:

- If $\dim V = 2$, the number of sing. HS in $\mathbb{P}$
- If $\dim V = 3$, the number of sing HS in $\mathbb{P}$ with non-isolated sing.

$\vdots$

Recall (Poonen)

Thm: $X \subset \mathbb{P}^n$ sm. proj. , $S = \oplus S_k$ coord. ring

density

$$\mu\left(\left\{ f \in S_d \;\middle|\; X \cap V(f) \text{ is smooth and of expected dimension} \right\} \right)$$

$$= Z(X, \dim X + 1)^{-1}$$

$$\mu\left(\left\{ f \in S_d \;\middle|\; X \cap V(f) \text{ is smooth or has isol. sing. \& of expected dim} \right\} \right) = 1$$

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Prop:

- $$\mu\left(\left\{ (A,B) \in H^0(\mathcal{L}^{-4k}) \times H^0(\mathcal{L}^{-6k}) \;\middle|\; \underbrace{-Y^2Z + X^3 + AXZ^2 + BZ^3}_{W_{A,B}} \text{ is smooth} \right\} \right)$$

$$= Z(V, \dim V + 1)^{-1}$$

- $$\mu\left(\left\{ \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad \text{or has isolated singularities} \right\} \right)$$

$$= 1$$

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Cor: If $\dim V \geq 3 \Rightarrow \mu_0 = 1$

$\dim V = 2 \Rightarrow \mu_0 \geq Z(V, 3)^{-1}$

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What about $\mathcal{M}_0$ in case $\dim V = 2$?

$$\{(A,B) \mid W_{A,B} \text{ is minimal}\}$$
$$\cup$$
$$\{(A,B) \mid W_{A,B} \text{ is singular}\} \supset \{(A,B) \mid h^4_{rig}(W_{A,B}) > h^2_{rig}(W_{A,B})\}$$
$$\cup \qquad\qquad\qquad \cup$$
$$\{(A,B) \mid W_{A,B} \text{ has a divisor that is not } \mathbb{Q}\text{-Cartier}\}$$
$$\cup$$
$$\{(A,B) \mid \operatorname{rank}(E(\mathbb{F}_q(v))) > 0\}$$

Experiment: $V = \mathbb{P}^2$, $k = 1$, $\mathcal{L} = \mathcal{O}(1)$

Given a pair (A,B) calculate $h^4_{rig}(W_{A,B})$:

To simplify, $W \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3))$

take $W \subset \mathbb{P}(1,1,1,2,3) =: \mathbb{P}$, & assume $\dim W_{sing} \leq 0$

$\cancel{H^3(W \setminus W_{sing}) \to H^3(W)}$

$$H^3(W) \to H^3(W \setminus W_{sing}) \to H^4_{W_{sing}}(W) \to H^4(W) \xrightarrow{\text{not zero map}} H^4(W \setminus W_{sing})$$

$$H^3(W \setminus W_{sing}) \overset{Res}{\cong} H^4((\mathbb{P} \setminus W_{sing}) \setminus (W \setminus W_{sing})) = H^4(\mathbb{P} \setminus W)$$

$H^4(W \setminus W_{sing})$ is 1-dim

$\mathbb{P} \setminus W$ is affine. Griffiths-Dwork-Steenbrink method finds a generating set for $H^4(\mathbb{P} \setminus W)$.

Assume that each sing. P of W is weighted homogeneous.

$$H^4_p(W) \cong H^2(S)_{prim}(-1)$$

10^5 E.C. $/ \mathbb{F}_5(s,t)$ s.t. $A = 0$

None of these had positive rank

Conj. $\mu_0 = 1$ if $\dim V = 2$

Expectation:

Largest family in $H^0(\mathbb{P}^2, \mathcal{O}(2k)) \times H^0(\mathbb{P}^2, \mathcal{O}(3k))$ with positive rank has codim $\geq rk^2$.