Alan Lauder 19/03/10 2:00

Degenerations of limit Frobenius structures in rigid cohomology

p prime,

$X_0/\mathbb{F}_p$ hypersurface of degree d in $\mathbb{P}^{n+1}$

$$Z(X_0,T) = \exp\Big(\sum_{k=1}^{\infty} |X_0(\mathbb{F}_{p^k})| \frac{T^k}{k}\Big) \underset{\text{Dwork}}{\in} \mathbb{Q}(T)$$

$$X_0 \text{ smooth} \quad = \frac{P_0(T)^{(-1)^{n+1}}}{(1-T)(1-pT)\dots(1-p^nT)}, \quad P_0(T) = \prod_i (1-\alpha_i T)$$

$\|\alpha_i\| = p^{n/2}$, deg P_0:

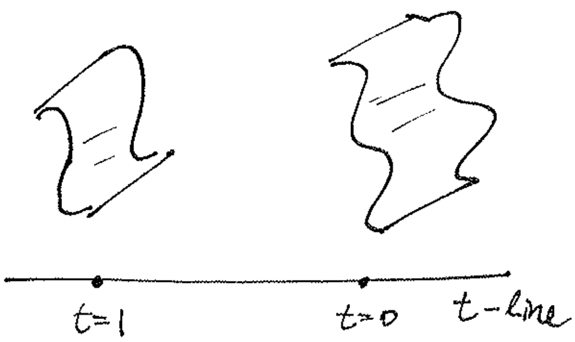
$n \backslash d$	3	4	5	6
1	2	6	12	20
2	6	21	...	
3	10	....		

deformation method

$X_0 : f_0(x_0, x_1, \dots, x_{n+1}) = 0$

$X_1 : f_1 = x_0^d + \dots + x_{n+1}^d = 0 \qquad (p \nmid d)$

$X : (1-t)f_0 + tf_1 = 0$



What if X_0 singular?

One can compute a "$P_0(T)$" (to any precision)

eg. i) Quartic to double conic

$$f_1 := X^4 + Y^4 + Z^4$$
$$f_0 := (X^2 + Y^2 + \dots)^2$$
$$p := 5$$

$$P_0(T) = 1 - 6T + 23T^2 - 58T^3 + 115T^4 - 150T^4 + 125T^6$$

but $\quad Z(X_0, T) = \dfrac{1}{(1-T)(1-5T)}$.

ii) Quartic curve to 3-cuspidal quartic

$$f_1 := X^4 + Y^4 + Z^4$$
$$f_0 := X^2Y^2 + Y^2Z^2 + \dots$$
$$p = 13$$

$$P_0(T) = (1 - 5T + 13T^2)^3$$

but $\quad Z(X_0, T) = \dfrac{1}{(1-T)(1-13T)}$

iii) Quintic curve to quintic with ordinary triple pt.

get $\quad P_0(T) = (1-T)(1+T) \dots$ but degree 11, not 12

... more on slides ...

<u>Thm</u>: Assume (*). Then one can compute to any finite precision a quadruple $(H, F, N; e)$ s.t. $pNF = FN$ called the <u>limiting Frobenius structure</u> at $t=$ and $e \in \mathbb{Z}$ (see later).

$$\underset{\text{Frobenius}}{F} \circlearrowright H \circlearrowleft \overset{N}{\ } \text{ monodromy} \quad \text{f.d.v.s.}/\mathbb{Q}_p, \quad P_0(T) := \det(1 - TF \mid \ker(N)).$$

<u>Conjecture</u>: There is a "Clemens-Schmidt" exact sequence in rigid cohomology

$$\cdots \to H^n_{rig}(\underset{\uparrow \text{"semistable limit"}}{X_0'}) \to \overset{\circlearrowright F}{H} \xrightarrow{N} \overset{\circlearrowright F}{H(-1)} \to \cdots$$

<u>Note</u>: X_0 smooth $\Rightarrow H = H^n_{rig}(X_0)$, $N = 0$, $e = 1$

(*) $X/\mathbb{F}_p$ "lifts" to $\mathfrak{X}/\mathbb{Q}_p$ which is smooth in the punctured p-adic open unit disk around $t=0$.

Back to examples: i) Pull-back $t \mapsto \underset{\boxed{e=2}}{t^2}$ gives trivial monodromy

$$X_0' : y^2 = 2(x^8 + 3x^7 + 2x^6 + 4x^3 + x^2 + 2x + 1)$$

$$Z(X_0', T) = \frac{P_0(T)}{(1-T)(1-5T)}$$

... more on slides ...

Motivation (for the Clemens-Schmidt sequence)

$$\Delta = \{ z \in \mathbb{C} \mid \|z\| < 1 \}, \quad \Delta^* := \Delta \setminus \{0\}$$

$$\begin{tikzcd} \mathcal{X}_t \arrow[r, hook] \arrow[d, mapsto] & \mathcal{X}^* \arrow[d, "\text{smooth}"] & \subset \mathcal{X} \arrow[d, "\text{proj}"] \\ t \arrow[r, hook] & \Delta^* & \subset \Delta \end{tikzcd}$$

$H^m(*) := H^m_{sing}(*, \mathbb{Q})$

Picard-Lefschetz transformation

$$T : H^m(\mathcal{X}_t) \longrightarrow H^m(\mathcal{X}_t)$$

Local monodromy theorem:

$\exists\ e, f \in \mathbb{N}$ s.t. $(T^e - I)^f = 0$, $f \leq m+1$

- by Hironaka can assume $\mathcal{X}$ is smooth and $\mathcal{X}_0 := e_1 \mathcal{X}_1 + \dots + e_s \mathcal{X}_s$ a strict n/c divisor, $e = \mathrm{lcm}(e_1, \dots, e_s)$
- ~~replace~~ pull-back $t \mapsto t^e$ and normalisation, gives

$\mathcal{X}'$,

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \Delta \arrow[r, "t \mapsto t^e"] & \Delta \end{tikzcd}$$

s.t. $\mathcal{X}'_0 = \mathcal{X}'_1 + \dots + \mathcal{X}'_{s'}$ is a reduced s. n/c. d.

(semistable).

$\Rightarrow$ Clemens-Schmidt:

$$\dots \to H^m(\mathcal{X}'_0) \longrightarrow H^m(\mathcal{X}_t) \xrightarrow{\log T} H^m(\mathcal{X}_t) \to \dots$$

want to $\rightsquigarrow \Delta \subset \mathbb{P}^1_{\mathbb{F}_p}$ through 0 & $H^m(*) =$ rigid cohomology