

# Degenerations of limit Frobenius structures in rigid cohomology

$p$  prime,

$X_0/\mathbb{F}_p$  hypersurface of degree  $d$  in  $\mathbb{P}^{n+1}$

$$Z(X_0, T) = \exp\left(\sum_{k=1}^{\infty} \frac{|X_0(\mathbb{F}_{p^k})|}{k} \frac{T^k}{k} \right) \in \mathbb{Q}(T)$$

Dwork

$$X_0 \text{ smooth} = \frac{P_0(T)^{(-1)^{n+1}}}{(1-T)(1-pT)\dots(1-p^n T)}, \quad P_0(T) = \prod_i (1-\alpha_i T)$$

$$\|\alpha_i\| = p^{n/2}$$

deg  $P_0$ :

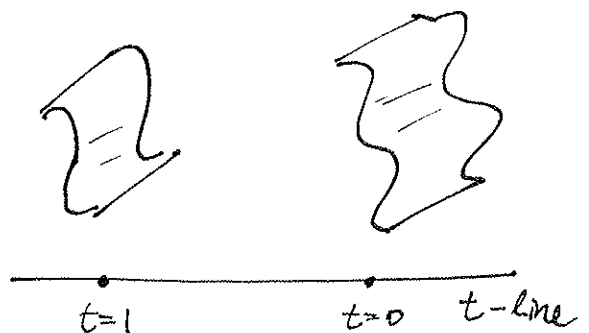
$n \backslash d$	3	4	5	6
1	2	6	12	20
2	6	21	...	
3	10	...		

## deformation method

$$X_0 : f_0(x_0, x_1, \dots, x_{n+1}) = 0$$

$$X_1 : f_1 = x_0^d + \dots + x_{n+1}^d = 0 \quad (p \nmid d)$$

$$X : (1-t)f_0 + tf_1 = 0$$



What if  $X_0$  singular?

One can compute a " $P_0(T)$ " (to any precision)

eg. i) Quartic to double conic

$$f_1 := X^4 + Y^4 + Z^4$$

$$f_0 := (X^2 + Y^2 + \dots)^2$$

$$p := 5$$

$$P_0(T) = 1 - 6T + 23T^2 - 58T^3 + 115T^4 - 150T^5 + 125T^6$$

$$\text{but } z(X_0, T) = \frac{1}{(1-T)(1-5T)}$$

ii) Quartic curve to 3-cuspidal quartic

$$f_1 := X^4 + Y^4 + Z^4$$

$$f_0 := X^2 Y^2 + Y^2 Z^2 + \dots$$

$$p = 13$$

$$P_0(T) = (1 - 5T + 13T^2)^3$$

$$\text{but } z(X_0, T) = \frac{1}{(1-T)(1-13T)}$$

iii) Quintic curve to quintic with ordinary triple pt.

get  $P_0(T) = (1-T)(1+T) \dots$  but degree 11, not 12

... More on slides ...

(2)

Thm: Assume (\*). Then one can compute to any finite precision a quadruple  $(H, F, N; e)$  s.t.  $pNF = FN$  called the limiting Frobenius structure at  $t=0$  and  $e \in \mathbb{Z}$  (see later).

Frobenius  $F \hookrightarrow H \xleftarrow{N}$  f.d.v.s. /  $\mathbb{Q}_p$ ,  $P_0(\tau) := \det(1 - \tau F | \ker(N))$ .  
*N monodromy*

Conjecture: There is a "Clemens-Schmidt" exact sequence in rigid cohomology

$$\dots \rightarrow H_{\text{rig}}^n(X'_0) \rightarrow H \xrightarrow{N} H(-1) \rightarrow \dots$$

$\uparrow$   
 "semistable limit"

Note:  $X_0$  smooth  $\Rightarrow H = H_{\text{rig}}^n(X_0)$ ,  $N=0$ ,  $e=1$

(\*)  $X/\mathbb{F}_p$  "lifts" to  $\mathcal{X}/\mathbb{Q}_p$  which is smooth in the punctured  $p$ -adic open unit disk around  $t=0$ .

Back to examples: i) Pull-back  $t \mapsto t^2$  gives trivial monodromy

$$X'_0: y^2 = 2(x^8 + 3x^7 + 2x^6 + 4x^3 + x^2 + 2x + 1)$$

$$Z(X'_0, \tau) = \frac{P_0(\tau)}{(1-\tau)(1-5\tau)}$$

... more on slides...

# Motivation (for the Clemens-Schmidt sequence)

$$\Delta = \{z \in \mathbb{C} \mid \|z\| < 1\}, \quad \Delta^* := \Delta \setminus \{0\}$$

$$\begin{array}{ccc} \mathcal{X}_t \hookrightarrow \mathcal{X}^* \subset \mathcal{X} & & H^m(\mathcal{X}^*) := H_{\text{sing}}^m(\mathcal{X}^*, \mathbb{Q}) \\ \downarrow & \begin{array}{c} \downarrow \text{smooth} \\ \downarrow \text{proj} \end{array} & \\ t \hookrightarrow \Delta^* \subset \Delta & & \text{Picard-Lefschetz transformation} \\ & & T: H^m(\mathcal{X}_t) \longrightarrow H^m(\mathcal{X}_t) \end{array}$$

Local monodromy theorem:

$$\exists e, f \in \mathbb{N} \text{ s.t. } (T^e - I)^f = 0, \quad f \leq m+1$$

- by Hironaka can assume  $\mathcal{X}$  is smooth and  $\mathcal{X}_0 := e_1 \mathcal{X}_1 + \dots + e_s \mathcal{X}_s$  a strict n/c divisor,  $e = \text{lcm}(e_1, \dots, e_s)$

- ~~pull-back~~ pull-back  $t \mapsto t^q$  and normalisation, gives

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{t \mapsto t^q} & \Delta \end{array}$$

s.t.  $\mathcal{X}'_0 = \mathcal{X}'_1 + \dots + \mathcal{X}'_s$  is a reduced s. n/c. d.

(semistable)

⇒ Clemens-Schmidt:

$$\dots \rightarrow H^m(\mathcal{X}'_0) \rightarrow H^m(\mathcal{X}_t) \xrightarrow{\log T} H^m(\mathcal{X}_t) \rightarrow \dots$$

want to  $\rightsquigarrow \Delta \subset \mathbb{P}_{\mathbb{C}}^1$  through 0 &  $H^m(\mathcal{X}^*) = \text{rigid cohomology}$