Towards an overconvergent Deligne-Kashiwara correspondence

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Connections and local systems

The derived Riemann-Hilbert correspondence

Kashiwara's correspondence

Grothendieck's infinitesimal site

Deligne's correspondence

Berthelot's correspondence

The mixed characteristic situation

The overconvergent site

An overconvergent Deligne-Kashiwara correspondence

Connections and local systems

Theorem (analytic Riemann-Hilbert)

If X is a complex analytic manifold, we have

$$\begin{aligned} MIC(X) &\xrightarrow{\simeq} LOC(X) \\ \mathcal{F} &\longmapsto \mathcal{H}om_{\nabla}(\mathcal{F}, \mathcal{O}_X). \end{aligned}$$

Here, $MIC(X)$ denotes the category of coherent modules with an integrable connection; and $LOC(X)$ denotes the category of local systems of finite dimensional vector spaces on X (locally constant sheaves of finite dimensional vector spaces).

Proof.

Straightforward. $\square$

Algebraic case

Theorem (algebraic R-H)

If X is a smooth complex algebraic variety, we have

$$\begin{aligned} \mathit{MIC}_{\mathrm{reg}}(X) &\xrightarrow{\ \simeq\ } \mathit{LOC}(X^{\mathrm{an}}) \\ \mathcal{F} &\longmapsto \mathcal{H}om_{\nabla}(\mathcal{F}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}). \end{aligned}$$

Now, $\mathit{MIC}_{\mathrm{reg}}(X)$ denotes the category of coherent modules with a regular integrable connection.

Proof.

The point is to show that $\mathit{MIC}_{\mathrm{reg}}(X)$ is equivalent to $\mathit{MIC}(X^{\mathrm{an}})$: see Deligne's book [Deligne] or Malgrange's lecture in [Borel]. $\square$

Derived Riemann-Hilbert correspondence

Theorem (derived R-H)

If X is a complex analytic manifold, we have

$$\begin{aligned} D^{\mathrm{b}}_{\mathrm{reg,hol}}(X) &\xrightarrow{\ \simeq\ } D^{\mathrm{b}}_{\mathrm{cons}}(X) \\ \mathcal{F} &\longmapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}, \mathcal{O}_X) \end{aligned}$$

Here, $D^{\mathrm{b}}_{\mathrm{reg,hol}}(X)$ denotes the category of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology; and $D^{\mathrm{b}}_{\mathrm{cons}}(X)$ denotes the category of bounded complexes of $\mathbf{C}_X$-modules with constructible cohomology.

Proof.

Beautiful theorem of Kashiwara ([Kashiwara1]). $\square$

Some remarks

1. The categories $MIC(X)$ and $LOC(X)$ have to be enlarged in order to get stability under standard operations.
2. The derived Riemann-Hilbert correspondence does not send regular holonomic $\mathcal{D}_X$-modules to $\mathbf{C}_X$-modules but we really do get complexes.
3. Conversely, constructible $\mathbf{C}_X$-modules do not come from $\mathcal{D}_X$-modules, but from complexes.

This is where "perversity" enters in the game. We will now recall the classical answer to 2) and the recent analogous answer to 3).

Perverse sheaves

Theorem (Perverse R-H)

If X is a complex analytic manifold, we have

$$(\mathcal{D}_X - \mathrm{mod})^{\mathrm{reg,hol}} \xrightarrow{\simeq} D^{\mathrm{perv}}_{\mathrm{cons}}(X)$$

(actually, we obtain an equivalence of t-structures).

$D^{\mathrm{perv}}_{\mathrm{cons}}(X)$ denotes the category of perverse sheaves: bounded complexes of $\mathbf{C}_X$-modules with constructible cohomology satisfying

$$\begin{cases} \dim \operatorname{supp} \mathcal{H}^n(\mathcal{F}) \leq -n \text{ for } n \in \mathbf{Z} \\ \mathcal{H}^n_Z(\mathcal{F})_{|Z} = 0 \text{ for } n < -\dim Z. \end{cases}$$

Proof.

See for example Beilinson-Bernstein-Deligne ([B-B-D]). □

THEOREM (KASHIWARA'S CORRESPONDENCE)

X is a smooth algebraic variety, we have

$$D^{\mathrm{perv}}_{\mathrm{reg,hol}}(X) \xrightarrow{\simeq} \mathrm{Cons}(X^{\mathrm{an}}).$$

Now, $D^{\mathrm{perv}}_{\mathrm{reg,hol}}(X)$ denotes the category of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology satisfying

$$\operatorname{codim} \operatorname{supp} \mathcal{H}^n(\mathcal{F}) \geq n \text{ for } n \geq 0 \text{ and } \mathcal{H}^n_Z(\mathcal{F}) = 0 \text{ for } n < \operatorname{codim} Z.$$

And $\mathrm{Cons}(X^{\mathrm{an}})$ denotes the category of constructible sheaves of $\mathbf{C}$-vector spaces on X^{an}.

PROOF.

Recent result from Kashiwara ([Kashiwara 2]). □

The infinitesimal site

Grothendieck introduced in [Grothendieck] the infinitesimal site $\mathrm{Inf}(X/\mathbf{C})$ of a complex algebraic variety.
This is the category of thickenings $U \hookrightarrow T$ of open subsets of X (i.e. locally nilpotent immersions) endowed with the Zariski topology.
A sheaf E is given by a compatible family of sheaves E_T on each thickening $U \hookrightarrow T$ (its realizations).
For example, the structural sheaf $\mathcal{O}_{X/\mathbf{C}}$ corresponds to the family $\{\mathcal{O}_T\}_{U \subset T}$.
An $\mathcal{O}_{X/\mathbf{C}}$-module E is called a crystal if $u^* E_T = E_{T'}$ whenever $u : T' \to T$ is a morphism of thickenings.
For example, a finitely presented $\mathcal{O}_{X/\mathbf{C}}$-module is a crystal with coherent realizations.

Finitely presented crystals

Theorem (finite Grothendieck correspondence)

When X is a smooth algebraic variety over $\mathbf{C}$, there is an equivalence

$$\begin{aligned} \mathrm{Mod_{fp}}(X/\mathbf{C}) &\xrightarrow{\simeq} MIC(X) \\ E &\longmapsto E_X \end{aligned}$$

Here $\mathrm{Mod_{fp}}(X/\mathbf{C})$ denotes the category of finitely presented $\mathcal{O}_{X/\mathbf{C}}$-modules.

Proof.

Since X is smooth, any thickening $U \hookrightarrow T$ has locally a section $s : T \to U$ and we set $E_T = s^*\mathcal{F}_{|U}$. Then, use the Taylor isomorphism to show that it is a crystal. $\square$

THEOREM (G-R-H CORRESPONDENCE)

If X is a smooth complex algebraic variety, we there is an equivalence

$$\begin{aligned} \mathrm{Mod}_{\mathrm{fp,reg}}(X/\mathbf{C}) &\xrightarrow{\simeq} LOC(X^{\mathrm{an}}) \\ E &\longmapsto \mathcal{H}om_{\nabla}(E_X, \mathcal{O}_X) \end{aligned}$$

$\mathrm{Mod}_{\mathrm{fp,reg}}(X/S)$ denotes the category of finitely presented $\mathcal{O}_{X/\mathbf{C}}$-module that give rise to a regular connection on X/S.

PROOF.

This is the composition of Grothendieck's equivalence and Riemann-Hilbert. □

Constructible crystals

Theorem (Deligne correspondence)

If X is a smooth algebraic variety, we have

$$\begin{aligned} \mathrm{Cons}_{\mathrm{reg}}(X/\mathbf{C}) &\xrightarrow{\simeq} \mathrm{Cons}(X^{\mathrm{an}}) \\ E &\longmapsto \mathcal{H}om_{\nabla}(E_X, \mathcal{O}_X) \end{aligned}$$

Here $\mathrm{Cons}_{\mathrm{reg}}(X/\mathbf{C})$ denotes the category of constructible pro-coherent crystals on $X/\mathbf{C}$ whose definition is left to the imagination of the reader.

Proof.

Proved by Deligne in an unpublished note called "Cristaux discontinus". He describes an explicit quasi-inverse. $\square$

Deligne-Kashiwara correspondence

Theorem (Deligne-Kashiwara correspondence)

If X is a smooth algebraic variety over $\mathbf{C}$, we have

$$\mathrm{Cons}_{\mathrm{reg}}(X/\mathbf{C}) \simeq D^{\mathrm{perv}}_{\mathrm{reg,hol}}(X).$$

Proof.

Composition of Deligne and Kashiwara correspondences. □

It would be interesting to give an algebraic proof of this equivalence; and derive Deligne's theorem from Kashiwara's. We quickly sketch how this could be done.

Actually, the above equivalence between finitely presented $\mathcal{O}_{X/\mathbf{C}}$-modules and coherent modules with integrable connection comes from a more general correspondence:

THEOREM (GROTHENDIECK'S CORRESPONDENCE)

If X is a smooth algebraic variety over $\mathbf{C}$, we have

$$\begin{aligned} \mathrm{Cris}(X/\mathbf{C}) &\xrightarrow{\simeq} \mathcal{D}_X - \mathrm{Mod} \\ E &\longmapsto E_X \end{aligned}$$

PROOF.

Exactly as before. □

Theorem (Berthelot's correspondence)

If X is a smooth algebraic variety over $\mathbf{C}$, we have

$$D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_X) \xrightarrow{\simeq} D^{\mathrm{b,crys}}_{\mathrm{qc}}(\mathcal{O}_{X/\mathbf{C}})$$

Here, $D^{\mathrm{b}}_{\mathrm{qc}}(\mathcal{D}_X)$ denotes the category of bounded complexes of $\mathcal{D}_X$-modules with quasi-coherent cohomology. $D^{\mathrm{b,crys}}_{\mathrm{qc}}(\mathcal{O}_{X/\mathbf{C}})$ is the category of crystalline bounded complexes of $\mathcal{O}_{X/\mathbf{C}}$-modules that are quasi-coherent on thickenings. A complex E of $\mathcal{O}_{X/\mathbf{C}}$-modules is said to be crystalline if $Lu^*E_T = E_{T'}$ whenever $u : T' \to T$ is a morphism of thickenings.

Sketch of proof

Proof.

The proof is sketched in [Berthelot]. We first consider the left exact and fully faithful functor

$$C_X : \mathcal{D}_X - \mathrm{Mod} \simeq \mathrm{Cris}(X/\mathbf{C}) \hookrightarrow \mathcal{O}_{X/\mathbf{C}} - \mathrm{Mod}$$

and derive it in order to get

$$CR_X := LC_X[d_X] : D^-(\mathcal{D}_X) \to D^-(\mathcal{O}_{X/\mathbf{C}}).$$

The next point is to study the behavior of local hom under this functor. $\square$

Note that the theory works in a very general situation (log scheme in any characteristic $p \geq 0$).

The arithmetic case

Assume now that K is a complete ultrametric field of characteristic 0, with valuation ring $\mathcal{V}$ and residue field k (of positive characteristic p).

We want to replace $\mathcal{D}$-modules with $\mathcal{D}^\dagger$-modules and the infinitesimal site with the overconvergent site (see [Le Stum 1] and [Le Stum 2]).

Let us be more explicit:

We assume that we are given a locally closed embedding $X \hookrightarrow P$ of an algebraic k-variety over into a formal $\mathcal{V}$-scheme. We assume that P is smooth (in the neighborhood of X) and that the locus at infinity $\infty_X := \overline{X} \setminus X$ has the form $T \cap \overline{X}$ where T is a divisor on P.

Then, we may consider the category of $\mathcal{D}^\dagger_P({}^\dagger T)_{\mathbf{Q}}$-modules with support on X. On the other hand, we may consider the small overconvergent site $\mathrm{an}^\dagger(X_P/K)$ that we will describe now.

The overconvergent site

The objects are (small) overconvergent varieties over X_P/K made of a locally closed embedding $X \hookrightarrow Q$ into a formal scheme Q over P and a (good) open subset V of Q_K.
Recall that Q_K is the generic fiber of Q which is a Berkovich analytic variety and that there is a specialization map $\mathrm{sp} : Q_K \to Q$. We will denote by $]X[_V$ the analytic domain of points in V that specialize to X and by $i_X :]X[_V \hookrightarrow V$ the inclusion map.
A morphism between overconvergent varieties is simply a morphism $u : V' \dashrightarrow V$ defined on some neighborhood of the tube that is compatible with specialization. The topology is induced by the analytic topology. A sheaf E is given by a compatible family of sheaves E_V on $]X[_V$ for each overconvergent variety V over X_P.
For example, we will consider the structural sheaf $\mathcal{O}^\dagger_{X_P/K}$ whose realization on V is $i_X^{-1}\mathcal{O}_V$.

Overconvergent isocrystals

Theorem

With the above notations, there is an equivalence

$$\begin{aligned} \mathrm{Mod}^{\dagger}_{\mathrm{fp}}(X_P/K) &\xrightarrow{\simeq} \mathrm{MIC}^{\dagger}(X \subset P/K) \\ E &\longmapsto E_{P_K} \end{aligned}$$

$\mathrm{Mod}^{\dagger}_{\mathrm{fp}}(X_P/K)$ denotes the category of finitely presented $\mathcal{O}^{\dagger}_{X_P/K}$-modules. $\mathrm{MIC}^{\dagger}(X \subset P/K)$ is the category of overconvergent isocrystals on $X \subset P/K)$ (coherent $i_X^{-1}\mathcal{O}_{P_K}$-modules with an integrable connection whose Taylor series converges on a neighborhood of the diagonal).

Proof.

Analogous to Grothendieck's proof. $\square$

The specialization functor

Theorem (Berthelot-Caro)

With the above notations, when $\overline{X}$ is smooth, there is a fully faithful functor

$$\begin{aligned} \mathrm{MIC}^{\dagger}(X \subset P/K) &\longrightarrow D^{\mathrm{b}}_{\mathrm{coh}}(X \subset P) \\ E &\longmapsto \mathrm{sp}_{+} E_{P_K}. \end{aligned}$$

$D^{\mathrm{b}}_{\mathrm{coh}}(X \subset P)$ denotes the category of bounded complexes of $\mathcal{D}^{\dagger}_{P}({}^{\dagger}T)_{\mathbf{Q}}$ with support in X and coherent cohomology.

Proof.

Stated and proved in [Caro] by Daniel Caro. $\square$

A first step

According Caro, the smoothness condition on $\overline{X}$ in the previous result can be removed.

Theorem

With the above notations, there is a fully faithful functor

$$\begin{aligned} \mathrm{Mod}^{\dagger}_{\mathrm{fp}}(X_P/K) &\longrightarrow D^{\mathrm{b}}_{\mathrm{coh}}(X \subset P) \\ E &\longmapsto \mathrm{sp}_+ E_{P_K} \end{aligned}$$

Proof.

It is sufficient to compose Caro's functor on the left with our equivalence above. □

What do we expect now ?

We want to extend specialization to a functor

$$\begin{aligned} \mathrm{Cons}^\dagger(X_P/K) &\longrightarrow D^b_{\mathrm{coh}}(X \subset P) \\ E &\longmapsto \mathrm{sp}_+ E_{P_K}. \end{aligned}$$

Here, $\mathrm{Cons}^\dagger(X_P/K)$ denotes the category of constructible overconvergent crystals, defined as one may think on the overconvergent site.
Ultimately, we are looking for an overconvergent Deligne-Kashiwara correspondence

$$\mathrm{Cons}^\dagger_{\mathrm{reg}}(X_P/K) \xrightarrow{\simeq} D^{\mathrm{perv}}_{\mathrm{reg,hol}}(X \subset P).$$

The Frobenius version should be more tractable:

$$F - \mathrm{Cons}^\dagger(X_P/K) \xrightarrow{\simeq} F - D^{\mathrm{perv}}_{\mathrm{hol}}(X \subset P)$$

with perversity defined as above.

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