Fast reduction in the algebraic de Rham cohomology of projective hypersurfaces

Sebastian Pancratz

University of Oxford

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Outline

Introduction

The aim of this talk is to describe a fast reduction procedure in the de Rham cohomology of (families of) smooth projective hypersurfaces, leading to a *practical* improvement in the computation of Gauss–Manin connections.

Example

Consider the family of projective hypersurfaces over $\mathbf{Q}$ given by

$$P(W, X, Y, Z) = W^4 + X^4 + Y^4 + Z^4 + t(WX^3 + W^3Y + W^3Z + WX^2Y).$$

Introduction

Remark

While the above example concerns a family of projective hypersurfaces containing a diagonal fibre, the techniques used to obtain a computational improvement are more naturally explained in the case of a (diagonal) projective hypersurface.

Notation

Let X be a smooth hypersurface in $\mathbf{P}^n(K)$, where K is a field of characteristic zero, defined by a homogeneous polynomial $P \in K[x_0, \dots, x_n]$ of degree d, and let $U = \mathbf{P}^n(K) - X$. Moreover, assume that $n \geq 2$, and that $d \geq 2$ whenever n is odd and $d \geq 3$ whenever n is even.

Introduction

For $i \geq 0$, let $H^i_{dR}(X/K)$ denote the ith algebraic de Rham cohomology vector space of X over K. We now follow the explicit description in Abbott–Kedlaya–Roe (2006; §3):

- For $0 \leq i \leq 2n$ with $i \neq n-1$,

$$\dim_K H^i_{dR}(X/K) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

  Thus, $H^{n-1}_{dR}(X/K)$ is the only cohomology group that remains to be computed.
- Using exact sequences from Griffiths (1969; (10.16)), one can shift attention to $H^n_{dR}(U/K)$.

Introduction

- Defining the n-form

$$\Omega = \sum_{i=0}^{n} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

it can be shown as in Griffiths (1969; §4) that $H^n_{dR}(U/K)$ is isomorphic as a K-vector space to the quotient of the group of n-forms $Q\Omega/P^k$ with $k \in \mathbf{N}$ and $Q \in K[x_0, \ldots, x_n]$ homogeneous of degree $kd - (n+1)$ by the subgroup generated by

$$\frac{(\partial_i Q)\Omega}{P^k} - k \frac{Q(\partial_i P)\Omega}{P^{k+1}},$$

for all $0 \leq i \leq n$.

Reduction of poles

Now $H^n_{dR}(U/K)$ can be equipped with a filtration whose ith part consists of all $Q\Omega/P^k$ with $\deg Q = kd - (n+1)$ and $1 \leq k \leq i+1$. We obtain a basis respecting this filtration as follows:

- For $k \in \mathbf{N}$, we find a basis B_k of polynomials of degree $kd - (n+1)$ for the quotient of the space of all such polynomials by the Jacobian ideal $(\partial_0 P, \ldots, \partial_n P)$.
- This yields a basis $\bigcup_{k\in\mathbf{N}} \mathcal{B}_k$ for $H^n_{dR}(U/K)$, where $\mathcal{B}_k = \{Q\Omega/P^k : Q \in B_k\}$. By a theorem of Macaulay (see Griffiths (1969; (4.11))), the set $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ already forms a basis.

Reduction of poles

To obtain a representative for the class of $Q\Omega/P^k$ in terms of elements of the above basis elements, we first express Q in the form

$$Q = Q_0 \partial_0 P + \cdots + Q_n \partial_n P + \gamma_k$$

where $Q_0, \ldots, Q_n$ are homogeneous polynomials in $K[x_0, \ldots, x_n]$ and γ_k is in the K-span of B_k. Continuing iteratively with the element

$$(k-1)^{-1} \Big(\sum_{i=0}^{n} \partial_i Q_i \Big) \Omega / P^{k-1},$$

we eventually obtain an expression for $Q\Omega/P^k$ as a sum of the form

$$\gamma_1 \Omega / P^1 + \cdots + \gamma_k \Omega / P^k$$

with γ_i in the K-span of B_i for all $1 \leq i \leq k$.

Co-ordinates in the Jacobian ideal

Problem

Given a homogeneous polynomial $Q \in K[x_0, \ldots, x_n]$ of degree $kd - (n+1)$ for some $k \in \mathbf{N}$, we try to find homogeneous polynomials $Q_0, \ldots, Q_n$ in $K[x_0, \ldots, x_n]$ such that

$$Q = Q_0 \partial_0 P + \cdots + Q_n \partial_n P + \gamma_k,$$

where $Q_0, \ldots, Q_n$ are homogeneous polynomials, necessarily zero or of degree $(k-1)d - n$, and γ_k is in the K-span of B_k.

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Remark

Some recommendations in the literature at this step suggest computations relying on a Gröbner basis computation. This has negative implications, both practical (in terms of the run-time) and theoretical (for a meaningful complexity analysis).

Co-ordinates in the Jacobian ideal

Definition

For $k \in \mathbf{N}$, let

$$B_k = \{x^i : \deg(x^i) = kd - (n+1) \text{ and } i_j < d - 1 \text{ for } 0 \leq j \leq n\},$$

where $i \in \mathbf{N}^{n+1}$ and $x^i = x_0^{i_0} \cdots x_n^{i_n}$. Also, $\mathcal{B}_k = \{x^i \Omega / P^k : x^i \in B_k\}$. Then the corresponding set $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ forms a basis of $H^n_{dR}(U/K)$.

Remark

The above problem is now to find the co-ordinates of $Q - \gamma_k$ in the ideal $(\partial_0 P, \dots, \partial_n P)$, letting γ_k be the sum of all monomial terms in Q with monomials in B_k.

Our approach is based on a generalisation of Sylvester matrices from two to $n+1$ polynomials, following Macaulay (1916; republished 1994).

Co-ordinates in the Jacobian ideal

For diagonal hypersurfaces X, we can further restrict the polynomials $Q_0, \ldots, Q_n$ that we seek.

Problem

Given a homogeneous polynomial $Q \in K[x_0, \ldots, x_n]$ of degree $kd - (n+1)$ for some $k \in \mathbf{N}$, we try to find homogeneous polynomials $Q_0, \ldots, Q_n$ in $K[x_0, \ldots, x_n]$ such that

$$Q \equiv Q_0 \partial_0 P + \cdots + Q_n \partial_n P$$

modulo the K-span of B_k. Moreover, for each $1 \leq j \leq n$, the polynomial Q_j may only contain non-zero coefficients for monomials of degree $(k-1)d - n$ that are not divisible by any of the monomials $x_0^{d-1}, \ldots, x_{j-1}^{d-1}$.

Co-ordinates in the Jacobian ideal

Definition

For $k \in \mathbf{N}$, define sets of monomials

$$\mathcal{R}_k = \{x^i : \deg(x^i) = kd - (n+1) \text{ and } \exists j \;\; i_j \geq d-1\},$$
$$\mathcal{C}_k^{(j)} = \{x^i : \deg(x^i) = (k-1)d - n \text{ and } i_0, \dots, i_{j-1} < d-1\},$$

for $j = 0, \dots, n$.

Theorem

Suppose that X is diagonal and let $k \in \mathbf{N}$. For $0 \leq j \leq n$, let $V_k^{(j)}$ be the K-vector space with basis $\mathcal{C}_k^{(j)}$ and let $V_k = V_k^{(0)} \times \cdots \times V_k^{(n)}$. Let W_k be the K-vector space with basis $\mathcal{R}_k$. Then

$$\phi_k \colon V_k \to W_k, (Q_0, \dots, Q_n) \mapsto Q_0 \partial_0 P + \cdots + Q_n \partial_n P$$

is an isomorphism.

Co-ordinates in the Jacobian ideal

Remark

The proof is an explicit computation matrix representing ϕ_k and its determinant. In particular, it generalises to the case of families of smooth projective hypersurfaces containing a diagonal fibre via specialisation to this fibre.

Sparse linear algebra

The above decomposition problem of finding polynomials $Q_0, \ldots, Q_n$, given a representative $Q\Omega/P^k$, such that

$$Q \equiv Q_0 \partial_0 P + \cdots + Q_n \partial_n P$$

modulo the K-span of B_k can thus be treated as a linear algebra problem: Let w be the vector of $Q - \gamma_k$ in W_k, let $v = (v_0, \ldots, v_n)$ denote the vector of $(Q_0, \ldots, Q_n)$ in V_k. If A is the matrix of ϕ_k w.r.t. the earlier choice of bases, then the above problem is precisely that of solving

$$Av = w.$$

Sparse linear algebra

To take advantage of the (typical) sparsity of the matrix A, we can use methods of Duff (1981) and Duff & Reid (1978).

1. First, find a permutation P such that PA has a zero-free diagonal. Then find another permutation Q such that $QPAQ^t$ is block lower triangular,

$$QPAQ^t = \begin{pmatrix} A^{(11)} & & & \\ A^{(21)} & A^{(22)} & & \\ \vdots & & \ddots & \\ A^{(N1)} & A^{(N2)} & \dots & A^{(NN)} \end{pmatrix},$$

   where each $A^{(kk)}$ square and can itself not be symmetrically permuted to block lower triangular form.

2. Thus, we solve the (typically much smaller) linear systems with matrices $A^{(kk)}$ for $k = 1, \dots, N$, using e.g. sparse LUP-decomposition.

Complexity

- The computation and pre-processing of the matrices for ϕ_k, $k = 2, \ldots, n+1$, is dominated by the LUP-decomposition.
- The LUP-decomposition can be arranged to require $\mathcal{O}\big(M\big(\binom{kd-1}{n}\big)\big)$ arithmetic operations in the base field, where $M(-)$ is the complexity of matrix multiplication.
  Since when computing Gauss–Manin connection matrices we may assume that $k \leq n+1$, this is in $\mathcal{O}(M((de)^n))$ where $e = \sum_{j=0}^{\infty} 1/j!$.
- In order to reduce a representative $Q\Omega/P^k$ with $k \leq n+1$, we have to solve at most one linear system at each step, corresponding to $\phi_{n+1}, \ldots, \phi_2$. Since we assume the matrices to be LUP-decomposed, this can then each be done in quadratic time, amounting to a total of $\mathcal{O}(n(de)^{2n})$ arithmetic operations.

Examples

- For the earlier example,

$$P(W,X,Y,Z) = W^4 + X^4 + Y^4 + Z^4 \\ + t(WX^3 + W^3Y + W^3Z + WX^2Y),$$

previous code implemented by Lauder using Magma requires about 26.5 minutes and just under 100MB of memory.

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previous code implemented by Lauder using Magma requires about 26.5 minutes and just under 100MB of memory.
The new implementation requires only 12.5s and 17MB of memory.

Examples

- Consider the family

$$P(W,X,Y,Z) = W^4 + X^4 + Y^4 + Z^4 + t(-3W^3X + 5W^3Y$$
$$+ 7W^2XY - 23WX^2Y - 29X^2YZ + 31Y^2Z^2 - 37WXYZ).$$

Here, the previous implementation requires 34 days and 12.5GB of memory, whereas the new implementation takes 530s and 127MB.

Examples

- The following example begins to show the limitations of the new approach,

$$P(W,X,Y,Z) = (1-t)(W^5+X^5+Y^5+Z^5) \\ + t\big(WXZ+Y^3)(W^2+XY+Z^2)+X^5+Z^5-3W^4X\big)$$

Here, the new implementation requires about 5.8h and 1.6GB. The Gauss–Manin connection matrix is a 52×52 matrix over $\mathbf{Q}(t)$ with numerators and denominators in $\mathbf{Z}[t]$ of degrees up to 461 and coefficients of size up to 360 decimal digits.

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