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Hypergeometric motives

$X_\psi : x_1^5 + \dots + x_5^5 - 5\psi x_1 \cdots x_5 = 0$, Calabi-Yau 3-fold

$H^3(X_\psi)^A \qquad A = \{(\zeta_1, \dots \zeta_5) \mid \zeta_i^5 = 1, \ \zeta_1 \cdots \zeta_5 = 1\}$

dim 4

Take $\psi \in \mathbb{Q} \setminus \{0, 1\}$

$$\sum_{n \geq 0} \frac{(5n)!}{n!^5} (5\psi)^{-5n} \quad \text{around } \psi = \infty$$

Hypergeometric equation

$$_4F_3\left(\begin{matrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1 \end{matrix} \,\middle|\, \psi^{-5} \right)$$

V space of local solutions to the diff. equation, $t = \psi^{-5}$

has regular singularities at $t = 0, 1, \infty$.

$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \longrightarrow GL(V)$ monodromy repn.

This is <u>rigid</u> : i.e. has local monodromies

$0 \qquad\qquad 1 \qquad\qquad \infty$

$$\gamma_0 = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \gamma_\infty = \begin{pmatrix} \zeta_5 & & & \\ & \zeta_5^2 & & \\ & & \zeta_5^3 & \\ & & & \zeta_5^4 \end{pmatrix}$$

$\chi_{\gamma_0}(\gamma_0) = m(\gamma_0)$ $\qquad\qquad$ $\chi(\gamma_\infty) = m(\gamma_\infty)$

$q_0 := (T-1)^4 \qquad\qquad q_\infty := T^4 + \dots + T + 1$

transvection or reflection : $\operatorname{rk}(\gamma_1 - I) = 1$

and these determine the <u>whole</u> of the representation.

L-functions

$$\begin{array}{ccc} \text{Autom. forms} & \longleftrightarrow \ \text{motives} \ \longleftrightarrow & \text{pieces of cohom. of alg. varieties} \\ & \rightsquigarrow \text{L-function} & \end{array}$$

Expect: Start with q_0, q_∞

$$\frac{q_\infty}{q_0} = \prod_{\nu \geq 1} (1 - T^\nu)^{\gamma_\nu} \quad , \quad \gamma_\nu \in \mathbb{Z}$$

$$\sum_{\nu \geq 1} \gamma_\nu = 0$$

eg quintic : $\dfrac{T^5 - 1}{(T-1)^5}$

$$\gamma := \sum_{\nu \geq 1} \gamma_\nu T^\nu$$

$$\gamma = T^5 - 5T$$

Obtain a family of motives over $\mathbb{Q}$

V_t , $t \in \mathbb{Q}$, each of pure weight w

Landau function

$$\mathcal{L}(x) := - \sum_{\nu \geq 1} \gamma_\nu \{\nu x\} \quad , x \in \mathbb{R} \, , \quad \{\cdot\} = \text{fractional part}$$

- periodic function of period 1
- locally constant
- right continuous

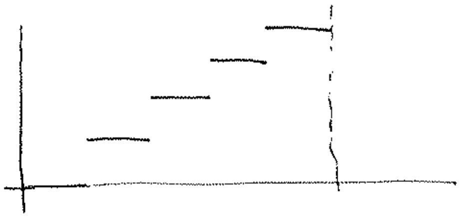
$$w + 1 = \max_x \mathcal{L}(x) - \min_x \mathcal{L}(x)$$

If $u_n = \prod_{\nu \geq 1} (\nu n)!^{\gamma_\nu}$, then $v_p(u_n) = \sum_{k \geq 1} \mathcal{L}\left(\frac{n}{p^k}\right)$

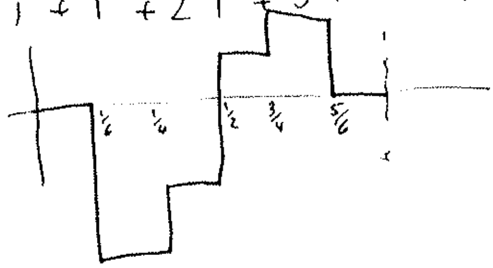
Landau proved: $u_n \in \mathbb{Z}$ for all n $\iff$ $\mathcal{L}(x) \geq 0$

[Chebyshev : $\frac{(30n)!\,n!}{(15n)!\,(10n)!\,(6n)!} \in \mathbb{Z}$]

eg 1) quintic



2) $\gamma = -2T^6 + T^4 + 2T^3 + 3T^2 - 4T$



Conj : (Conti - Gorbyshev)

Hodge polynomial

$$\sum_{i=0}^{w} h^{i,w-i} T^i = T^\delta \sum_x T^{\mathcal{L}^-(x)} [\ell(x)]$$

where $\mathcal{L}^-(x) = \lim_{y \to x^-} \mathcal{L}(y)$ $\qquad [n] := \begin{cases} 1+T+\dots+T^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

$\ell(x) = \mathcal{L}(x) - \mathcal{L}^-(x)$

NB. egs 1) & 2) have same Hodge numbers

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Good primes : Hypergeometric sum defined by Katz ($p \nmid \mathrm{denom}(x)\ \gamma_v \neq 0$, $t, t-1, t^{-1}$)

$t=1$ $\qquad \begin{pmatrix} 1\,1 & & \\ 0\,1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\qquad$ Assume ~~inertia~~ monodromy is inertia

$p \mid t-1$ $\qquad \Rightarrow$ fixed space : $2 + 1$

wts : $3 \quad 2$

$\therefore$ Euler factor $(1 - a_p T + p^3 T^2)(1 - \chi(p) p T)$

What character is χ ?

Appears to be a quadratic character associated to $\sqrt{(-1)^{d/2} D}$ where $u_n = \prod_{\nu \geq 1} (n\nu)!^{\gamma_\nu} \sim \frac{\sqrt{D}}{(2\pi n)^{d/2}} K^n$